PHIL 500 Introduction to Logic

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Solutions for the Sample Final Exam

Part 1: Translations

To make things more readable, I'm going to abbreviate the individual constants 'smith', 'jones', 'rogers' and 'phil101' as 's', 'j', 'r' and 'p'. Feel free to introduce similar abbreviations yourself!

1. $(\mathsf{Passed}(s,p) \land \mathsf{Passed}(j,p)) \lor (\neg\mathsf{Passed}(s,p) \land \neg\mathsf{Passed}(j,p))$
2. $\mathsf{Student}(j) \land \neg\mathsf{Professor}(j)$
3. $\neg(\mathsf{Advised}(r,s) \land \mathsf{Advised}(r,j))$
4. $\neg(\mathsf{Advised}(r,s) \lor \mathsf{Advised}(r,j))$
5. $\mathsf{SameAge}(s,j) \land \mathsf{EnrolledIn}(j,p)$
6. $\mathsf{EnrolledIn}(s,p) \rightarrow \mathsf{EnrolledIn}(j,p)$
7. $((\mathsf{EnrolledIn}(s,p) \land \mathsf{EnrolledIn}(j,p)) \rightarrow (\mathsf{Passed}(s,p) \lor \mathsf{Passed}(j,p))$
8. $\neg\mathsf{Taught}(r,p) \rightarrow (\mathsf{Enrolled}(s,p) \rightarrow \mathsf{Passed}(s,p))$
9. $\mathsf{Passed}(s,p) \rightarrow (\mathsf{Advised}(r,s) \land \mathsf{Taught}(r,p))$
10. $\neg\forall x(\mathsf{Enrolled}(x,p) \rightarrow \mathsf{Passed}(x,p)) \rightarrow \neg\mathsf{Passed}(s,p)$
11. $\forall x(\mathsf{Student}(x) \rightarrow \mathsf{SameAge}(x,x))$
12. $\neg\forall x(\mathsf{Advised}(r,x) \rightarrow \mathsf{EnrolledIn}(x,p))$
13. $\mathsf{Advised}(r,s) \land \neg\exists x(\mathsf{Course}(x) \land \mathsf{Taught}(r,x) \land \mathsf{EnrolledIn}(s,x))$
14. $\neg\exists x(\mathsf{Student}(x) \land \mathsf{Advised}(r,x) \land \mathsf{EnrolledIn}(x,p))$
    $\lor\forall x((\mathsf{Student}(x) \land \mathsf{Advised}(r,x)) \rightarrow \mathsf{EnrolledIn}(x,p))$
15. $\exists x(\mathsf{Course}(x) \land \mathsf{Taught}(r,x) \land \mathsf{EnrolledIn}(j,x))$
16. $\forall x((\mathsf{Course}(x) \land \mathsf{EnrolledIn}(s,x) \land \mathsf{EnrolledIn}(j,x)) \rightarrow (\mathsf{Passed}(s,x) \land \mathsf{Passed}(j,x)))$
17. $\neg\mathsf{Passed}(s,p) \land \mathsf{Student}(s) \land \mathsf{EnrolledIn}(s,p)$
    $\land\forall x((x \neq s \land \mathsf{Student}(x) \land \mathsf{EnrolledIn}(x,p)) \rightarrow \mathsf{Passed}(x,p))$

    (*Note:* This was a difficult one: it's hard to know whether the given sentence would be false (as opposed to having a false implicature) if Smith weren't a student, or didn't enroll in the course, or did pass it. So I'd also give full marks to someone who dropped any or all of the first three conjuncts.)

18. $\forall x(\mathsf{Professor}(x) \rightarrow \exists y(\mathsf{Course}(y) \land \mathsf{Taught}(x, y)))$

19. $\forall x((\mathsf{Professor}(x) \land \exists y(\mathsf{Student}(y) \land \mathsf{Advised}(x, y) \land \mathsf{EnrolledIn}(y, p)))$
$\rightarrow \exists y(\mathsf{Student}(y) \land \mathsf{Advised}(x, y) \land \mathsf{Passed}(y, p)))$

20. $\neg\exists x\exists y\exists z(\mathsf{Student}(x) \land \mathsf{Professor}(y) \land \mathsf{Professor}(z) \land \mathsf{Advised}(y, x) \land \mathsf{Advised}(z, x) \land y \neq z)$

Part 2: Truth-tables

You can check these yourself using Boole.

Part 3: Informal proofs

(4) The argument is valid. Suppose Smith is a professor. By the third premise, it follows that he is not a student. So it follows by *modus tollens* from the second premise that it's not the case that (Rogers is a professor or a student, and Jones is a professor). But the first premise is equivalent to the claim that Rogers is a professor or a studnet, so we can conclude that Jones is not a professor.

(5) The argument is invalid. Suppose that Rogers, Smith and Jones are all students, and none of them are professors. Then the conclusion is false, although the first premise is true since it has a true consequent, and the second and third premises are true since they have false antecedents.

(6) The argument is valid. Consider a world where there are just four blocks: two cubes at opposite corners of a rectangle, and two tetrahedra at the other two corners. Then each of the premises is true, but the conclusion is false, since the two cubes are neither in the same row nor in the same column as one another.

(7) The argument is valid. By the third premise, there is at least one cube; so let $c1$ be any cube. By the first premise, there is a tetrahedron in the same column as $c1$; call it $t1$. By the second premise, there is a cube in the same row as $t1$; call it $c2$. By the first premise again, there is a tetrahedron in the same column as $c2$; call it $t2$. $t1$ must be distinct from $t2$. For suppose that $t1$ were identical to $t2$. Then $t1$ would be both in the same column and in the same row as $c2$, which is impossible, since distinct things can't be both in the same row and in the same column, and nothing is both a tetrahedron and a cube. Thus we can conclude that there are two distinct tetrahedra.
(*Note:* This was pretty hard — the informal proofs on the exam will be considerably easier.)

Part 3: Formal proofs

1	$A \lor \neg B \lor C$	
2	$B \land \neg C$	
3	B	$\land$Elim, 2
4	$\quad A$	
5	$\quad A \land B$	$\land$Intro, 3, 4
6	$\quad \neg B$	
7	$\quad \bot$	$\bot$Intro, 3, 6
8	$\quad A \land B$	$\bot$Elim, 7
9	$\quad C$	
10	$\quad \neg C$	$\land$Elim, 3
11	$\quad \bot$	$\bot$Intro, 9, 10
12	$\quad A \land B$	$\bot$Elim, 11
13	$A \land B$	$\lor$Elim, 1, 4–5, 6–8, 9–12

1	$\neg(\neg A \lor B)$	
2	$\quad \neg A$	
3	$\quad \neg A \lor B$	$\lor$Intro, 2
4	$\quad \bot$	$\bot$Intro, 1, 3
5	A	$\neg$Intro, 2–4
6	$\quad B$	
7	$\quad \neg A \lor B$	$\lor$Intro, 6
8	$\quad \bot$	$\bot$Intro, 1, 7
9	$\neg B$	$\neg$Intro, 6–8
10	$A \land \neg B$	$\land$Intro, 5, 9

1	$A \leftrightarrow B$	
2	$A \rightarrow (C \lor B)$	
3	$\quad B$	
4	$\quad A$	$\leftrightarrow$Elim, 1, 3
5	$\quad C \lor B$	$\rightarrow$Elim, 2, 4
6	$\quad\quad C$	
7	$\quad\quad C \lor A$	$\lor$Intro, 6
8	$\quad\quad B$	
9	$\quad\quad A$	$\leftrightarrow$Elim, 1, 8
10	$\quad\quad C \lor A$	$\lor$Intro, 9
11	$\quad C \lor A$	$\lor$Elim, 5, 6–7, 8–10
12	$B \rightarrow (C \lor A)$	$\rightarrow$Intro, 3–11

1	$\mathsf{Tall(jones)}$	
2	$\forall \mathsf{x} \forall \mathsf{y}(\mathsf{SameHeight(x, y)} \to (\mathsf{Tall(x)} \leftrightarrow \mathsf{Tall(y)}))$	
3	$a \quad \mathsf{SameHeight(a, jones)}$	
4	$\quad \mathsf{SameHeight(a, jones)} \to (\mathsf{Tall(a)} \leftrightarrow \mathsf{Tall(jones)}$	$\forall$Elim, 2
5	$\quad \mathsf{Tall(a)} \leftrightarrow \mathsf{Tall(jones)}$	$\to$Elim, 3, 4
6	$\quad \mathsf{Tall(a)}$	$\leftrightarrow$Elim, 1, 5
7	$\forall \mathsf{x}(\mathsf{SameHeight(x, jones)} \to \mathsf{Tall(x)})$	$\forall$Intro, 3–6

1	$\forall \mathsf{x}(\mathsf{Planet(x)} \to \mathsf{Spherical(x)})$	
2	$\quad \exists \mathsf{x}(\mathsf{Planet(x)} \land \neg \mathsf{Spherical(x)})$	
3	$\quad m \quad \mathsf{Planet(m)} \land \neg \mathsf{Spherical(m)}$	
4	$\quad\quad \mathsf{Planet(m)} \to \mathsf{Spherical(m)}$	$\forall$Elim, 1
5	$\quad\quad \mathsf{Planet(m)}$	$\land$Elim, 3
6	$\quad\quad \mathsf{Spherical(m)}$	$\to$ELim, 3, 5
7	$\quad\quad \neg \mathsf{Spherical(m)}$	$\land$Elim, 3
8	$\quad\quad \bot$	$\bot$Intro, 6, 7
9	$\quad \bot$	$\exists$Elim, 2, 3–8
10	$\neg \exists \mathsf{x}(\mathsf{Planet(x)} \land \neg \mathsf{Spherical(x)})$	$\neg$Intro, 2–9

1	$(A \to B) \to A$	
2	$\quad \neg A$	
3	$\quad \neg(A \to B)$	Modus Tollens, 1, 2
4	$\quad \neg(\neg A \lor B)$	Definition of $\to$, 3
5	$\quad A \land \neg B$	De Morgan, 4
6	$\quad A$	$\land$Elim, 5
7	$\quad \bot$	$\bot$Intro, 2, 6
8	A	$\neg$Intro, 2–7

1		
2	$\neg A \lor A$	Excluded Middle
3	$\neg A \lor \neg B \lor A$	$\lor$Intro, 2
4	$\neg A \lor \neg(B \land \neg A)$	De Morgan, 3

1		
2	A	
3	$A \vee B$	$\vee$Intro, 2
4	$A \vee \neg B$	$\vee$Intro, 2
5	$(A \vee B) \wedge (A \vee \neg B)$	$\wedge$Intro, 3, 4
6	$(A \vee B) \wedge (A \vee \neg B)$	
7	$\neg A$	
8	$A \vee B$	$\wedge$Elim, 6
9	B	Disjunctive Syllogism, 7, 8
10	$A \vee \neg B$	$\wedge$Elim, 6
11	$\neg B$	Disjunctive Syllogism, 7, 10
12	$\bot$	$\bot$Intro, 9, 11
13	A	$\neg$Intro, 7–12
14	$A \leftrightarrow ((A \vee B) \wedge (A \vee \neg B))$	$\leftrightarrow$Intro, 2–5, 6–13

1		$\mathsf{Small(a)} \vee \exists \mathsf{x}(\mathsf{Cube(x)} \wedge \mathsf{Small(x)})$	
2		$\mathsf{Small(a)}$	
3		$\mathsf{a} = \mathsf{a}$	$=$Intro
4		$\mathsf{a} = \mathsf{a} \vee \mathsf{Cube(a)}$	$\vee$Intro, 3
5		$(\mathsf{a} = \mathsf{a} \vee \mathsf{Cube(a)}) \wedge \mathsf{Small(a)}$	$\wedge$Intro, 2, 4
6		$\exists \mathsf{x}((\mathsf{x} = \mathsf{a} \vee \mathsf{Cube(x)}) \wedge \mathsf{Small(x)})$	$\exists$Intro, 5
7		$\exists \mathsf{x}(\mathsf{Cube(x)} \wedge \mathsf{Small(x)})$	
8	c	$\mathsf{Cube(c)} \wedge \mathsf{Small(c)}$	
9		$\mathsf{Cube(c)}$	$\wedge$Elim, 8
10		$\mathsf{c} = \mathsf{a} \vee \mathsf{Cube(c)}$	$\vee$Intro, 9
11		$\mathsf{Small(c)}$	$\wedge$Elim, 8
12		$(\mathsf{c} = \mathsf{a} \vee \mathsf{Cube(c)}) \wedge \mathsf{Small(c)}$	$\wedge$Intro, 10, 11
13		$\exists \mathsf{x}((\mathsf{x} = \mathsf{a} \vee \mathsf{Cube(x)}) \wedge \mathsf{Small(x)})$	$\exists$Intro, 12
14		$\exists \mathsf{x}((\mathsf{x} = \mathsf{a} \vee \mathsf{Cube(x)}) \wedge \mathsf{Small(x)})$	$\exists$Elim, 7, 8–13
15		$\exists \mathsf{x}((\mathsf{x} = \mathsf{a} \vee \mathsf{Cube(x)}) \wedge \mathsf{Small(x)})$	$\vee$Elim, 1, 2–6, 7–14

1	$\forall x \forall y(((\mathsf{Cube}(x) \land \mathsf{Cube}(y)) \rightarrow \neg \mathsf{FrontOf}(x, y)))$	
2	$\forall x(\mathsf{Cube}(x) \rightarrow \exists y(\mathsf{FrontOf}(y, x)))$	
3	$\forall x(\mathsf{Cube}(x) \lor \mathsf{Small}(x))$	
4	c	
5	$\mathsf{Cube}(c) \lor \mathsf{Small}(c)$	
6	$\mathsf{Cube}(c)$	
7	$\mathsf{Cube}(c) \rightarrow \exists y(\mathsf{FrontOf}(y, c))$	∀Elim, 2
8	$\exists y(\mathsf{FrontOf}(y, c))$	→Elim, 6, 7
9	a $\mathsf{FrontOf} a, c$	
10	$\neg \mathsf{Small}(a)$	
11	$\mathsf{Cube}(a) \lor \mathsf{Small}(a)$	∀Elim, 3
12	$\mathsf{Cube}(a)$	Disjunctive Syllogism, 10, 11
13	$\mathsf{Cube}(a) \land \mathsf{Cube}(c)$	∧Intro, 6, 12
14	$(\mathsf{Cube}(a) \land \mathsf{Cube}(c)) \rightarrow \neg \mathsf{FrontOf}(a, c)$	∀Elim, 1
15	$\neg \mathsf{FrontOf}(a, c)$	→Elim, 13, 14
16	$\perp$	⊥Intro, 9, 15
17	$\mathsf{Small}(a)$	¬Intro, 10–16
18	$\exists x \mathsf{Small}(x)$	∃Intro, 17
19	$\exists x \mathsf{Small}(x)$	∃Elim, 8, 9–18
20	$\mathsf{Small}(c)$	
21	$\exists x \mathsf{Small}(x)$	∃Intro, 20
22	$\exists x \mathsf{Small}(x)$	∨Elim, 5, 6–19, 20–21
23	$\forall y \exists x \mathsf{Small}(x)$	∀Intro, 4–22
24	$\exists x \mathsf{Small}(x)$	∀Elim, 23

Oops — that last proof was pretty tricky. I promise there won't be anything nearly that hard on the real final!