Vagueness and many-valued logic

Cian Dorr

September 13, 2005

1 Three-valued logic

1.1 Negation

The following is uncontroversial:

(~a) The negation of any sentence is true iff the sentence is false.

Indeed, 'is false' is often defined as 'has a true negation', which makes (~a) true by definition. The following is a bit more controversial—intuitionists refuse to accept it—but accepted by those we are talking about today:

(~b) The negation of any sentence is false iff the sentence is true.

The information presented in (~a) and (~b) can be presented in the form of a truth-table:

P	Not-P
T	F
–	–
F	T

1.2 Disjunction

The "three-valued" approach to disjunction is summed up in the following two claims:

(∨a) A disjunction is true iff either of its disjuncts is true

(∨b) A disjunction is false iff both of its disjuncts are false

(∨b) and the right-to-left direction of (∨a) are relatively uncontroversial (again leaving intuitionism aside). An argument for the left-to-right direction of (∨a): if a disjunction could ever be true without having a true disjunct, this would happen for 'the patch is red or orange' or 'the patch is red or not red', when the patch in question is borderline red-orange. But these are very strange things to assert about a borderline red-orange patch! They seem not to be true (or at least not to be determinately true).

(∨a) and (∨b) can be summed up in the following truth-table:

		Q		
	$P \vee Q$	T	–	F
	T	T	T	T
P	–	T	–	–
	F	T	–	F

- But note that if we drop the assumption that every sentence is either true, false, or neither, the truth-table doesn't contain all the information that (∨a) contains. See section **??** below for more discussion of what happens when we drop this assumption.

1.3 Conjunction

Here the three-valued approach is characterised by the following claims:

($\wedge$a) A conjunction is true iff both of its disjuncts is true.

($\wedge$b) A conjunction is false iff one of its conjuncts is false.

($\wedge$a) and the right-to-left direction of ($\wedge$b) are (relatively) uncontroversial. To defend the left-to-right direction of ($\wedge$b), one can appeal to one of De Morgan's laws, which says that '$\sim (P \wedge Q)$' is logically equivalent to '$\sim P \vee \sim Q$'. Presumably that means that if the former is true, the latter is. So by ($\vee$a), either $\sim P$ is true or $\sim Q$ is. So by ($\sim$a), either P is false or Q is.

($\wedge$a) and ($\wedge$b) can (with the same caveat as for disjunction) be summed up in a truth-table:

	$P \wedge Q$	Q: T	$-$	F
	T	T	$-$	F
P	$-$	$-$	$-$	F
	F	F	F	F

1.4 'If' and 'Iff'

Candidate three-valued truth-tables for 'If P then Q':

	$P \supset Q$	Q: T	$-$	F
	T	T	$-$	F
P	$-$	T	$-$	$-$
	F	T	T	T

Material conditional

	$P \rightarrow Q$	Q: T	$-$	F
	T	T	$-$	F
P	$-$	T	T	$-$
	F	T	T	T

"Łukasiewicz" conditional

	$P \Rightarrow Q$	Q: T	$-$	F
	T	T	F	F
P	$-$	T	T	F
	F	T	T	T

"Strong" conditional

Corresponding accounts of 'P if and only if Q':

	$P \equiv Q$	Q: T	$-$	F
	T	T	$-$	F
P	$-$	$-$	$-$	$-$
	F	T	$-$	T

Material biconditional

	$P \leftrightarrow Q$	Q: T	$-$	F
	T	T	$-$	F
P	$-$	$-$	T	$-$
	F	F	$-$	T

"Łukasiewicz" biconditional

	$P \Leftrightarrow Q$	Q: T	$-$	F
	T	T	F	F
P	$-$	F	T	F
	F	F	F	T

"Strong" biconditional

All three conditionals coincide with the material conditional (Either not-P or Q) when both antecedent and consequent are either true or false. Since, even leaving vagueness out of account, it's already a minority position that the English 'if…then' is truth-functional, why bother?

- Translating restricted universal quantifiers. Even if the conditional we study in logic is no good as an account of 'if then', '$\forall x Fx \leftarrow Gx$' might be just right as an account of 'All Fs are Gs'.

- Maybe it would be useful to study the logic of a conditional '$\leftarrow$' for which 'Necessarily $P \leftarrow Q$' is equivalent in some sense to 'Necessarily if P then Q', for some reading of 'Necessarily'.

1.5 Validity

There are several possible definitions of validity that would coincide under the assumption that every sentence is true or false, but that come apart when we allow for sentences that are neither.

(i) Validity as preservation of truth. (a) 'For an argument to be valid is for it to be impossible for its premises to be true while its conclusion is not true.' (b) 'For an argument to be valid is for it to be necessary that if its premises are true, its conclusion is true.'

- Contraposition fails: an argument from P (and possibly some other premises) to Q can be valid while the argument from not-Q (and the other premises) to not-P is not valid.
- *Reductio ad absurdum* fails. It can happen that an argument from P and some other premises to a contradiction is valid, while the argument from those premises to not-P is not valid. (Take P to be the conjunction of some necessarily-borderline sentence with its negation.)

(ii) Validity as preservation of lack of falsity. (a) 'For an argument to be valid is for it to be impossible for it to have no false premises and a false conclusion.' (b) 'For an argument to be valid is for it to be necessary that if it has no false premises, it does not have a false conclusion.'

- Contraposition and *Reductio* fail, again.
- Disjunctive syllogism (from $P \vee Q$ and $\sim P$, infer Q) is invalid, since it could have borderline premises and a false conclusion.
- Modus ponens is invalid for the material and Łukasiewicz conditionals: if 'P' is neither true nor false and 'Q' is false, the argument from '$P \rightarrow Q$' and 'P' to 'Q' has non-false premises and a false conclusion.
- This conception seems a bit eccentric. However, the predictions about which arguments are in fact valid that we get from this definition are the same predictions made by dialethists who think some sentences are both true and false, and understand validity as preservation of truth. (Logic 'LC').

(iii) Validity as preservation of both truth and lack of falsity (van Inwagen). Equivalent to the conjunction of (i) and (ii).

- Contraposition holds (assuming that every sentence is either true, false or neither).
- *Reductio* and disjunctive syllogism fail.
- *Modus ponens* fails.

(iv) Validity as impossibility of going from truth to falsity. (a) 'For an argument to be valid is for it to be impossible for all its premises to be true and its conclusion false.' (b) 'For an argument to be valid is for it to be necessary that if all its premises are true, its conclusion is not false.'

— This is really crazy: we can no longer chain together valid arguments to get valid arguments ('cut'). The arguments from P to Q and from Q to R could both valid while the argument from P to R was not valid.

1.6 Quantification

Existential quantifications behave logically like the disjunctions of their instances, and universal quantifications behave like conjunctions. So the views about disjunction and conjunction which we have been discussing have analogues for the quantifiers:

(∃a) '$\exists x F x$' is true iff 'F' is true of something.

(∃b) '$\exists x F x$' is false iff 'F' is false of everything.

(∀a) '$\forall x F x$' is true iff 'F' is true of everything.

(∀b) '$\forall x F x$' is false iff 'F' is false of something.

- Note the following consequence. Suppose (reaonsably, given the identification of borderlineness with neither-truth-nor-falsity) that 'there is a planet inhabited by human beings' is true, and that 'there is a planet inhabited by human beings contains more than n atoms' and 'there is a planet inhabited by human beings that does not contain more than n atoms' are untrue. Then there must be an object such that 'planet inhabited by human beings' is true of it while 'contains more than n atoms' and 'does not contain more than n atoms' are both untrue of it. Thus, 'contains more than n atoms' must be neither true nor false of it: it is a borderline case of this predicate.
- Intuitively, 'contains more than n atoms' is a precise predicate. (Ignore quantum mechanics.) Thus, proponents of the generalised truth-functional approach to the logic of vagueness are under pressure to adopt something other than the standard analysis of 'vague' as 'allows for borderline cases'.
- One candidate meaning for 'vague object': something like 'object x such that there is at least one atom y such that it is indeterminate whether y is part of x.'

The analogical step from the generalised truth-functional account of disjunction and conjunction to (∃a)–(∀b) is not irresistable, however. It's plausible that disjunction and conjunction aren't themselves sources of vagueness; this is less plausible for the quantifiers. It's not at all plausible for the contextually restricted quantifiers ('there is no beer') we seem to use most of the time in ordinary life. Perhaps there is a distinguished, "absolutely unrestricted" sense of the quantifiers that is not a source of vagueness in the same way: but this would need to be argued for on philosophical grounds.

- However, the argument that even precise-looking predicates like 'contains more than n atoms' have borderline cases will still go through. If we want to allow for vague quantifiers, presumably we will treat them in the same way as we are already committed to treating quantifiers with vague restrictions. And it is not hard to convince oneslef that for quantifiers with vague restrictions, we should hold onto the left-to-right directions of (∃a)–(∀b), which are all that the argument requires.

1.7 The problem of penumbral connections

The proposal counts as indeterminate sentences that are intuitively false: ‘Frank is bald and Frank is not bald’; ‘Borderline Bob is married and Borderline Bob is a bachelor’; ‘some bachelors are married’; ‘the left tower is tall and the right tower is not tall’ (said of towers of equal height).

The proposal counts as indeterminate (hence not true) sentences that are intuitively true: ‘It is not the case that Frank is both bald and not bald’; ‘It is not the case that Borderline Bob is a married bachelor’; ‘There are no married bachelors’; ‘It is not the case that the left tower is tall and the right tower is not tall.’

Further problems arise if we adopt one of the truth-conditional approaches to the conditional or to restricted universal quantification. If restricted universal quantification is understood as involving the material conditional, ‘all bachelors are married’ is no longer counted as true. If the conditional is understood according to truth table (ii) or (iii), ‘If Frank is bald, he is not bald’ is counted as true.

Possible response: although sentences like ‘It is not the case that Frank is both bald and not bald’ are not true, they are nevertheless correctly asserted—we use them to assert something that is not

1.8 The Sorites

The versions with many premises of the form ‘either n grains can make a heap, or $n+1$ grains can’t make a heap’ or ‘it is not the case that n grains can’t make a heap and $n+1$ grains can make a heap’ are treated the same by the three-valued theory. If we assume that each premise is either true, false or borderline (i.e. ignore higher-order vagueness), we will say that there is an initial run of true premises, then a run of borderline premises, then another run of true premises; the conclusion is false. The argument is valid on conceptions (i) and (iv) of validity, invalid on conceptions (ii) and (iii) (since it takes one from borderline premises to a false conclusion).

The version with conditional premises (‘If n grains can make a heap, then $n + 1$ grains can make a heap’) is treated differently, if we adopt a non-material reading of the conditional. If we adopt the second or third truth-table for ‘if’, we will say that all but two of the premises are true; the remaining two are indeterminate (if we adopt the second truth-table) or false (if we adopt the third truth-table).

Quantified versions: Given the left-to-right direction of (∀a) and (∀b), we must say that the quantified premises are not true.

1.9 The problem of higher-order vagueness

When one reasons from truth-tables to the conclusion that some given argument-form is valid, one relies on the premise that the truth-tables cover all the cases: in the present case, that each sentence is either true, false, or neither true nor false.

But there is *exactly as much reason* to think that sentences of the form ‘S is true’ are sometimes borderline as there is to think that sentences of the form ‘x is a heap’ are sometimes borderline. A sorites series for ‘true’ can easily be constructed from a sorites series for ‘heap’: just replace each clause of the form ‘n grains can make a heap’ with ‘ “n grains can make a heap” is true’.

- This phenomenon is called ‘second-order vagueness’: ‘*P*’ is second-order vague iff either ‘ “*P*” is true’ or
  ;“*P*” is false’ (or alternatively, ‘Determinately *P*’ or ‘Determinately not-*P*’) is vague.
  — I find this name a bit misleading: it suggests that phenomenon of ‘ “*P*” is true’ (or ‘Determinately *P*’) admitting of borderline cases is due entirely to some distinctive characteristic of ‘*P*’, as opposed to ‘true’ (or ‘determinately’).
- There may be other reasons for recognising some sentences as indeterminate that have nothing to do with the Sorites, and hence do not correspond to reasons for regarding any sentences as second-order indeterminate.

If disjunctions of borderline sentences are themselves borderline, that means that some sentences of the form ‘Either S is true or S is not true’ are borderline, hence not true. Likewise for ‘Either S is true or S is false or S is neither true nor false’. So we presumably shouldn’t reason from these sentences!

We must also come to terms with the fact that sentences like ‘Some disjunctions with two true disjuncts are untrue’ and ‘Some disjunctions with two untrue disjuncts are true’ are going to be counted as borderline, hence not true.

— Suppose that ‘ “*P*” is true’ and ‘ “*Q*” is true’ are borderline. Then ‘ “$P \vee P$” is an untrue disjunction with two true disjuncts’ is also borderline.

OK; but can we still get sentences like ‘All disjunctions with at least one true disjunct are true’ and ‘All true disjunctions have at least one true disjunct’ to come out true, by understanding these in terms of a non-material conditional?

The answer turns out to be that if similar principles are true of the conditional in question, the sentences attributing truth to these sentences are not true. Suppose that

(a) Any conditional with a true antecedent and an untrue consequent is untrue.

is a true sentence. And suppose further that

(b) Any true conditional with a non-false antecedent has a non-false consequent.

Given (a) and (b), we can argue that whenever ‘ “*P*” is true’ is neither true nor false, ‘ “If *P* then *P*” is true’ is not true. For if ‘ “*P*” is true’ is neither true nor false, by (∼a) and (∼b) ‘ “*P*” is not true’ is neither true nor false, so by (∧a) and (∧b), ‘ “*P*” is true and “*P*” is not true’ is neither true nor false. Hence

(1) ‘ “If *P* then *P*” is a conditional with a true antecedent and an untrue consequent’ is neither true nor false

But by the truth of (a), and (∀a) and universal instantiation,

(2) ‘If “If *P* then *P*” is a conditional with a true antecedent and an untrue consequent, then “If *P* then *P*” is untrue’ is true

It follows that

(3) ‘If “If *P* then *P*” is a conditional with a true antecedent and an untrue consequent, then “If *P* then *P*” is untrue’ is a true conditional with a non-false antecedent.

So by (b),

(4) ‘ “If P then P” is untrue’ is not false.

and hence, by (~a) and (~b),

(5) ‘ “If P then P” is true’ is not true.

Or in operator terms: not definitely definitely (if P then P).

We can run the same argument ascending one level. Suppose ‘ “ ‘P’ is true” is true’ is neither true nor false (P is third-order vague). ‘ “If ‘P’ is true then ‘P’ is true” is true’ is not true. (‘Not definitely definitely (if definitely P then definitely P)’).

For exactly similar reasons, if ‘ “ ‘P’ is true” is true’ and ‘ “ ‘$P \vee Q$’ is true” is true’ are both neither true nor false, ‘ “If ‘P’ is true then ‘$P \vee Q$’ is true” is true’ is not true. Hence ‘ “Every disjunction with a true first disjunct is true” is true’ is not true.

But claims like this one are the heart of this whole approach to vagueness. What is one doing in asserting them, if they are not definitely true?

Moral: if one wants to be able to assert such claims, one must construe them in terms of a *radically* non-truth-functional conditional—one whose truth values aren’t *definitely* constrained in any way at all by any assignment of truth-values to its antecedent and consequent. Weird!

1.10 Higher-order vagueness and logic

Once we start allowing for higher-order vagueness, it makes a big difference whether we define ‘valid’ using conjunction and disjunction (as in the (a) definitions) or using conditionals (as in the (b) definitions) which we can understand as non-material.

If we use the (a) definitions, it will turn out that almost no interesting argument or argument-form is determinately valid. Take Reiteration (from P infer P), for example. If it is indeterminate whether P is true, then it is indeterminate whether P is both true and untrue, hence indeterminate whether the argument from P to P is an argument with a true premise and an untrue conclusion, hence indeterminate whether some instances Reiteration have true premises and untrue conclusions.

- Similar arguments can be given for the other (a) definitions of validity, and for other argument forms.
- To establish that it’s indeterminate whether Reiteration is valid on definition (iv.a) of validity, we would need an example of a sentence for which ‘P is true’ and ‘P is false’ are both borderline. But it is quite plausible that there are such sentences!

We can avoid these problems by adopting one of the (b)-definitions of validity, and understanding the conditional non-materially. But as in the last section, it will turn out that if (a) from the last section is true of the conditional we’re using, this will merely postpone the problem: when we take account of third-order indeterminacy, it will turn out that Reiteration (and just about any other argument-form) aren’t definitely definitely valid. So again, if we want to use the conditional to construe validity in such a way that we get to assert, e.g., that Reiteration is valid, it needs to be a conditional whose truth-value isn’t definitely determined by any given assignment of truth-values to antecedent and consequent.

1.11 A note on Williamson's argument that the three-valued approach cannot allow for higher-order vagueness

The argument depends on the assumption that it is possible to introduce an operator '$*$' by stipulating that '$*P$' shall be true iff 'P' is indeterminate, 'else some constraint on meaning has been unaccounted for'.

By why should that be possible, given the vagueness of 'indeterminate'? Compare: 'I hereby stipulate that "is a quasiheap" shall be true of any object x iff x is a heap'. A three-valued theorist might well think that this stipulation cannot succeed. If it succeeds, it is presumably by causing us to use 'quasiheap' in a certain way: but any use that might be induced by that stipulation will certainly allow for borderline cases.

1.12 Giving up the equation between 'borderline' and 'neither true nor false'

One could develop something isomorphic to the three-valued theory without identifying 'borderline' with 'neither true nor false', and thus without having to give up the intersubstitutability of 'P' and ' "P" is true'.

Just call the three statuses 'borderline', 'true and not borderline', 'false and not borderline'. When S is borderline, 'S is true' will be borderline. It will be indeterminate whether any borderline sentences are true. It will also be indeterminate whether all borderline sentences are neither true nor false, and indeterminate whether all borderline sentences are both true and false.

Given intersubstituability, there is a direct argument that sentences of the form 'S is true' are sometimes borderline. We no longer need a separate argument for the existence of second-order vagueness to raise problems for the definitions of validity and so forth.

1.13 Three-valued logic and the nature of indefiniteness

On a linguistic theory, it seems like it should be a contingent matter whether our language is one that obeys the central principles of "three-valued logic". Some people could speak such a language, although it might be very awkward to do so.

On a psychological theory, it is less clear. Could some beings have a language of thought that obeyed the principles?

It is hard to see what could motivate the principles if an epistemic theory were true. Most obstacles to knowledge do not obey the principle that when there is an obstacle to knowledge whether P and an obstacle to knowledge whether Q, there is an obstacle to knowledge whether $P \vee Q$; why should the obstacle characteristic of vagueness be any different?

On a primitive-operator theory, there is little to be said in favour of analysing 'borderline' as 'neither true nor false'. The principles will boil down to schemas for logical truths involving the 'definitely' operator: it commutes with conjunction, disjunction and perhaps also quantification, though not with negation.

Some think that the three-valued approach is required by a metaphysical theory of vagueness; I have no idea why.