Formal proofs and the quantifiers

Straightforward rules

Universal Elimination (∀ Elim):

$$\begin{array}{l|l} & \forall x\, S(x) \\ & \vdots \\ \triangleright & S(c) \end{array}$$

- Here x can be any variable, c can be any individual constant [or variable-free term] in the language, S(x) stands for any formula whose only free variable is x, and S(c) stands for the result of replacing all occurrences of x in S(x) with c.
  - Generous use: you can eliminate more than one universal quantifier at once.

Existential Introduction ($\exists$ Intro):

$$
\begin{array}{l|l}
 & S(c) \\
 & \vdots \\
\triangleright & \exists x\, S(x)
\end{array}
$$

- Here again, x may be any variable; c may be any individual constant (or variable-free term); S(c) may be any sentence containing zero or more occurrences of c; S(x) is the result of replacing some or all of these occurrences of c with occurrences of x.
  - Generous use: you can introduce more than one existential quantifier at once.

Methods of proof

Existential Elimination (∃ Elim):

$\exists x\, S(x)$
$\vdots$
  $\boxed{c}$ $S(c)$
  $\vdots$
  Q
$\triangleright$ Q

Where c does not occur outside the subproof where it is introduced.

- Generous interpretation: eliminate more than one at once, using more than one boxed constant

- $\exists$-elim corresponds to the informal method of existential generalisation. This lets us, having established something of the form $\exists x S(x)$, introduce a new 'dummy' name c and assert S(c).

- Example: suppose we are given the premises $\exists x(\text{Cube}(x) \land \text{Small}(x))$ and $\forall x(\text{Cube}(x) \rightarrow (\text{Small}(x) \rightarrow \text{Larger}(a,x)))$; we want to prove that $\exists x \text{Larger}(a,x)$. We can argue as follows: by the first premise, there is a small cube. Call it Tiny. By the second premise, if Tiny is small, a is larger than it; but since Tiny is small, it follows that a is larger than Tiny. Hence a is larger than something.

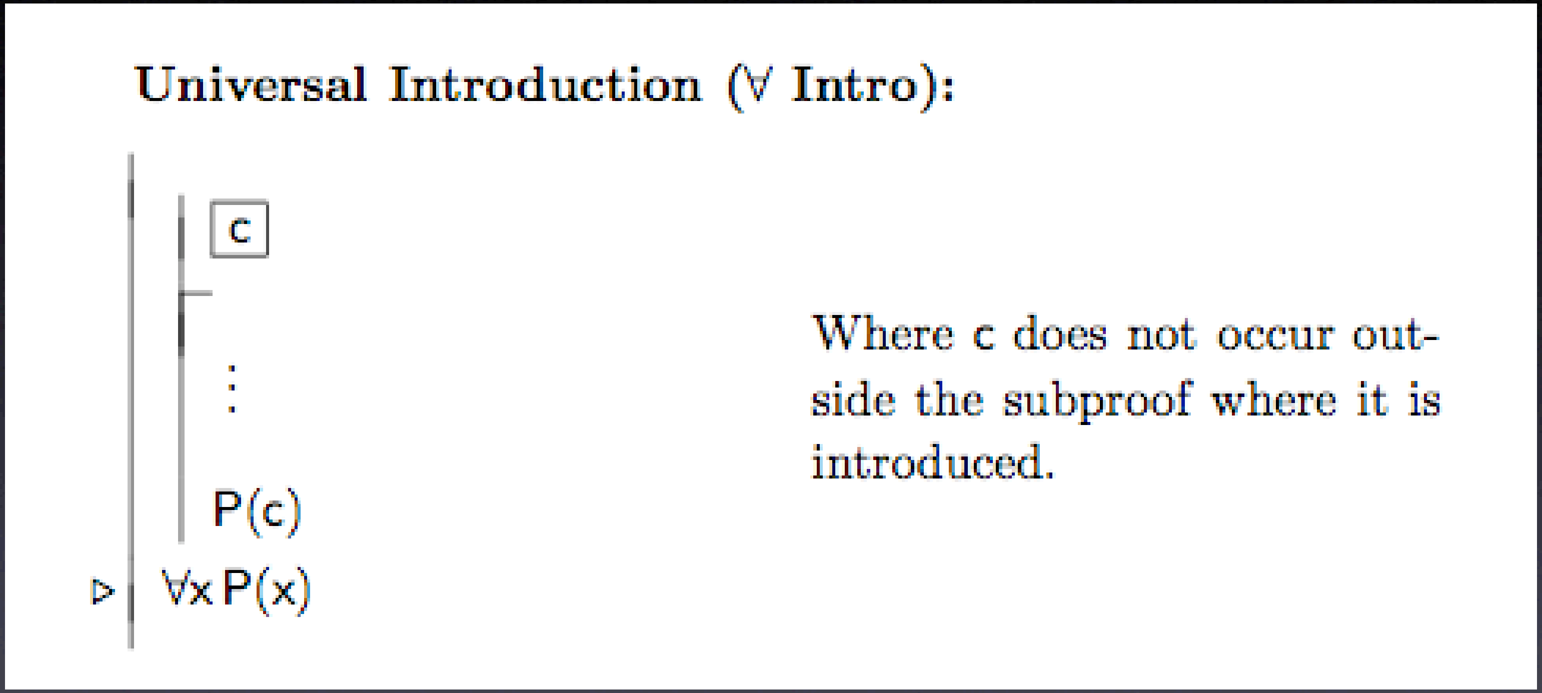
General Conditional Proof (∀ Intro):

$$
\begin{array}{l|l}
 & \begin{array}{|l} \boxed{c}\ P(c) \\ \hline \vdots \\ Q(c) \end{array} \\
\triangleright & \forall x\,(P(x) \to Q(x))
\end{array}
$$

Where c does not occur outside the subproof where it is introduced.

- Corresponds to the informal method of *general conditional proof*. Suppose that, just from the assumption that P(c), where c is some new dummy name, you can derive (given other things you've established) that Q(c). Then you're entled to conclude that all P's are Q's.
    - Examples are very common in mathematics.

- As a special case, we allow this rule to be used with an empty assumption, thus:



- As with the other rules, has a generous use, on which you can introduce more than one universal quantifier, using more than one boxed constant.

For next week

- Read: through chapter 13
- Do: 13.2, 13.3, 13.4, 13.7, 13.11, 13.12, 13.13, 13.16; (8% each); 12.2, 12.3 (10% each); write informal proofs based on your formal proofs of 13.2, 13.7, 13.13 and 13.16 (4% each).