Vagueness and many-valued logic: part 2

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September 14, 2005

Matters arising

1.1 The argument from last week, spelled out somewhat more rigorously

(1)	$T\ulcorner\forall s_1\forall s_2(Ts_1\wedge\sim Ts_2)\rightarrow\sim T\ulcorner s_1\rightarrow s_2\urcorner\urcorner$	Theory
(2)	$\forall s_1\forall s_2(T\ulcorner s_1\rightarrow s_2\urcorner\wedge\sim T\ulcorner\sim s_1\urcorner)\rightarrow\sim T\ulcorner\sim s_2\urcorner$	Theory
(3)	$\forall s(\sim T\wedge\sim T\ulcorner\sim s\urcorner\rightarrow\sim T\ulcorner\sim(s\wedge\sim s)\urcorner)$	Theory
(4)	$\sim T\ulcorner Tp\urcorner\wedge\sim T\ulcorner\sim Tp\urcorner$	Premise
(5)	$\sim T\ulcorner\sim(Tp\wedge\sim Tp)\urcorner$	4, 3
(6)	$T\ulcorner(Tp\wedge\sim Tp)\rightarrow\sim T\ulcorner p\rightarrow p\urcorner\urcorner$	4, semantics for $\forall$
(7)	$T\ulcorner(Tp\wedge\sim Tp)\rightarrow\sim T\ulcorner p\rightarrow p\urcorner\urcorner\wedge\sim T\ulcorner\sim(Tp\wedge\sim Tp)\urcorner$	5,6
(8)	$\sim T\ulcorner\sim\sim T\ulcorner p\rightarrow p\urcorner\urcorner$	1,7
(9)	$\sim T\ulcorner T\ulcorner p\rightarrow p\urcorner\urcorner$	8, semantics for $\sim$

2 Degrees of truth

2.1 Motivation

We do occasionally speak of one sentence as 'truer' than another, although it's not clear how we think of this as working.

- Test: rate the following sentences in order from truest to least true: 'A 6-foot 6 person is tall'; "A 7-foot person is tall'; '1+1=2'.

The appeal of talking about degrees of truth in the context of vagueness is that it fits with our sense that *in all really important respects*, each sentence in the Sorites is very similar to its successor.

How are degrees of truth supposed to fit with belief or assertion? If I'm wondering whether to assert or believe 'P', how are my beliefs about the degree of truth of 'P' supposed to bear on my decision?

2.2 How does 'true to degree n' relate to 'borderline'?

Two competing answers:

(i) 'Borderline' means 'not true to degree 1 and not true to degree 0'.

(ii) 'Borderline' is vague (even if we ignore any vagueness in 'true to degree n'). The degree to which 'S is borderline' is true is some interesting function f of the degree to which S is true. Presumably f is a smooth function such that $f(0) = f(1) = 0$, which takes larger values in the middle—perhaps even $f(\frac{1}{2}) = 1$.

— 'So what is it for a sentence to be borderline?' Vagueness often seems to be an obstacle to giving definitions/analyses: it seems that we are not in a position to say what it is to be bald, for example. We can't do so in precise terms, and it's not clear how vague terms would help.

Perhaps the lesson to be drawn from this is that requests for definitions of vague predicates are generally misconceived. To give understanding, we should instead give some other sort of account of the predicate—capturing some centrally important thing about its use.

The continuum-valued theorist has a natural suggestion about how to do this: we convey understanding by saying what it takes for the sentence to have a given degree of truth.

2.3 How does 'true to degree *n*' relate to 'true' and 'false'?

Three competing answers:

(i) 'True' means 'true to degree 1'; 'false' means 'true to degree 0'. (C.f. Machina's talk of sentences with a noninteger degree of truth as 'not taking either of the classical truth values'—what are these if not truth and falsity?)

(ii) 'True' and 'false' are vague and work disquotationally: the degrees which 'S is true' is true is the same as the degree to which S is true; the degree to which 'S is false' is true is the same as the degree to which Not-S is true.

(iii) 'True' and 'false' are vague and work non-disquotationally. The degrees of truth of 'S is true' and 'S is false' are some *interesting* functions g and h of the degree of truth of S. Perhaps one should set $f + g + h = 1$, on the grounds that this (in some very non-obvious sense!) corresponds to thinking of borderlineness, truth and falsity as mutually exclusive.

2.4 Łukasiewicz logic

(i) The degree of truth of '$\sim P$' = 1 - the degree of truth of 'P'.

(ii) The degree of truth of '$P \vee Q$' = whichever is the larger of the degrees of truth of P and Q.

(iii) The degree of truth of '$P \wedge Q$' = whichever is the smaller of the degrees of truth of P and Q.

(iv) The degree of truth of '$P \rightarrow Q$ = 1 if the degree of truth of P is not less than the degree of truth of Q; otherwise = 1 - (the amount by which the degree of truth of P exceeds the degree of truth of Q) [or equivalently: the degree of truth of $\sim P$ + the degree of truth of Q]

- This is the clause that makes the system technically interesting to logicians.
- Compare the definition of the "Łukasiewicz conditional" from three-valued logic.
- Note that the degree of truth of '$P \rightarrow Q$' can never be less than that of the material conditional '$\sim P \vee Q$'.
- Note also that the degree of truth of '$P \rightarrow (Q \rightarrow R)$' can be greater than that of '$(P \wedge Q) \rightarrow R$'.

2.5 Validity

Many different conceptions:

(i) Validity as "preservation" of degree 1.

- As we already saw in the context of 3-valued logic, claims about 'preservation' like this one can be spelled out either using conditionals or using conjunction/disjunction; these may come apart once we entertain higher-order vagueness. In the present case, the definitions would be as follows:
  (a) For an argument to be valid is for it to be the case that it can never happen that all its premises are true to degree 1 and its conclusion is not true to degree 1.
  (b) For an arguement to be valid is for it to be necessary that if its premises are all true to degree 1, its conclusion is true to degree 1.

(ii) Validity as preservation of degree above some chosen threshold.

- Valid arguments can't be chained together!

(iii) Validity as preservation of minimum degree (Machina)

- What does that mean? (a) An argument is valid iff its conclusion can't be less true than each of its premises / (b) an argument is valid iff for all n, if each of its premises is true to degree at least n, its conclusion is true to degree at least n.
- Modus ponens not valid!

(iv) Validity as a vague notion (even ignoring any vagueness in 'true to degree n'). A sentence attributing validity to an argument is true to degree at least $1-n$ iff its conclusion can't be less true by more than n than each of its premises.

- 'Modus ponens is valid' is true only to degree 0.5!

(v) (Edgington:) An argument is valid iff the degree of falsity of its conclusion can't exceed the sum of the degrees of falsity of all its premises. (Wild!)

How do these connect to the standard definition of validity as preservation of truth? This is expressed by (i) if truth is degree-1. It is expressed by (iv) if we understand truth disquotationally, and understand 'the argument preserves truth' using the conditional ('necessarily, if all the premises of the argument are true, the conclusion is').

- Could a valid argument have only true premises and an untrue conclusion? If 'true' means 'true to degree 1', (i) and (iii) answer 'no' (bracketing worries about higher-order vagueness), while on (iv) the answer 'yes' is true to a very high degree, perhaps 1. If 'true' is disquotational, the answer 'yes' is true to a very high degree on (i), and to a degree close to 0.5 on (iii) and (iv). [Check!]
- Could a valid argument have only true premises and a false conclusion? Same answer if 'true' and 'false' are disquotational; 'no' across the board if they mean 'true to degree 1' and 'true to degree 0'.
- Are all valid arguments such that if all their premises are true, their conclusion is also true? If 'true' means 'true to degree 1', (i) and (iii) answer 'yes' (bracketing higher-order vagueness), while on (iv) the answer 'no' is true to a very high degree. And if

'true' is disquotational. . . ?

2.6 Quantification

2.7 The Sorites

Disjunction / conjunction versions: premises start off very true, go down to about half true, go back up to very true; conclusion is very false.

Conditional version: premises are all very true, though not all perfectly true; conclusion is very false (perfectly false?)

Quantified conditional version: both premises are very true, though the quantified premise is not perfectly true.

Quantified conjunction/ disjunction version: quantified premise is about half true.

Are the arguments valid? Yes on conception (i) (bracketing worries about higher-order vagueness). No on conception (iii), and on most versions of conception (ii). "Only to a very small degree" on concepion (iv). . . .

Why are we puzzled? One possible explanation, relevant to the conditional versions: we mistake *almost*-perfectly-true sentences for perfectly true sentences. But why would we make such a mistake?

- And why are the premises of the negated-conjunction version so compelling to us, if they're only about half true?
- And why, if the arguments are not valid, do they strike us as valid?

2.8 The problem of penumbral connections, redux

All the same problems as the three-valued theory.

2.9 The problem of higher-order vagueness, redux

If the sorites sequence of heaps gives us reason to think that sentences of the form 'x is a heap' are sometimes true to intermediate degrees, then by parity of reasoning sorites sequences of sentences will give us reason to think that 'x is true to degree at least n' and 'x is true to degree at most n' are sometimes true to intermediate degrees.

If 'true to degree at least n' and 'true to degree and most n' are vague, how should we speak of degrees of truth? There is an analogous question about with 'at least n metres tall' and 'at most n metres tall', which are also vague (considering stoops and the elasticity of living matter). I see two possibilities.

(i) Analyse 'exactly n metres tall' as 'at least n metres tall and at most n metres tall'. Result: since there will presumably be a range of ns for which 'the tree is at least n metres tall' and 'the tree is at most n metres tall' are both roughly half-true, 'the tree is exactly n metres tall' will be roughly half-true when ns in that range. It will be less than half-true elsewhere. So 'there is an n such that the tree is exactly n metres tall' will be roughly half-true.

(ii) Analyse ‘at least n metres tall’ as ‘exactly m metres tall, for some $m \geq n$.

- Given the treatment of quantification, this will mean that even ‘the tree is at least 0 metres tall’ is only as true as the truest sentence of the form ‘the tree is exactly m metres tall’.
- How true is that? A hard question. If we say it’s low, we will have the (bad?) consequence that ‘the tree is not at least 0 metres tall’ is true to a high degree. If we say it’s high, we’ll have the (bad?) consequence that ‘there are distinct n and m such that the tree is both exactly n metres tall and exactly m metres tall’ is true to a high degree. So perhaps the best thing to say is that it’s somewhere closer to being half-true?
- In any case, proposal (i) seems preferable: surely sentences like ‘the tree is between 1 and 1000 metres tall’ should be true to a high degree.

Whichever option we take, recognising the vagueness of ‘true to degree exactly n’ and ‘true to degree at least n’ and ‘true to degree at most n’ lands the degree-theorist in trouble of a similar sort to the trouble we talked about last week in the three-valued case. Given the vagueness of these expressions, we can argue that the central tenets of the degree theory are themselves not true to degree 1. Indeed, given some further plausible claims about the extent to which these expressions are vague, we can argue that some of the central tenets of the theory are not even close to having degree 1.

Let’s take as our example the axiom governing the degrees of truth of conditionals:

(→) The degree of truth of $P \to Q$ is 1, or 1 - the degree of truth of P + the degree of truth of Q, whichever is lower’

The initial difficulty in thinking through how the degree-theory should evaluate (→) on the assumption that it contains vague vocabulary has to do with the definite descriptions. If we think of this Russell-style, as implying uniqueness, we’ll be faced with the problem that it’s not even close to being fully true that each sentence has a unique degree of truth; so if (→) carries an implication of uniqueness, it too is not even close to being fully true.

Let me propose the following substitute for (→), which seems to get across the central idea without bringing in implications of uniqueness.

(→A) For all sentences S_1 and S_2 and numbers m_1, m_2: if S_1 is true to degree at most m_1 and S_2 is true to degree at least m_2, then the conditional whose antecedent is S_1 and whose consequent is S_2 is true to degree at most $1 - m_2 + m_1$.

(→B) For all sentences S_1 and S_2 and numbers m_1, m_2: if S_1 is true to degree at least m_1 and S_2 is true to degree at most m_2, then the conditional whose antecedent is S_1 and whose consequent is S_2 is true to degree at least $min(1, 1 - m_2 + m_1)$.

Or in symbols:

(→A) $\forall S_1 \forall S_2 \forall m_1 \forall m_2 ((S_1 \leq m_1 \wedge S_2 \geq m_2) \to (\ulcorner S_1 \to S_2 \urcorner \leq 1 - m_2 + m_1))$

(→B) $\forall S_1 \forall S_2 \forall m_1 \forall m_2 ((S_1 \geq m_1 \wedge S_2 \leq m_2) \to (\ulcorner S_1 \to S_2 \urcorner \geq 1 - m_2 + m_1))$

To see what recognition of the vagueness of ‘≤’ and ‘≥’ together with acceptance of the degree theory forces us to say about these sentences, it will help to turn to a simpler example claim:

(*) For any sentence S and number n, if S is true to degree at least n, then S is true to degree at least n.

Suppose that we have some sentence S that is subject to a serious dose of higher-order vagueness, so that

(10) 'S is true to degree at least 0.7' is true to degree at least 0.4. ($\ulcorner S \geq 0.7 \urcorner \geq 0.4$)

(11) 'S is true to degree at most 0.3' is true to degree at least 0.4 ($\ulcorner S \leq 0.3 \urcorner \geq 0.4$)

Suppose further that (→A) and (→B) are true to degree at least 0.8.

(12) $\ulcorner \forall S_1 \forall S_2 \forall m_1 \forall m_2 ((S_1 \leq m_1 \wedge S_2 \geq m_2) \rightarrow (\ulcorner S_1 \rightarrow S_2 \urcorner \leq 1 - m_2 + m_1)) \urcorner \geq 0.8$

(13) $\ulcorner \forall S_1 \forall S_2 \forall m_1 \forall m_2 ((S_1 \geq m_1 \wedge S_2 \leq m_2) \rightarrow (\ulcorner S_1 \rightarrow S_2 \urcorner \geq 1 - m_2 + m_1)) \urcorner \geq 0.8$

By the principles for conjunction (set up in a style analogous to (→A) and (→B), we can infer from (10) and (11) that

(14) $\ulcorner S \geq 0.7 \wedge S \leq 0.3 \urcorner \geq 0.4$

We want to get from this to a conclusion that sets a lower limit to the degree of truth to a sentence setting an upper limit to the degree of truth of $\ulcorner S \rightarrow S \urcorner$. (→A) is of no help to us directly. But we can use the claim that (→A) is true to degree at least 0.8. By the semantics for the universal quantifier, this entails that the relevant instance of (→A) is also true to degree at least 0.8:

(15) $\ulcorner (S \leq 0.3 \wedge S \geq 0.7) \rightarrow (\ulcorner S \rightarrow S \urcorner \leq 0.6) \urcorner \geq 0.8$

Now we can apply the following principle, a contraposed version of (→A):

(16) $\forall S_1 \forall S_2 (\ulcorner S_1 \rightarrow S_2 \urcorner \geq 0.8 \wedge S_1 \geq 0.4) \rightarrow S_2 \geq 0.2)$

to conclude that

(17) $\ulcorner \ulcorner S \rightarrow S \urcorner \leq 0.6 \urcorner \geq 0.2$

The claim that $\ulcorner S \rightarrow S \urcorner$ is true only to a moderate degree is true to a fairly significant degree.

Of course this argument works even when S itself is a sentence describing the degree of truth of some other sentence.

And we can use a very similar argument to derive a similar conclusion about the universally conditional sentences that comprise the theory themselves. We can't conclude that they are not true to a high degree, but we can conclude that if they are true to a high degree, then sentences saying that they are not are themselves true to a fairly significant degree.

The precise nature of these conclusions depends on the maximum extent of higher order vagueness. If there is some S that is REALLY seriously higher-order vague, in that $\ulcorner S \geq 0.9 \urcorner$ and $\ulcorner S \geq 0.1 \urcorner$ are both true to at least degree 0.4, then that will mean that it is true to a fairly high degree that the axioms are true only to a very low degree.

What is the significance of all this? It's not clear; because it's not clear how claims about the degree of truth of a sentence are supposed to bear on the claim expressed by the sentence itself.

2.10 Higher-order vagueness and validity, redux