On the Nature of Policy Functions of Dynamic Optimization Models*

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1. INTRODUCTION

Dynamic optimization models are currently in use in a number of different areas in economics, to address a wide variety of issues. The relationship between the dynamic optimization model and the (optimal) policy function associated with it is central to such applications. The purpose of this paper is to present a selective survey of results relevant to understanding this relationship.

As a backdrop to this survey, one should recall that two well-known characterizations of optimality have figured prominently in the literature on dynamic optimization. The first uses a primal approach, and characterizes optimality in terms of the existence of *value function* satisfying the functional equation of dynamic programming (often referred to as *Bellman's optimality principle*). The second (known as the *price characterization of optimality*) is based on a dual approach, and is developed for convex structures, where separation theorems for convex sets play a crucial role. Here an optimal programme is characterized in terms of existence of a sequence of dual variables or shadow prices, in terms of which (generalized) profit is maximized at the programme at each date compared to any alternative activity available at that date and, in addition, an asymptotic transversality condition is satisfied.

These characterizations are certainly useful in some respects. However, if we are given a function, h, from the state space, X, to itself,

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and want to know whether h could be the policy function of some dynamic optimization model, we do not see how to answer this directly with Bellman's principle of optimality. Similarly, given an intertemporal sequence $(x_0, x_1, \ldots)$ of the state variable (with x_t in X for $t \geq 0$), if we want to know whether $(x_0, x_1, \ldots)$ is the optimal programme from x_0 generated by some dynamic optimization model, the price characterization of optimality result does not help us to resolve this issue.

The kinds of questions posed above arise quite naturally in the literature on *chaotic* economic dynamics, which has been developed primarily over the last decade. This literature has been concerned with the issue of whether chaotic behaviour is possible in the context of dynamic optimizaton models (and, if so, how 'likely' is it that it would occur). One could answer the possibility question in the affirmative if one could show, for example, that the *logistic* map $h(x) = 4x(1 - x)$ on $X = [0, 1]$ is the optimal policy function of *some* dynamic optimization model, since the logistic map is well-known to generate chaotic dynamics. Similarly, one could answer the question in the affirmative if one could show that the sequence $(x_0, x_1, x_2, x_0, x_1, x_2, \ldots)$, with x_0, x_1 and x_2 distinct, is the optimal programme from x_0, generated by some dynamic optimization model, since this establishes the existence of a period-three cycle, and the Li–Yorke theorem tells us that the model must generate optimal programmes (from other initial states) which exhibit (topological) chaos.

At the same time, it is worth emphasizing that while the question under discussion arose specifically in the context of studying chaotic economic dynamics, it is clearly of broader interest. It is useful, therefore, to pose the more general problem at this point. It is known that, given a standard dynamic optimization model, the policy function, h (from the state space X to itself) is continuous on X (and satisfies $h(0) = 0$, if one assumes, as we do, that the dynamic optimization model allows inaction and postulates impossibility of free production). We can now ask whether there are any *additional* restrictions that the exercise of dynamic optimization imposes on the policy function, or equivalently, whether any continuous function is *rationalizable* as a policy function of some dynamic optimization model. The problem is, of course, similar in spirit to the issue resolved in the Sonnenschein–Mantel–Debreu result in general equilibrium theory, where it has been shown that, given any continuous function, f, satisfying homogeneity of degree zero and Walras Law (and a compact subset, A, of the set of positive price vectors), one can construct an exchange economy so that the market excess-demand of this economy coincides with f (on the set A of positive price vectors).

Boldrin and Montrucchio (1986) took a significant step in addressing the problem posed above when they showed that any *twice continuously differentiable* function can be obtained as a policy function of an appropriate dynamic optimization model. Their demonstration raises two issues which we will discuss in this paper.

First, since policy functions are known to be continuous, the question arises whether their result can be extended to the class of all *continuous* functions. Neumann *et al.* (1988) showed that any continuously differentiable function, whose derivative satisfies a Lipschitz condition, can be rationalized as a policy function of a dynamic optimization model, thereby obtaining a slight weakening of the C^2 restriction used by Boldrin and Montrucchio. However, such a result clearly falls short of the mark since there are robust examples of dynamic optimization models which generate policy functions with a kink in the interior of the state space (similar to *tent map*). Specifically, Nishimura and Yano (1994) have shown that if the (reduced-form) utility function, u, is twice continuously differentiable in the interior of the transition possibility set, Ω, and the cross partial (u_{12}) of the utility function is negative throughout, then policy functions will have a shape similar to a tent-map. Further, a standard two-sector model of optimal growth, with Cobb–Douglas production functions (and fully depreciating capital), such that the consumption-goods sector is always more capital intensive than the investment-goods sector, will give rise to such a reduced-form utility function [see Benhabib and Nishimura (1985)]. Thus, differentiability (and so, continuous differentiability and Lipschitz continuity of the derivative) is not a general property of the policy function. [The contributions of Araujo (1991) and Santos (1991), which demonstrate differentiability of the policy function, but only under additional restrictive assumptions on the dynamic optimization model, complement this observation.] Thus, if one is interested in a *complete* characterization, the extension of the Boldrin-Montrucchio result must be addressed for a broader class of functions, which are continuous, but not necessarily differentiable.

We provide two results in this context that show that the required extension is not to be had free of charge. First, we establish a result on the behaviour of the optimal policy function at a boundary fixed point, from which it follows that the class of continuous functions is given by,

$$h(x) = \mu x^{\alpha}(1 - x^{\alpha}) \quad \text{for } x \text{ in } [0, 1] \tag{1}$$

(where the class is parametrized by μ and α, with $0<\alpha<1$, and $0<\mu\leq 4$) is not rationalizable. Since differentiability of h fails here, precisely at the boundary, one may be tempted to conclude that the obstacle one encounters here is a *boundary problem*, reminiscent of a similar difficulty in the Sonnenschein–Mantel–Debreu result in general equilibrium theory. However, we establish another result on the behaviour of the optimal policy function at an interior fixed point, which compels us to be more cautious. From this second result, it follows that the continuous, *increasing*, function, given by,

$$h(x) = 1 + (x-1)^{1/3} \quad \text{for } x \text{ in } [0, 2] \tag{2}$$

is not rationalizable. This example shows that the obstacle with the extension is not a boundary problem. It also shows that the problem is not intrinsically related to *chaotic* functions, since an increasing, continuous function can only display *simple dynamics*.The two results convincingly demonstrate that dynamic optimization places non-trivial restrictions on the nature of policy functions, besides continuity.

A similarity between the two examples mentioned above is that these continuous functions are infinitely steep at a fixed point. It might, therefore, be conjectured that the 'correct' extension of the Boldrin–Montrucchio result should be to the class of Lipschitz-continuous functions. While we leave this as an open question, we note that even the simple one-sector model of capital accumulation (*á la* Cass-Koopmans) can generate robust examples, in which the policy function is infinitely steep at a fixed point and is, therefore, not Lipschitz continuous.

The second issue stemming from the Boldrin–Montrucchio result is concerned with the restriction on the magnitude of the discount factor at which a given C^2 function is rationalized. This turns out to be particularly important in rationalizing chaotic functions, the context in which these problems arose in the first place.

Clearly, the Boldrin–Montrucchio result can be applied to rationalize the quadratic family of functions, given by,

$$h(x) = \mu x(1-x) \quad \text{for } x \text{ in } [0, 1] \tag{3}$$

where the family is parametrized by μ, with $0<\mu\leq 4$. But, it turned out that the dynamic optimization model (Ω, u, δ) constructed by them, to yield the logistic map [given by (3) when $\mu = 4$] as its policy function, had a very low discount factor (about 0.01). This raised the question of whether chaotic optimal policy functions can be ruled out when more 'reasonable' discount factors prevailed, or whether this feature of the dynamic optimization model was simply a shortcoming of the particular method used in its construction.

It is now known that these discount factors restrictions observed in the Boldrin–Montrucchio construction are, in fact, inescapable in rationalizing chaotic functions. The logistic map, for instance, cannot be rationalized at any discount factor exceeding 0.25, and the tent-map, given by,

$$h(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 0.5 \\ 2-2x & \text{for } 0.5 \leq x \leq 1 \end{cases} \tag{4}$$

cannot be rationalized for any discount factor exceeding 0.5. Both of these observations can be obtained quite simply from the two results mentioned above, as we show in this paper.

More generally, following the contributions of Sorger (1992a, 1992b, 1994), it has been demonstrated by Mitra (1996) and Nishimura and Yano (1996) that if a continuous function exhibits Li–Yorke chaos (equivalently a period-three cycle), then it can be rationalized only if the associated discount factor satisfies,

$$\delta < [(\sqrt{5} - 1)/2]^2 \tag{5}$$

Furthermore, if the discount factor satisfies the restriction in (5), then one can construct a transition possibility set, Ω, and a reduced-form utility function, u, such that the dynamic optimization model (Ω, u,δ) exhibits Li–Yorke chaos.

It is worth making a remark about our approach to the class of problems discussed above. We indicated that two distinct characterizations (a 'primal' and a 'dual') have figured prominently in the literature on dynamic optimization models. Weitzman (1973) combined the two approaches to show the existence of shadow prices associated with an optimal programme, such that (generalized) profit was maximized at each date at the optimal program, a transversality condition was satisfied, and in addition, these shadow prices also supported the value function at each date. The results established in this paper follow quite simply from an application of his characterization of optimality.

2. PRELIMINARIES

2.1 DYNAMICAL SYSTEMS

Let I be an interval in $\Re$, the set of reals. Let $f: I \rightarrow I$ be a continuous map of the interval I into itself. The pair (I, f) is called a *dynamical system*; I is called the *state space* and f the *law of motion* of the dynamical system.

We write $f^0(x) = x$ and for any integer $n \geq 1, f^n(x) = f[f^{n-1}(x)]$. If $x \in I$, the sequence $\tau(x) = \{f^n(x)\}_0^\infty$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) = \{y : y = f^n(x)$ for some $n \geq 0\}$.

A point $x \in I$ is a *fixed point* of f if $f(x) = x$. A point $x \in I$ is called a *periodic point* of f if there is $k \geq 1$ such that $f^k(x) = x$. The smallest such k is called the *period* of x. [In particular, if $x \in I$ is a fixed point of f, it is periodic with period 1.] If $x \in I$ is a periodic point with period k, we also say that the orbit of x (or trajectory from x) is periodic with period k.

The following fundamental result on the existence of periodic orbits is due to Sarkovskii (1964).

PROPOSITION 1 (Sarkovskii): Let the positive integers be totally ordered in the following way:

$$3 \prec 5 \prec 7 \prec 9 \prec \ldots \prec 2 \bullet 3 \prec 2 \bullet 5 \prec \ldots \prec 2^2 \bullet 3 \prec 2^2 \bullet 5 \prec \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

If f has a periodic orbit of period n and if $n \prec m$, then f also has a periodic orbit of period m.

In order to study the nature of trajectories which are not periodic, it is useful to define a *scrambled* set. A set $S \subset I$ is called a *scrambled set* if it possesses the following two properties.

(i) If $x, y \in S$ with $x \neq y$, then,

$$\limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0$$

(ii) If $x \in S$ and y is any periodic point of f,

$$\limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0$$

Thus trajectories starting from points in a scrambled set are not even 'asymptotically periodic'. Furthermore, for any pair of initial states in the scrambled set, the trajectories move apart and return close to each other infinitely often.[1]

The following theorem, due to Li and Yorke (1975), is fundamental in establishing a connection between the existence of period-three cycles and the existence of an uncountable scrambled set.

PROPOSITION 2. (Li and Yorke): Assume that there is some point x^* in I such that:

$$f^3(x^*) \leq x^* < f(x^*) < f^2(x^*) \text{ (or } f^3(x^*) \geq x^* > f(x^*) > f^2(x^*)) \quad (6)$$

Then (i) for every positive integer $k = 1, 2, \ldots$, there is a periodic point of period k;
(ii) there is an uncountable scrambled set $S \subset I$.

We will say that the dynamical system (I, f) exhibits *Li–Yorke chaos* if conditions (i) and (ii) of Proposition 2 are satisfied.[2] It is easy to check that (I, f) exhibits Li-Yorke chaos if and only if (I, f) has a periodic point of period three.

[1]Property (ii) is meant to capture the aspect of 'irregular' or 'erratic' behaviour. It means that long-run behaviour of this dynamical system cannot be approximated by regular periodic motion, however long the period of the cycles. Property (i), on the other hand, is meant to capture the aspect of sensitive dependence of the system to initial conditions. Thus, small (computation or estimtion) errors in initial conditions can be transformed into large errors over time, making 'intermediate-run' predictions inaccurate.

The reader is referred to Block and Coppel (1992) and Devaney (1989) for further discussion of these issues.

[2]The reason for including condition (i) in the definition of chaos should be explained. The point of view being expressed here (as in much of the literature on 'topological' chaos) is that the concept of chaos should involve 'complicated behaviour' but also a certain amount of 'regularity'. The existence of a scrambled set is meant to capture the first aspect, and the presence of the periodic points the second.

There are, of course, variations on condition (i) that have been used. It has been suggested that one could use, instead of (i), 'there is a positive integer k^* such that for every positive integer $k \geq k^*$, there is a periodic point of period k'. Or, one could use, instead of (i), 'there are infinitely many periodic points of different periods'. From the point of view of obtaining discount factor restrictions assocated with chaotic optimal behaviour (discussed in Section 5), these alternate definitions do, of course, make a difference. The first variation mentioned above continues to impose strong discount factor restrictions for chaotic optimal behaviour [see Mitra (1996)], the second variation does not [see Nishimura and Yano (1995)].

2.2 Dynamic Optimization

The standard framework of dynamic optimization is described by a triplet (Ω, u, δ), where Ω, a subset of $\Re_+ \times \Re_+$, is a 'transition possibility set', $u : \Omega - \Re$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$.

The transition possibility set describes the states $z \in \Re_+$ that it is possible to go to tomorrow, if one is in the state $x \in \Re_+$ today. We define a correspondence $\Gamma : \Re_+ \to \Re_+$ by $\Gamma(x) = \{y \in \Re_+ : (x, y) \in \Omega\}$ for each $x \in \Re_+$.

A programme $\{x_t\}_0^\infty$ from $x \in \Re_+$ is a sequence satisfying,

$$x_0 = x \text{ and } (x_t, x_{t+1}) \in \Omega \quad \text{for } t \geq 0$$

If one is in state x today and one moves to state z tomorrow [with $(x, y) \in \Omega$] then there is an immediate utility (or 'reward' or 'return') generated, measured by the utility function, u.

The discount factor, δ, is the weight assigned to tomorrow's utility (compared to today's) in the objective function. The *discount rate* (associated with the discount factor, δ) is given by $\rho = (1/\delta) - 1$.

The following assumptions are imposed on the transition possibility set, Ω:

(A.1) (i) $(0, 0) \in \Omega$, (ii) $(0, z) \in \Omega$ implies $z = 0$.
(A.2) Ω is (i) closed, and (ii) convex.
(A.3) There is $\xi > 0$ such that $(x, z) \in \Omega$ and $x \geq \xi$ implies $z < x$.
(A.4) If $(x, z) \in \Omega$ and $x' \geq x$, $0 \leq z' \leq z$ then $(x', z') \in \Omega$.

Clearly, we can pick $0 < \zeta < \xi$, such that if $x > \zeta$ and $(x, z) \in \Omega$, then $z < x$. It is straightforward to verify that if $(x, z) \in \Omega$, then $z \leq \max(\zeta, x)$. It follows from this that if $\{x_t\}_0^\infty$ is a programme from

[3]The distinction between ξ and ζ should be explained. Clearly ζ can be a 'maximum sustainable stock', while ξ cannot. Thus $Y = [0, \xi]$ is a somewhat larger closed interval than the state space $X = [0, \zeta]$, where the important dynamics will take place.

The distinction is made with assumption (A.7) in mind. It would have been nice to simply assume the monotone property of u on Ω. But this creates problems in establishing (the sufficiency part of) Theorem 3. The assumption (A.7) states that the monotone property of u holds on Ω, when x, x' are restricted to the closed interval, Y. The point about Y instead of X appearing in this definition is that the sufficiency part of Theorem 3 can be established even when u is required to be monotone on Y, rather than merely on X. [The necessity part of Theorem 3 is valid even when X is used instead of Y in the statement of assumption (A.7).]

$x \in \Re_+$ then $x_t \leq \max(\zeta, x)$ for $t \geq 0$. In particular, if $x \leq \zeta$, then $x_t \leq \zeta$ for $t \geq 0$. This leads us to choose the closed interval $[0, \zeta]$ as the natural state space of our model, which we will denote by X. We denote the interval $[0, \xi]$ by Y.[3]

The following assumptions are imposed on the utility function, u:

(A.5) u is concave on Ω; further if (x, z) and (x', z') are in Ω, and $x \neq x'$, then for every $0 < \lambda < 1$, $u[\lambda(x, z) + (1 - \lambda)(x', z')] > \lambda u(x, z) + (1 - \lambda)u(x', z')$.

(A.6) u is upper semi-continuous in Ω.

(A.7) If $x, x' \in Y$, $(x, z) \in \Omega$, $x' \geq x$ and $0 \leq z' \leq z$, then $u(x', z') \geq u(x, z)$.

We will refer to a triplet (Ω, u, δ) satisfying (A.1)–(A.7) as a *dynamic optimization model*. A programme $\{\hat{x}_t\}_0^\infty$ from $x \geq 0$ is an 'optimal programme' if,

$$\sum_0^\infty \delta^t u(x_t, x_{t+1}) \leq \sum_0^\infty \delta^t u(\hat{x}_t, \hat{x}_{t+1})$$

for every programme $\{x_t\}_0^\infty$ from x.

2.3 Two Examples

We now provide two examples of dynamic optimization problems in economics which can be studied by converting them to the 'reduced-form' problem described in the previous sub-section.

EXAMPLE 1. Optimal Exploitation of a Renewable Resource

The state variable, x, here is to be interpreted as the *resource stock*. The future population of the stock is determined from the present population by means of a 'reproduction' or 'recruitment' function, f.

If we denote the harvest of the resource (the control variable) by c, then the *return* at any date is dependent on the harvest, c, and the stock, x, through a function, w. Given a discount factor, δ, the problem of resource management is to determine the sequence of harvests which will maximize the discounted sum of returns.

Formally, the model[4] is specified by (f, w, δ), where,

(a) $f: \Re_+ \to \Re_+$ is a *stock-recruitment function* satisfying, $f(0) = 0$, f is increasing, strictly concave and continuous with

$$\lim_{x \to \infty} [f(x)/x] < 1;$$

[4]Majumdar and Mitra (1994) have established the possibility (and robustness) of chaotic behaviour in this aggregative model.

(b) $w : \Re_+^2 \to \Re$ is a *return function* satisfying,
  (i) $w(x, c)$ is continuous in (x, c);
  (ii) $w(x, c)$ is non-decreasing in c (given x) and in x (given c); also it is increasing in c if $x > 0$;
  (iii) $w(x, c)$ is concave in (x, c); also it is strictly concave in c if $x > 0$;

(c) $0 < \delta < 1$ is a *discount factor.*

A *programme* from x is a sequence $\{x_t\}$ such that:

$$x_0 = x,\ 0 \le x_{t+1} \le f(x_t) \quad \text{for } t \ge 0$$

Associated with a programme $\{x_t\}$ is a sequence of harvests $\{c_{t+1}\}$:

$$c_{t+1} = f(x_t) - x_{t+1} \quad \text{for } t \ge 0$$

A programme $\{x_t^*\}$ from x is optimal if,

$$\sum_0^\infty \delta^t w(x_t^*, c_{t+1}^*) \le \sum_0^\infty \delta^t w(x_t, c_{t+1})$$

for every programme $\{x_t\}$ from x.

If we define $\Omega = \{(x, z) \text{ in } \Re_+^2 : 0 \le z \le f(x)\}$, and for all (x, z) in Ω, $u(x, z) = w(x, f(x) - z)$, then the *reduced-form model* (Ω, u, δ) satisfies assumptions (A.1)–(A.7), and programmes (optimal programmes) in the reduced-form model correspond exactly to programmes (optimal programmes) in the primitive form of the model.

Notice that we allow for a 'stock effect' in the return function, but our assumptions allow us to consider the special case in which the return, w, is independent of the stock, x. In that case, the above model is formally equivalent to the one-sector neoclassical optimal growth model, *à la* Cass–Koopmans.

EXAMPLE 2. Two Sector Model of Optimal Economic Growth

Formally, the model is specified by (f, g, μ, δ) where,

(a) the *production function* in the *consumption good sector,* $f : \Re_+^2 \to \Re_+$, satisfies;
  (i) f is continuous on $\Re_+^2$,
  (ii) f is non-decreasing on $\Re_+^2$ and increasing on $\Re_{++}^2$,
  (iii) f is concave on $\Re_+^2$, and strictly concave in the first argument (capital) when the second argument (labour) is positive;

(b) the *production function* in the *investment good sector*, $g : \Re_+^2 \to \Re_+$, satisfies;

(i) g is continuous on $\Re_+^2$,

(ii) g is non-decreasing on $\Re_+^2$ and increasing on $\Re_{++}^2$

(iii) g is concave on $\Re_+^2$, and strictly concave in the first argument (capital) when the second argument (labour) is positive,

(iv) $\lim_{K \to \infty} [g(K, 1)/K] < 1$;

(c) the *depreciation factor*, μ, satisfies $0 \leq \mu < 1$;

(d) the *discount factor*, δ, satisfies $0 < \delta < 1$.

A *programme* from x is a sequence $\{k_t, n_t, x_t\}$ such that,

$$x_0 = x, \text{ and } 0 \leq n_t \leq 1, 0 \leq k_t \leq x_t \quad \text{for } t \geq 0$$

$$x_{t+1} = \mu x_t + g(x_t - k_t, 1 - n_t) \quad \text{for } t \geq 0$$

Associated with a programme $\{k_t, n_t, x_t\}$ from x is a *consumption sequence* $\{c_{t+1}\}$ given by,

$$c_{t+1} = f(k_t, n_t) \quad \text{fo } t \geq 0$$

A programme $\{k_t^*, n_t^*, x_t^*\}$ is *optimal* if

$$\sum_0^\infty \delta^t c_{t+1}^* \geq \sum_0^\infty \delta^t c_{t+1}$$

for every programme $\{k_t, n_t, x_t\}$ from x.

If we define,

$$\Omega = \{(x, z) \text{ in } \Re_+^2 : z \leq \mu x + g(x, 1)\},$$

and for all (x, z) in Ω, define,

$$u(x, z) = \max f(k, n)$$

subject to

$$0 \leq k \leq x$$

$$0 \leq n \leq 1$$

$$z \leq \mu x + g(x - k, 1 - n)$$

then the 'reduced form model' (Ω, u, δ) satisfies assumptions (A.1)–(A.7) and optimal programmes in the reduced-form model correspond exactly to optimal programmes in the primitive form of the model.[5]

[5]Boldrin and Montrucchio (1986) have established the possibility of chaotic behaviour in a two-sector model similar to this one.

3. DYNAMIC PROGRAMMING AND DUALITY THEORY

3.1 The Dynamic Programming Approach

Under the maintained assumptions of the previous section [(A.1)–(A.7)], it is standard exercise to show that there is a unique optimal programme from every $x \in \Re_+$. This leads us to define the value and policy functions as follows.

The *value function* $V : \Re_+ \to \Re$ is defined by,

$$V(x) = \sum_{0}^{\infty} \delta^t u(\hat{x}_t, \hat{x}_{t+1})$$

where $\{\hat{x}_t\}_0^\infty$ is the optimal programme from $x \in \Re_+$.

The 'policy function' $h : \Re_+ \to \Re_+$ is defined by,

$$h(x) = \hat{x}_1$$

where $\{\hat{x}_t\}_0^\infty$ is the optimal program from $x \in \Re_+$.

The properties of the value and policy functions can be summarized in the following result. This is based on Dutta and Mitra (1989) and Stokey, Lucas and Prescott (1989).

Proposition 3:

(i) The value function V is strictly concave and continuous on $\Re_+$ and non-decreasing on Y. Further, V is the unique continuous function on $Y \equiv [0, \xi]$ which satisfies the functional equation of dynamic programming,

$$V(x) = \max_{y \in \Gamma(x)} [u(x, y) + \delta V(y)]$$

(ii) The policy function h satisfies the following property: for each $x \in \Re_+$, $h(x)$ is the unique solution to the constrained maximization problem,

$$\begin{aligned} &\text{Maximize} && u(z, y) + \delta V(y) \\ &\text{Subject to} && y \in \Gamma(x) \end{aligned}$$

Further, h is continuous on $\Re_+$, and $h(0) = 0$.

Remarks:

(i) In view of the definition of the policy function h, the optimal programme from $x \in X$ is the trajectory $\{h^t(x)\}_0^\infty$ generated by the policy function. Thus, an optimal programme from $x \in X$ can be called *periodic* (with period k) if x is a periodic point of h (with period k).

(ii) Since V is concave on $\Re_+$, it has well-defined left-hand and right-hand derivatives for all $x > 0$, which we denote by $\mu(x)$ and $\nu(x)$ respectively. If $x > y > 0$, then we have $\mu(y) \geq \nu(y) > \mu(x) \geq \nu(x)$, the strict inequality following from the strict concavity of V.

We will find it convenient to state a result here [see Mitra (1996) for a proof] which compares the value and policy functions of two dynamic optimization models. To proceed more formally, let us refer to a triplet (Ω, u, δ) satisfying (A.1)–(A.7) as a *dynamic optimization model*. Now, let $\mathfrak{I} = (\Omega, u, \delta)$ be a dynamic optimization model, with value function V and policy function h. We can then construct another dynamic optimization model $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$ with $\delta^* = \delta^2$, such that the policy function, h^* of $\mathfrak{I}^*$ is h^2 and the value function, V^*, of $\mathfrak{I}^*$ is V.

PROPOSITION 4: Let $\mathfrak{I} = (\Omega, u, \delta)$ be a dynamic optimization model, with value function V and policy function h. Then, there exists Ω^* and u^* satisfying (A.1)–(A.7) such that with $\delta^* = \delta^2$, (i) the value function V^*, of the dynamic optimization model $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$ is given by V, and (ii) the policy function, h^*, of $\mathfrak{I}^*$ is given by h^2.

3.2 Duality Theory

Optimality can be characterized in terms of dual variables or shadow prices. At the shadow prices supporting an optimal programme, there is no activity which yields a higher 'generalized profit' at any date (value of utility plus value of terminal stocks minus value of initial stocks at that date) than the activity chosen along the optimal programme at that date. It was observed by Weitzman (1973) that these shadow prices support the value function as well.[6] The basic result of the theory, describing this characterization, can be stated as follows. [A full discussion can be found in Weitzman (1973) and McKenzie (1986).]

PROPOSITION 5: (Weitzman)

(a) If $\{x_t\}_0^\infty$ is an optimal programme from $x \in X$ and $x > 0$, and there is some $(\bar{x}, \bar{y}) \in \Omega$ with $\bar{y} > 0$ then there is a sequence $\{p_t\}_0^\infty$ of non-negative prices such that for $t \geq 0$,

(i) $\delta^t V(x_t) - p_t x_t \geq \delta^t V(x) - p_t x$ for all $x \geq 0$

[6]Weitzman's *proof* of this price characterization of optimality also fully exploited Bellman's principle of optimality.

(ii) $\delta^t u(x_t, x_{t+1}) + p_{t+1}x_{t+1} - p_t x_t \geq \delta^t u(x, y) + p_{t+1}y - p_t x$ for all $(x, y) \in \Omega$

(iii) $\lim_{t \to \infty} p_t x_t = 0$

(b) If $\{x_t\}_0^\infty$ is a programme from $x \geq 0$, and there is a sequence $\{p_t\}_0^\infty$ of non-negative prices such that for $t \geq 0$, (ii) and (iii) above are satisfied, then $\{x_t\}_0^\infty$ is an optimal programme from x.

If $\{x_t\}_0^\infty$ is a programme from $x \geq 0$, and $\{p_t\}_0^\infty$ is non-negative sequence of prices satisfying (i), (ii) and (iii) of Proposition 5(a), we will say that the program $\{x_t\}_0^\infty$ is price supported by $\{p_t\}_0^\infty$. When $\{x_t\}_0^\infty$ is price supported by $\{p_t\}_0^\infty$, we refer to $\{p_t\}_0^\infty$ as a sequence of *present-value prices*. Associated with $\{p_t\}_0^\infty$ is a sequence $\{p_t\}_0^\infty$ of *current value prices* defined by,

$$P_t = (p_t/\delta^t) \quad \text{for } t \geq 0$$

This price characterization of optimality leads to a basic tool for analyzing the nature of policy functions which we state in the following proposition.[7]

PROPOSITION 6: Let (Ω, u, δ) be a dynamic optimization model. Suppose $\{x_t\}_0^\infty$ is an optimal programme with price support $\{p_t\}_0^\infty$ and $\{y_t\}_0^\infty$ is an optimal programme with price support $\{q_t\}_0^\infty$. Denoting (p_t/δ^t) by P_t and (q_t/δ^t) by Q_t for $t \geq 0$, we have,

(i) $\delta(P_{t+1} - Q_{t+1})(y_{t+1} - x_{t+1}) \leq (P_t - Q_t)(y_t - x_t)$ for $t \geq 0$

(ii) $(P_t - Q_t)(y_t - x_t) \geq 0$ for $t \geq 0$

Furthermore, if $y_t \neq x_t$ for some t, then the inequalities in (i) and (ii) are strict for that t.

Proof: Let $\{x_t\}_0^\infty$ be an optimal programme with price support $\{p_t\}_0^\infty$, and $\{y_t\}_0^\infty$ be an optimal programme with price support $\{q_t\}_0^\infty$. Then using (ii) of Proposition 5, we get,

$$\delta^t u(x_t, x_{t+1}) + p_{t+1}x_{t+1} - p_t x_t \geq \delta^t u(y_t, y_{t+1}) + p_{t+1}y_{t+1} - p_t y_t \quad (6)$$

[7]The inequalities of Proposition 6 are very familiar objects from the point of view of the turnpike theory literature, where they suggest a natural choice of a Lyapunov function for the study of global asymptotic stability of optimal growth paths; see especially Cass and Shell (1976) and McKenzie (1986). The same inequalities have figured prominently in the literature on the intertemporal decentralization of the transversality condition; see Brock and Majumdar (1988), and Dasgupta and Mitra (1988).

and

$$\delta^t u(y_t, y_{t+1}) + q_{t+1}y_{t+1} - q_t y_t \geq \delta^t u(x_t, x_{t+1}) + q_{t+1}x_{t+1} - q_t x_t \quad (7)$$

Adding (6) and (7) and cancelling common terms,

$$(q_t - p_t)(x_t - y_t) \geq (q_{t+1} - p_{t+1})(x_{t+1} - y_{t+1}) \quad (8)$$

Denoting (p_t/δ^t) by P_t and (q_t/δ^t) by Q_t, we get (i). By (A.5), the inequality in (i) is strict when $x_t \neq y_t$.

Using (i) of Proposition 5, we get,

$$\delta^t V(x_t) - p_t x_t \geq \delta^t V(y_t) - p_t y_t \quad (9)$$

and

$$\delta^t V(y_t) - q_t y_t \geq \delta^t V(x_t) - q_t x_t \quad (10)$$

Adding (9) and (10) and cancelling common terms,

$$(q_t - p_t)(x_t - y_t) \geq 0 \quad (11)$$

which yields (ii), upon dividing (11) by δ^t. By the strict concavity of V, the inequality in (ii) is strict when $y_t \neq x_t$.

4. THE NATURE OF POLICY FUNCTIONS

4.1 THE BOLDRIN–MONTRUCCHIO RESULT

We know (from Proposition 3) that given a dynamic optimization model (Ω, u, δ) satisfying (A.1)–(A.7), the policy function, $h : \mathfrak{R}_+ \to \mathfrak{R}_+$, is continuous on $\mathfrak{R}_+$ and $h(0) = 0$. We now ask the following question: Are these the only restrictions on the policy function that the exercise of dynamic optimization imposes?

The starting point of the discussion regarding an answer to this question is, of course, the result of Boldrin and Montrucchio (1986), who showed that if h is any, twice continuously differentiable function, then one can always construct a dynamic optimization model, such that h is the policy function of that model. We state the result here in a slightly modified form, since the maintained assumptions on a dynamic optimization model that we use differ from theirs.

PROPOSITON 7: (Boldrin and Montrucchio) Let h: $\mathfrak{R}_+ \to \mathfrak{R}_+$ be any C^2 function on $\mathfrak{R}_+$ satisfying $h(0) = 0$. Then, there is dynamic optimization model (Ω, u, δ), such that its policy function, H, coincides with h on the state space X, associated with it.

This result has sometimes been interpreted as showing that dynamic optimization imposes no non-trivial restrictions on the nature of the policy function. The rest of this section is devoted to examining the validity of this claim.

We should observe at this point that differentiability of the policy function of a dynamic optimization model has been obtained only under quite strong restrictions [see Araujo (1991), Santos (1991)]. This indicates that it is not a 'natural' restriction imposed by the optimization exercise. Indeed, if we consider the standard, two-sector model of optimal economic growth (see Example 2 in Section 'Preliminaries'), with fully-depreciating capital ($\mu = 0$) and with Cobb–Douglas production functions [$f(k, n) = k^a n^{1-a}$ and $g(K, N) = K^{\alpha} N^{1-\alpha}$], such that the consumption goods sector is more capital-intensive than the capital goods, sector [$a > \alpha$], then the policy function will be a tent-like map with a kink in the interior of the associated state space [see Nishimura and Yano (1994) for a demonstration of this interesting result].

This means that there is no hope of strengthening Proposition 3 to conclude differentiability (and certainly, twice continuous differentiability) of the policy function, *even if the primitives of the model (utility and production functions) were smooth.*[8]

Recognition of this fact compels us to consider the C^2 restriction in the Boldrin–Montrucchio result as a strong one, and consequently, it becomes important to examine whether their result can be extended to the class of continuous functions. Neumann et al. (1988) have shown that we can replace the C^2 restriction in Proposition 7 with the restriction that h be C^1 and the derivative of h be Lipschitz continuous.[9] As we have already argued, we have to address the problem where h is continuous, but not necessarily differentiable, and so, this extension result is not useful in this regard.

One route to a possible extension of the Boldrin–Montrucchio result is to uniformly approximate the given continuous function (using the famous Weierstrass theorem) by polynomials (which are, of course,

[8]The differentiability of the *value function* (in the interior of the state space) can be considered to be a 'natural restriction' imposed by the optimization exercise; see Benveniste and Scheinkman (1979).

[9]Unlike the Boldrin-Montrucchio paper, where the state space is a subset of $\mathbb{R}^n$, the paper of Neumann et al. (1988) deals entirely with the special framework, in which the state space is a subset of $\mathbb{R}$. However, the result of Neumann et al. has been established in the more general framework, by using a different method of proof, in Montrucchio (1994).

C^2 functions), and use the Boldrin–Montrucchio result on the sequence of approximating functions. But, obtaining a dynamic optimization model as the limit of the associated sequence of dynamic optimization models, which will yield the limiting function as its policy function, turns out to be a major obstacle, even under uniform approximation.

In fact, we will show below, by examining the nature of policy functions at their *fixed points* (boundary or interior) that the Boldrin–Montrucchio result *cannot* be extended to the class of continuous functions. We turn to this demonstration in the next sub-section.

4.2 Behaviour of a Policy Function Near a Fixed Point

We will present two results in this sub-section which will show that the optimization exercise imposes non-trivial restriction on the nature of the policy function near fixed points.[10] Our first result explores such restrictions at a boundary fixed point, and the second at an interior fixed point.[11]

Theorem 1: Suppose $h : \Re_+ \to \Re_+$ is a continuous function, and there is $b > 0$, such that,

$$h(0) = 0 = h(b) \tag{12}$$

If h is the optimal policy function of a dynamic optimization model (Ω, u, δ), then,

$$\delta\omega \leq 1 \tag{13}$$

where,

$$\omega \equiv \limsup_{x \to 0} [h(x)/x] \tag{14}$$

Proof: Since (13) is clearly satisfied when $\omega \leq 1$, we consider only the case $\omega > 1$. This means, in particular, that there is some $a > 0$ such that $h(a) > 0$.

Since h is the optimal policy function of a dynamic optimization model (Ω, u, δ), and $h(b) = 0$, the sequence $(b, 0, 0, \ldots)$ is the optimal

[10]By Proposition 4, and its obvious extension to (h^n, δ^n) where $n \geq 2$ [see Sorger (1992b) for a version of such a result], we conclude that the optimization exercise also imposes non-trivial restrictions on the nature of the policy function near periodic points.

[11]Sorger (1992a, 1992b) obtains a discount factor restriction under a condition like (12), but he also requires h to be surjective, which makes it less widely applicable than our result.

programme from b. By Proposition 5, it can be price-supported by $(r_0, r_1, r_2, \ldots)$ since $h(a) > 0$. Then the sequence $(0, 0, \ldots)$ is price supported by $(p_0, p_1, p_2 \ldots)$ where $p_t = r_{t+1}$ for $t \geq 0$.

We claim that there is $\hat{p}$ such that p_t can be chosen to be $\delta^t \hat{p}$ for $t \geq 0$. To see this, denote the current-value price sequence associated with $(p_0, p_1, p_2, \ldots)$ by $(P_0, P_1, P_2, \ldots)$ and note that for all $t \geq 0$,

$$u(x, y) + \delta P_{t+1} y - P_t x \leq u(0, 0) \quad \text{for all } (x, y) \in \Omega$$

and

$$V(x) - P_t x \leq V(0) \quad \text{for all } x \geq 0$$

We divide our analysis into two sub-cases.

Case (i): Suppose $P_{t+1} \geq P_t$ for some $t = \tau$. Then, it follows that,

$$u(x, y) + \delta P_\tau y - P_\tau x \leq u(0, 0) \quad \text{for all } (x, y) \in \Omega$$

and

$$V(x) - P_\tau x \leq V(0) \quad \text{for all } x \geq 0$$

Thus, choosing $\hat{P} = P_\tau$, the claim is established.

Case (ii): Suppose $P_{t+1} < P_t$ for all $t \geq 0$. Then P_t converges to some $P \geq 0$, and so,

$$u(x, y) + \delta P y - P x \leq u(0, 0) \quad \text{for all } (x, y) \in \Omega$$

and

$$V(x) - P x \leq V(0) \quad \text{for all } x \geq 0$$

Thus, choosing $\hat{P} = P$, the claim is established again.

Since $\omega > 1$, there is a sequence $x^s \to 0$ such that $h(x^s) > x^s > 0$ for $s = 1, 2, 3, \ldots$, and $\omega = \lim_{s \to \infty} [h(x^s)/x^s]$. Using Proposition 5, for each s, the optimal programme can be price-supported by $(p_0^s, p_1^s, \ldots)$, with associated current price sequence $(P_0^s, P_1^s, \ldots)$. Using Proposition 6, we then have,

$$\delta(\hat{P} - P_1^s)(h(x^s) - 0) \leq (\hat{P} - P_0^s)(x^s - 0)$$

Now, we have, by the strict concavity of V, $\hat{P} > \mu(x^s) \geq P_0^s \geq \nu(x^s) > \mu(h(x^s)) \geq P_1^s$, where $\mu(\nu)$ denotes the left (right) hand derivative of V. Thus, we get,

$$\delta[h(x^s)/x^s] \leq (\hat{P} - P_0^s)/(\hat{P} - P_1^s) < 1$$

and letting $s \to \infty$, we obtain $\delta\omega \leq 1$.

Theorem 1 allows us to write down a family of continuous functions, h, with $h(0) = 0$, which cannot be policy functions of any dynamic optimization model.

COROLLARY 1: Suppose $h(x) = \mu x^a(1 - x^a)$, where $0 < \mu \leq 4$, $0 < a < 1$, and $x \in [0, 1]$, then there is no dynamic optimization model (Ω, u, δ), whose optimal policy function is given by h on $[0, 1]$.

Corollary 1 raises the question of whether the difficulty in extending the Boldrin–Montrucchio result to continuous functions is a 'boundary phenomenon' similar to that in the Sonnenschein–Mantel–Debreu theorem on market excess-demand functions [see Shafer and Sonnenschein (1982)]. Our next result on the behaviour of a policy function near an interior fixed point indicates that this is not the case.[12]

Theorem 2: Suppose $h : \Re_+ \to \Re_+$ is a continuous function, and there is $b > 0$, such that,

$$b = h(b) \tag{15}$$

If h is the optimal policy function of a dynamic optimization model (Ω, u, δ), then,

$$\delta\theta \leq 1 \tag{16}$$

where,

$$\theta \equiv \limsup_{x \downarrow b} \{[h(x) - h(b)]/[x - b]\} \tag{17}$$

Proof: Since (16) is clearly satisfied when $\theta \leq 1$, we consider only the case in which $\theta > 1$. This means that there is a sequence $\{x^s\}_{s=1}^{\infty}$ such that $h(x^s) > x^s > b$ and $[h(x^s) - h(b)]/(x^s - b)$ converges to θ as $s \to \infty$.

Since $h(b) = b$, the sequence $(b, b, b, \ldots)$ is the optimal programme from b, and can be price-supported by $(p_0, p_1, \ldots)$. Using the method of Sutherland (1967) and McKenzie (1986), we can find $\hat{P}$ such that p_t can be chosen to be $\delta^t \hat{P}$ for $t \geq 0$.

Using Proposition 5, for each $s = 1, 2, 3, \ldots$, the optimal programme from x^s can be price-supported by $(p_0^s, p_1^s, \ldots)$, with associated current price sequence $(P_0^s, P_1^s, \ldots)$. Using Proposition 6,

$$\delta(\hat{P} - P_1^s)(h(x^s) - b) \leq (\hat{P} - P_0^s)(x^s - b)$$

[12]A similar result is established in Hewage and Neumann (1990), by using methods different from ours.

By the strict concavity of V, we have $\hat{P} > \mu(x^s) \geq P_0^s \geq \nu(x^s) > \mu(h(x^s)) \geq P_1^s$, where $\mu(\nu)$ denotes the left (right) hand derivative of V. Thus, we get,

$$\delta(h(x^s) - b)/(x^s - b) \leq (\hat{P} - P_0^s)/(\hat{P} - P_1^s) < 1$$

and so, letting $s \to \infty$, we obtain $\delta\theta \leq 1$.

Theorem 2 allows us to write down a continuous, increasing, function, h, with $h(0) = 0$, which cannot be the policy function of any dynamic optimization model, even on a restricted domain which excludes the boundary of the natural state space.

COROLLARY 2: Suppose $h(x) = 1 + (x - 1)^{1/3}$ for $x \in [0, 1]$. Given any $0 \leq \varepsilon < 1$, there is no dynamic optimization model (Ω, u, δ) whose optimal policy function is given by h on $[\varepsilon, 2 - \varepsilon]$.

Both Corollary 1 and Corollary 2 provide examples of continuous functions, with unbounded steepness at some point of the relevant domain. One might, therefore, conjecture that the Boldrin–Montrucchio result might be extended to the class of Lipschitz continuous policy functions. We leave this as an open question.

We note, however, that even Lipschitz-continuity of the policy function can be ensured only under fairly stringent conditions on the dynamic optimization model [see Montrucchio (1987)]. In fact, if we take the standard neoclassical one-sector model of economic growth [obtained as a special case of Example 1, where there is a function $g : \Re_+ \to \Re_+$ such that $w(x, c) = g(c)$ for all $(x, c) \in \Re_+^2$], it is fairly straightforward to obtain policy functions which do not satisfy Lipschitz-continuity.

To see this let $f(x) = 2x^{1/2}$ for $x \in \Re_+$; $g(c) = 2c^{1/2}$ for $c \in \Re_+$; and $0 < \delta < 1$. We claim that the policy function, h, of this dynamic optimization model satisfies,

$$\limsup_{x \to 0} [h(x)/x] = \infty \tag{18}$$

For, suppose there is $0 < M < \infty$, such that,

$$\limsup_{x \to 0} [h(x)/x] < M \tag{19}$$

Then, there is $\varepsilon_0 > 0$ such that on $Z = (0, \varepsilon_0)$, $[h(x)/x] \leq M$. Choose $0 < \varepsilon < \varepsilon_0$, such that $(1/\varepsilon^{1/2}) \geq \max\,[M, 2^{1/2}M^{3/4}/\delta]$.

Choose any $x \in (0, \varepsilon)$. We know that an optimal programme from x is interior ($x_t > 0$ and $c_{t+1} > 0$ for $t \geq 0$), and so, the following

Ramsey-Euler equation holds,

$$[g'(c_1)/g'(c_2)] = \delta f'(x_1) \qquad (20)$$

Using (19), we get

$$f(x_1) = f[h(x)] \leq f(Mx) \leq 2M^{1/2}x^{1/2},$$

so that

$$c_2 = f(x_1) - x_2 \leq f(x_1) \leq 2M^{1/2}x^{1/2}.$$

Also,

$$c_1 = f(x) - x_1 = f(x) - h(x) \geq 2x^{1/2} - Mx \geq x^{1/2} + (x^{1/2} - Mx) > x^{1/2}$$

since $(1/x^{1/2}) > M$. Thus, we get,

$$[g'(c_1)/g'(c_2)] < 2^{1/2}M^{1/4} \qquad (21)$$

Using (19), we also get $x_1 = h(x) \leq Mx$, so that,

$$\delta f'(x_1) \geq [\delta/x_1^{1/2}] \geq [\delta/M^{1/2}x^{1/2}] \qquad (22)$$

Combining (20), (21), and (22), we obtain,

$$2^{1/2}M^{1/4} > [\delta/M^{1/2}x^{1/2}]$$

so that,

$$[2^{1/2}M^{3/4}/\delta] > [1/x^{1/2}] > [1/\varepsilon^{1/2}]$$

But this contradicts the definition of ε, and establishes our claim (18).

5. DISCOUNT FACTOR RESTRICTIONS FOR CHAOTIC POLICY FUNCTION

When Boldrin and Montrucchio applied their result to the logistic function,

$$h(x) = 4x(1 - x) \quad \text{for } x \in [0, 1]$$

they found a dynamic optimization model $[\Omega, u, \delta]$ whose policy function coincides with h on [0, 1], and the discount factor, δ, of this constructed dynamic optimization model was approximately 0.01. This discount factor was considered to be 'low'. Let us follow Nishimura and Sorger (1996) to explain what we mean by this.

Let the model's time period be equal to $m > 0$ years. Let r be the long-run annual real interest rate. Then the discount factor, δ, is related to r and m by the usual formula,

$$\delta = [1/(1 + r)^m]$$

Suppose $r = 5\%$. Then the model's time period is 1 year if $\delta = 0.952$; 10 years if $\delta = 0.614$; 20 years if $\delta = 0.377$.

Thus, if $\delta < 0.5$, then the model's time period (which represents the decision-making time) is longer than 14 years, which is much too long to be realistic.

In the case of the logistic function, a very low discount factor of about 0.01 raised the question of whether the logistic function had actually been 'rationalized'. Of course, it was not clear whether this feature of the dynamic optimization model was simply a shortcoming of the particular method used in its construction, or whether chaotic policy functions could never be rationalized when 'reasonable' discount factors prevailed.

Sorger (1992a, 1992b) began a systematic study of discount factor restrictions that must be satisfied by any dynamic optimization model, which generates a chaotic policy function. These were followed by more definitive results by Sorger (1994), Mitra (1996) and Nishimura and Yano (1996).

We will not try to do justice to all the contributions. Instead, we will show how the two theorems of the previous section, which were used there to obtain restrictions on policy functions generated by dynamic optimization models, can be conveniently used also to obtain discount factor restrictions on the dynamic optimization model generating a logistic, or a tent map as its policy function.

COROLLARY 3: Suppose $h(x) = \mu x(1 - x)$, where $0 < \mu \leq 4$, and $x \in [0, 1]$, and h is the optimal policy function of any dynamic optimization model (Ω, u, δ), then $\delta \leq (1/\mu)$.

Corollary 3 readily follows from Theorem 1, since $h(0) = 0 = h(1)$, and $\omega = \limsup_{x \to 0} [h(x)/x] = \mu$.

The class of functions covered by Corollary 3 is called the 'quadratic family'. The logistic function is a member of this family (obtained when $\mu = 4$), and so, the discount factor restriction on any dynamic optimization model, which generates it as a policy function, is 0.25.

COROLLARY 4: Suppose h_a: [0, 1] – [0, 1] is given by the following family of tent-maps,

$$h_a(x) = \begin{cases} ax & \text{for } 0 \leq x \leq (1/a) \\ [a/(a-1)] - [a/(a-1)]x & \text{for } (1/a) \leq x \leq 1 \end{cases}$$

where $a > 1$, and h_a is the optimal policy function of any dynamic optimization model (Ω, u, δ), then,

$$\delta \leq \min\,[(a-1)/a,\ 1/a]$$

Corollary 4 follows from Theorems 1 and 2. Since $h(0) = 0 = h(1)$, and $\omega \equiv \limsup_{x \to 0}\,[h(x)/x] = (1/a)$, Theorem 1 implies that,

$$\delta \leq (1/a) \tag{23}$$

The interior fixed point of h_a is at $x_a = a/(2a-1)$. Clearly h_a is decreasing in x at this fixed point, with slope $= -[a/(a-1)]$. Thus, the second iterate of h_a, denoted by h_a^2, is increasing at x_a with slope $a^2/(a-1)^2 > 1$.

Since h_a is the policy function of the dynamic optimization model (Ω, u, δ), using Proposition 4, we can find a dynamic optimization model $(\Omega^*, u^*, \delta^*)$, such that $\delta^* = \delta^2$ and h_a^2 is the policy function of $(\Omega^*, u^*, \delta^*)$. Applying Theorem 2 to h_a^2, and noting that,

$$\theta \equiv \limsup_{x \downarrow x_a}\,\{[h_a^2(x) - h_a^2(x_a)]/[x - x_a]\} = a^2/(a-1)^2$$

we obtain $\delta^2[a^2/(a-1)^2] \leq 1$, so that,

$$\delta \leq (a-1)/a \tag{24}$$

Combining (23) and (24) yields the corollary.

Note that the symmetric tent-map,

$$h(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 0.5 \\ 2 - 2x & \text{for } 0.5 \leq x \leq 1 \end{cases}$$

is a member of the family of tent-maps covered by Corollary 4 (obtained when $a = 2$). Thus, the discount factor restriction on any dynamic optimization model, which generates the symmetric tent-map as its policy function, is 0.5. For the non-symmetric cases, the discount factor restriction is more severe than 0.5, for if $a > 2$, then $\delta \leq (1/a) < 0.5$, while if $1 < a < 2$, then $\delta \leq (a-1)/a = 1 - (1/a) < 1 - 0.5 = 0.5$.

While the discount factor restrictions obtained above are very strong indeed, they are derived for specific chaotic functions (logistic or tent-maps). The question arises whether there are uniformly strong restrictions on the discount factor for every dynamic optimization model, whose policy function exhibits Li-Yorke chaos (equivalently, an optimal period-three cycle). Sorger (1994) showed that 'period three implies heavy discounting', and his discount factor restriction was refined independently by Mitra (1996) and Nishimura and Yano (1996) to obtain the following definitive result.

THEOREM 3: Let (Ω, u, δ) be a dynamic optimization model, with a policy function h. If h exhibits Li–Yorke chaos, then,

$$\delta < [(\sqrt{5} - 1)/2]^2 \approx 0.3819 \tag{25}$$

Further, if $0 < \delta < [(\sqrt{5} - 1)/2]^2$, then one can construct (Ω, u) such that (Ω, u, δ) is a dynamic optimization model whose policy function exhibits Li–Yorke chaos.

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