PHIL 500 Introduction to Logic

Prof. Cian Dorr

Sample Final Exam

Part 1: Translations

Translate each of the following sentences into a dialect of FOL with the following vocabulary:

smith : Smith
jones : Jones
rogers : Rogers
phil101 : PHIL 101
Student(x) : x is a student
Professor(x) : x is a professor
Course(x) : x is a course
Enrolled(x, y) : x enrolled in y
Passed(x, y) : x passed y
Taught(x, y) : x taught y
Advised(x, y) : x advised y
SameAge(x, y) : x is the same age as y

Domain of quantification: professors, students, courses.

1. Either Smith and Jones both passed PHIL101, or neither of them did.
2. Smith is a student, not a professor.
3. Smith and Jones weren't both advised by Rogers.
4. Neither Smith nor Jones was advised by Rogers.
5. Smith is the same age as Jones, who enrolled in PHIL 101.
6. If Smith enrolled in PHIL 101, Jones did too.
7. If both Smith and Jones enrolled in PHIL 101, at least one of them passed it.
8. Provided that he enrolled in it, Smith passed PHIL 101—unless of course it was taught by Rogers.
9. Only if Rogers both advised Smith and taught PHIL 101 did Smith pass it.
10. Smith didn't pass PHIL 101 unless everyone who enrolled in it did.
11. Every student is the same age as him- or herself.
12. Not every student advised by Rogers took PHIL101.

13. Although Rogers advised Smith, Smith didn't enroll in any of the courses she taught.

14. Either no students advised by Rogers enrolled in PHIL 101, or all of them did.

15. Jones enrolled in a course taught by Rogers.

16. Smith and Jones passed every course both of them enrolled in.

17. Except for Smith, every student who enrolled in it passed PHIL 101.

18. Every professor taught at least one course.

19. Every professor who advised a student who enrolled in PHIL 101 advised a student who passed PHIL 101.

20. No student was advised by more than one professor.

Part 2: Truth-tables

(1) Use the truth-table method, determine whether the sentence '$\neg(A \lor B) \lor (A \land B)$' is a tautology.

(2) Use the truth-table method to determine whether the following sentences are tautologically equivalent:

$$P \to (Q \to R)$$
$$(P \land Q) \to R$$

(3) Use the truth-table method to determine whether the following argument is tautologically valid:

$$((A \lor B) \leftrightarrow (B \lor C)) \leftrightarrow (A \lor C)$$
$$\neg A$$
$$\overline{}$$
$$(B \land C)$$

Part 3: Informal proofs

Two of the following arguments are valid; two are invalid. Give informal proofs for the valid ones, and informal counterexamples for the invalid ones.

(4)

$$\begin{array}{|l}
\neg\mathsf{Professor(rogers)} \rightarrow \mathsf{Student(rogers)} \\
((\mathsf{Professor(rogers)} \vee \mathsf{Student(rogers)}) \wedge \mathsf{Professor(jones)}) \rightarrow \mathsf{Student(smith)} \\
\mathsf{Professor(smith)} \rightarrow \neg\mathsf{Student(smith)} \\
\hline
\mathsf{Professor(smith)} \rightarrow \neg\mathsf{Professor(jones)}
\end{array}$$

(5)

$$\begin{array}{|l}
\neg\mathsf{Professor(rogers)} \rightarrow \mathsf{Student(rogers)} \\
((\mathsf{Professor(rogers)} \vee \mathsf{Student(rogers)}) \wedge \mathsf{Professor(jones)}) \rightarrow \mathsf{Student(smith)} \\
\mathsf{Professor(smith)} \rightarrow \neg\mathsf{Student(smith)} \\
\hline
\mathsf{Professor(rogers)} \vee \mathsf{Professor(smith)} \vee \mathsf{Professor(jones)}
\end{array}$$

(6)

$$\begin{array}{|l}
\forall \mathsf{x}(\mathsf{Cube(x)} \rightarrow \exists \mathsf{y}(\mathsf{Tet(y)} \wedge \mathsf{SameCol(x,y)})) \\
\forall \mathsf{x}(\mathsf{Tet(x)} \rightarrow \exists \mathsf{y}(\mathsf{Cube(y)} \wedge \mathsf{SameRow(x,y)}) \\
\exists \mathsf{x}(\mathsf{Cube(x)}) \\
\hline
\forall \mathsf{x} \forall \mathsf{y}((\mathsf{Cube(x)} \wedge \mathsf{Cube(y)}) \rightarrow (\mathsf{SameRow(x,y)} \vee \mathsf{SameCol(x,y)})
\end{array}$$

(7)

$$\begin{array}{|l}
\forall \mathsf{x}(\mathsf{Cube(x)} \rightarrow \exists \mathsf{y}(\mathsf{Tet(y)} \wedge \mathsf{SameCol(x,y)})) \\
\forall \mathsf{x}(\mathsf{Tet(x)} \rightarrow \exists \mathsf{y}(\mathsf{Cube(y)} \wedge \mathsf{SameRow(x,y)}) \\
\exists \mathsf{x}(\mathsf{Cube(x)}) \\
\hline
\exists \mathsf{x} \exists \mathsf{y}(\mathsf{Tet(x)} \wedge \mathsf{Tet(y)} \wedge \mathsf{x} \neq \mathsf{y})
\end{array}$$

Part 3: Formal proofs

Section A: Proofs in $\mathcal{F}$

Provide formal derivations using just the introduction and elimination rules of $\mathcal{F}$ to establish the validity of each of the following arguments.

(8)

$$\begin{array}{|l}
\mathsf{A} \vee \neg\mathsf{B} \vee \mathsf{C} \\
\mathsf{B} \wedge \neg\mathsf{C} \\
\hline
\mathsf{A} \wedge \mathsf{B}
\end{array}$$

(9)

$$\begin{array}{|l}
\neg(\neg\mathsf{A} \vee \mathsf{B}) \\
\hline
\mathsf{A} \wedge \neg\mathsf{B}
\end{array}$$

(10)

$$\begin{array}{|l}
\mathsf{A} \leftrightarrow \mathsf{B} \\
\mathsf{A} \rightarrow (\mathsf{C} \vee \mathsf{B}) \\
\hline
\mathsf{B} \rightarrow (\mathsf{C} \vee \mathsf{A})
\end{array}$$

(11)

$$\begin{array}{|l}
\mathsf{Tall(jones)} \\
\forall \mathsf{x} \forall \mathsf{y}(\mathsf{SameHeight(x,y)} \rightarrow (\mathsf{Tall(x)} \leftrightarrow \mathsf{Tall(y)}) \\
\hline
\forall \mathsf{x}(\mathsf{SameHeight(x,jones)} \rightarrow \mathsf{Tall(x)})
\end{array}$$

(12)

$$\begin{array}{|l}
\forall \mathsf{x}(\mathsf{Planet(x)} \rightarrow \mathsf{Spherical(x)}) \\
\hline
\neg\exists \mathsf{x}(\mathsf{Planet(x)} \wedge \neg\mathsf{Spherical(x)})
\end{array}$$

Section B: Proofs in a more relaxed system

(13)

$$\begin{array}{|l} \\ \hline \neg A \lor \neg(B \land \neg A) \end{array}$$

(14)

$$\begin{array}{|l} (A \to B) \to A \\ \hline A \end{array}$$

(15)

$$\begin{array}{|l} \\ \hline A \leftrightarrow ((A \lor B) \land (A \lor \neg B)) \end{array}$$

(16)

$$\begin{array}{|l} \mathsf{Small}(a) \lor \exists x(\mathsf{Cube}(x) \land \mathsf{Small}(x)) \\ \hline \exists x((x = a \lor \mathsf{Cube}(x)) \land \mathsf{Small}(x)) \end{array}$$

(17)

$$\begin{array}{|l} \forall x \forall y((\mathsf{Cube}(x) \land \mathsf{Cube}(y)) \to \neg \mathsf{FrontOf}(x, y)) \\ \forall x(\mathsf{Cube}(x) \to \exists y(\mathsf{FrontOf}(y, x)) \\ \forall x(\mathsf{Cube}(x) \lor \mathsf{Small}(x)) \\ \hline \exists x \mathsf{Small}(x) \end{array}$$

Additional Inference Rules

Inference rules

(Disjunctive Syllogism)	$P \lor Q$ $\neg P$ ――― Q
(Modus Tollens)	$P \to Q$ $\neg Q$ ――― $\neg P$
(Transitivity of the conditional)	$P \to Q$ $Q \to R$ ――― $P \to R$
(Transitivity of the biconditional)	$P \leftrightarrow Q$ $Q \leftrightarrow R$ ――― $P \leftrightarrow R$
(Excluded Middle)	――― $P \lor \neg P$
(Substitution)	$P \leftrightarrow Q$ $\ldots P \ldots$ ――― $\ldots Q \ldots$

Equivalences

These may be used in two ways: either directly, to pass between sentences which are logically equivalent in one of the specified ways, or in conjunction with the principle of substitution, to pass from any sentence to another sentence derived from the first by replacing one or more formulae contained within the first sentence with formulae which are logically equivalent in one of the specified ways.

(Idempotence)	$P \wedge P \Leftrightarrow P$
	$P \vee P \Leftrightarrow P$
(Distribution)	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
(De Morgan)	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
(Definition of $\rightarrow$)	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$
(Definition of $\leftrightarrow$)	$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
(Contraposition)	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
	$P \leftrightarrow Q \Leftrightarrow \neg Q \leftrightarrow \neg P$
(Quantifier De Morgan)	$\forall v \neg P \Leftrightarrow \neg \exists v P$
	$\exists v \neg P \Leftrightarrow \neg \forall v P$
(Quantifier equivalences)	$\forall v (P \wedge Q) \Leftrightarrow \forall v P \wedge \forall v Q$
	$\exists v (P \vee Q) \Leftrightarrow \exists v P \vee \exists v Q$
(Double Negation*)	$P \Leftrightarrow \neg\neg P$
(Commutativity*)	$P \wedge Q \Leftrightarrow Q \wedge P$
	$P \vee Q \Leftrightarrow Q \vee P$
(Associativity*)	$(P \vee Q) \vee R \Leftrightarrow P \vee Q \vee R$
	$P \vee (Q \vee R) \Leftrightarrow P \vee Q \vee R$
	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge Q \wedge R$
	$P \wedge (Q \wedge R) \Leftrightarrow P \wedge Q \wedge R$

The final three rules—the ones marked with an asterisk (*)—are special. You may make inferences in accordance with these rules "silently", without citing the rule or adding an extra step to your derivation, *provided you combine these rules with one of the other rules listed on this page*. Thus the following derivations would be graded as correct:

1	$\mathsf{Dog(socrates)} \rightarrow \neg\mathsf{Man(socrates)}$	
2	$\mathsf{Man(socrates)} \rightarrow \neg\mathsf{Dog(socrates)}$	Contraposition, 1

1	$\mathsf{Mortal(socrates)} \vee \neg\mathsf{Man(socrates)}$	
2	$\mathsf{Man(socrates)} \rightarrow \mathsf{Mortal(socrates)}$	Definition of $\rightarrow$, 1

(Observe that the first of these derivations contains a "silent" application of Double Negation, the second contains a "silent" application of Commutativity.) The following derivation, by contrast, would not be counted as correct, since it uses one of the introduction and elimination rules rather than one of the additional rules from this page:

1	$\neg\neg\mathsf{Man}(\mathsf{socrates}) \land \mathsf{Woman}(\mathsf{xanthippe})$	
2	$\mathsf{Man}(\mathsf{socrates})$	$\land$Elim, 1