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A Characterization of Infinite Horizon Optimality in Terms of Finite Horizon Optimality and a Critical Stock Condition

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I. INTRODUCTION

The purpose of this note is to present a characterization of infinite-horizon optimality in an aggregative stationary model. Our characterization is in terms of *finite-horizon optimality* (for every finite horizon) and a condition which essentially says that the input level on the program must be, at all times, below a *critical stock* (which is below the maximum sustainable stock).

Two aspects of this result are worth emphasizing. First, out characterization is valid for arbitrary non-convexities in the technology set; the sufficiency part of our result does not even depend on concavity of the utility function. Thus, this aspect should be of particular interest when viewed in the light of contributions to the literature on optimal intertemporal allocation under non-convexities in production. (For this literature, see the papers by Clark 1971; Skiba 1978; Majumdar and Mitra 1982, 1983; Majumdar and Nermuth 1982; Dechert and Nishimura 1983; and Mitra and Ray 1984.) For a convex technology set, and a concave utility function, finite-horizon optimality (with positive consumption in some period) can itself be characterized in terms of the Ramsey-Euler conditions or the so-called competitive conditions.

Second, our characterization of finite-horizon optimal programs which are *not* infinite-horizon optimal is in terms of a "critical stock" being exceeded by the input level on the program in *some* period. This aspect should be viewed in the spirit of the recent literature on intertemporal decentralization, where an attempt has been made to replace the usual asymptotic (transversality) condition by period-by-period verifications to signal capital overaccumulation along competitive programs. (For this literature, see the papers by Majumdar 1988; Hurwicz-Majumdar 1988; Brock-Majumdar 1988; and Dasgupta and Mitra 1988.)

II. The Model

We consider a stationary aggregative model with discounting, characterized by a *production function* $f : R_+ \to R_+$, a *discount factor* $\delta \in (0,1)$, and a *utility function* $u : R_+ \to R$. Without loss of generality we take $u(0) = 0$.

On the production function, we make the assumption:

(F) $f(0) = 0$, *f is increasing and continuous, and there is* $K > 0$ *such that* $f(x) > x$ *for* $0 < x < K$, $f(x) < x$ *for* $x > K$.

Below, we will invoke one or both of the following assumptions on the utility function:

(U.1) *u is continuous.*

(U.2) *u is increasing and strictly concave.*

III. Feasible Programs

Programs start from an *initial stock a*. We will suppose that initial stocks may be drawn from the interval $A \equiv (0,\alpha]$ where $\alpha < K$.

A program (x,y,c) is *feasible* from (some initial stock) $a > 0$ if

$$x_o \le a \qquad (1)$$

$$x_t + c_t \le y_t \qquad \text{for } t \ge 1 \qquad (2)$$

$$y_t \le f(x_{t-1}) \qquad \text{for } t \ge 1 \qquad (3)$$

IV. Optimal Programs

A feasible program (x^*, y^*, c^*) from a is *optimal* from a if it solves

$$\max \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$$

subject to (x,y,c) feasible from a. Under (U.1) and (F), optimal programs exist.

We will also consider the following assumption:

(P) *For each* $a \in A$ *and each optimal program* $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{c}^*)$ *from* a, $x_t^* > 0$ *for all* $t \ge 0$.

Assumption (P) can be replaced by conditions on the utility and production functions that guarantee (P). These conditions do not necessarily require concavity of f or even u. For example, either of the following two conditions can be shown to guarantee (P), given (F) and (U.1):
(i) Unbounded steepness of u at the origin and no unbounded steepness anywhere else.
(ii) f is δ-productive near the origin and u is concave.

V. Finite Horizon Programs

Let T be an integer ≥ 1 (the *horizon*). Consider two stocks $(a,b) \geq 0$. The program $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)^T$ is *T-feasible* from a to b if

$$x_o \leq a, x_T \geq b \tag{4}$$

$$x_t + c_t \leq y_t \qquad \text{for } t = 1,\ldots,T \tag{5}$$

$$y_t \leq f(x_{t-1}) \qquad \text{for } t = 1,\ldots,T \tag{6}$$

We shall also use $(\mathbf{x},\mathbf{y},\mathbf{c})^T$ to denote the obvious restriction of a feasible program $(\mathbf{x},\mathbf{y},\mathbf{c})$ from a to its first T periods. That is, $(\mathbf{x},\mathbf{y},\mathbf{c})^T$ will be T-feasible from a to x_T.

A T-feasible program $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)$ is *T-optimal* from a to b if it solves

$$\max \sum_{t=1}^{T} \delta^{t-1} u(c_t)$$

subject to $(\mathbf{x},\mathbf{y},\mathbf{c})^T$ T-feasible from a to b.

VI. A Critical Stock

Under (F), define k^* by the condition

$$k^* \equiv \min\{s : f(s) - s \geq f(x) \dot{-} x \text{ for all } x \geq 0\} \tag{7}$$

It is obvious that k^* is well defined and that $0 < k^* < K$. Recalling that $A = (0,\alpha]$, choose some number, C, such that

$$K > C > \max\{k^*,\alpha\} \tag{8}$$

C can be interpreted as a *critical stock* in the results that follow.

VII. RESULTS

Proposition 1

Under (F), (U.1) *and* (U.2): *if* $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)$ *is optimal from a, then*
(1.a) $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)^T$ *is T-optimal from a to* x_T^* *for each* $T \geq 1$, *and*
(1.b) $x_t^* \leq C$ *for all* $t \geq 0$.

Proof Suppose $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)$ is optimal from a. Then (1.a) follows by a straightforward application of the Principle of Optimality.

To establish (1.b), note that by Mitra and Ray (1984, Proposition 4.1, Lemma 5.3 and Theorem 5.1), x_t^* converges *monotonically* to some $x^* \in [0,k^*]$. Using the definition of C, we are done. **(Q.E.D.)**

Proposition 2

Under (F), (U.1) *and* (P): *if* $(\overline{\mathbf{x}},\overline{\mathbf{y}},\overline{\mathbf{c}})$ *is feasible from a, and satisfies:*
(2.a) $(\overline{\mathbf{x}},\overline{\mathbf{y}},\overline{\mathbf{c}})^T$ *is T-optimal from a to* $\overline{x}_T$ *for each* $T \geq 1$,
(2.b) $\overline{x}_t \leq C$ *for all* $t \geq 0$,
then $(\overline{\mathbf{x}},\overline{\mathbf{y}},\overline{\mathbf{c}})$ *is optimal from a.*

Proof Suppose not. Let $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{c}^*)$ be an optimal program from a. (This exists, by (F) and (U.1)).

Now, there exists $\theta > 0$ such that

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t^*) > \sum_{t=1}^{\infty} \delta^{t-1} u(\overline{c}_t) + \theta$$

So (again using (U.1) and (F)), there exists T' such that for all $T \geq T'$

$$\sum_{t=1}^{T} \delta^{t-1} u(c_t^*) \geq \sum_{t=1}^{\infty} \delta^{t-1} u(\overline{c}_t) + \theta \tag{9}$$

Pick any $\hat{T} \geq T'$.

By (P), $x_{\hat{T}}^* > 0$. By (F), there is $S > \hat{T}$ such that $f^{(S-\hat{T})}(x_{\hat{T}}) \geq C$ (where $f^{(k)}$ is the k-fold composition of f with itself). Now define an S-feasible program from a to C as follows: $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{c}})^S$ is described by $\hat{x}_t = x_t^*$, $t = 0,\ldots,\hat{T}$; $(\hat{y}_t,\hat{c}_t) = (y_t^*,c_t^*)$, $t = 1,\ldots,\hat{T}$; $\hat{x}_t = f^{(t-\hat{T})}(x_{\hat{T}}^*)$, $t = \hat{T}+1,\ldots,S$, $\hat{y}_t = f(\hat{x}_{t-1})$, $t = \hat{T}+1,\ldots,S$; and $\hat{c}_t = 0$, t $= \hat{T}+1,\ldots,S$.

By our convention that $u(0) = 0$,

$$\sum_{t=1}^{S}\delta^{t-1}u(\hat{c}_t)) = \sum_{t=1}^{\hat{T}}\delta^{t-1}u(c_t^*)$$

$$\geq \sum_{t=1}^{\infty}\delta^{t-1}u(\overline{c}_t)) + \theta \qquad \text{(by (9))}$$

$$\geq \sum_{t=1}^{S}\delta^{t-1}u(\overline{c}_t)) + \theta \qquad \text{(by } u(0) = 0 \text{ and (2.a))} \qquad (10)$$

Now, $\hat{x}_s \geq C \geq \overline{x}_s$. So $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{c}})^S$ is also feasible to $\overline{x}_s$. But then (10) contradicts (2.a). **(Q.E.D.)**

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