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Noncooperative Resource Exploitation by Patient Players

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Abstract We consider a discrete-time dynamic game in which a finite number of players extract a non-renewable resource and derive consumption solely from the extracted amount (cake-eating game). Markov perfect Nash equilibria consisting of linear strategies can be constructed in this game not only if the players have time-preference factors that are smaller than 1, but also if these factors are equal to or even larger than 1. We demonstrate this result both for the case of identical players and for the case of heterogeneous players. In addition, we study the influence of the model parameters on the equilibrium.

Keywords Non-renewable resource · Common property resource · Dynamic game · Patient players · Markov perfect Nash equilibrium

JEL Classification C73 · Q30

1 Introduction

Problems caused by climate change and dwindling resources have become major challenges for mankind. Among the most fundamental stumbling blocks on the way toward a solution of these issues are the public good nature of the biosphere (especially the property of non-excludability) and the very long planning horizon that is characteristic of environmental problems. As for the latter, there is considerable controversy about the size of the discount factor that should be used in the assessment of different environmental policies; see e.g., [11]. Whereas this paper does not make any suggestions regarding the *appropriate* size of

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the discount factor, it adds a few observations about the set of *possible* discount factors that could be used in such an analysis. In particular, we show that equilibria exist and can therefore also be evaluated if there is *no discounting at all* so that all future generations are treated equally. In addition, we provide a comparative analysis of equilibrium solutions with respect to the key parameters of the model. Among other things, this sheds light on the influence of impatience on the speed of resource depletion.

There is no universal model to address issues of resource exploitation or environmental degradation. For that reason, we have decided to work with a very parsimonious model that still captures both the public good character and the intertemporal nature of environmental problems. This model is one in which $m \geq 2$ players have access to a common property resource stock and want to exploit it in an optimal way. Each player is an infinitely lived dynasty as in [1]. In every period, all players simultaneously harvest the resource and derive instantaneous utility from that amount of resource which they have extracted. The players want to maximize the sum of their discounted utilities over an infinite time horizon. To simplify even further, we consider only the case of a non-renewable resource so that our analysis fits the issues related to limited groundwater reserves better than those related to climate change. Models with the features described above are known as cake-eating problems; see [4,5] or [2].

If there is a single player and the discount factor $\rho \geq 0$ is strictly smaller than 1, then it is well known that a unique optimal extraction path exists. However, if the discount factor ρ is equal to 1, then there is no optimal solution: No matter how the single player decides to consume the resource, there is always a better way; see [4]. Theorem 1 below shows that this result no longer holds if there are $m \geq 2$ identical players and if utility maximization is interpreted in the sense of a symmetric Markov perfect Nash equilibrium (MPNE). As a matter of fact, we demonstrate that in the game with $m \geq 2$ players, there exists a unique symmetric MPNE that consists of linear strategies if the common discount factor ρ is less than *or equal to* 1, and there exist two different symmetric MPNE consisting of linear strategies if ρ is *greater than* but sufficiently close to 1. This result is derived under the assumption that the common utility function of the players has a constant elasticity of marginal utility which is not too large. Hence, the presence of strategic interaction resolves the nonexistence problem that arises in the case of a single player and no discounting. When there are two symmetric MPNE in linear strategies (which happens if the discount factor is greater than 1), then it holds that the MPNE along which the resource is depleted more slowly dominates the other MPNE in terms of welfare. Finally, we show that the aggregate propensity to extract the resource stock is a decreasing function of the discount factor and an increasing function of the number of players. This conforms to the intuition that higher impatience and stronger competition are bad from a conservationist's point of view.

The results mentioned above are shown for a model with identical players. In Sect. 5, we briefly study the case of two heterogeneous players and prove the existence of MPNE consisting of linear strategies also in this case. It can be shown that MPNE in linear strategies exist if the discount factors are smaller than, equal to, or even slightly larger than 1 provided that the elasticities of marginal utility are not too large. A comparative analysis with respect to all model parameters reveals that both a player's discount factor and the player's elasticity of marginal utility have a negative effect on this player's propensity to consume. As in the case of identical players, we find that higher impatience is bad from a conservationist's point of view.

The analysis of natural resource problems by means of dynamic games has a long tradition in economics. A quite recent survey of the literature is given in [7]. There is a huge variety of models that are considered. They can be classified along various dimensions, for example, whether the resource is renewable or non-renewable, whether the players derive utility directly from their own extraction or have to sell the extracted amount on a resource market, or whether strategies are assumed to depend on time alone (open-loop) or on the resource stock alone (Markovian). The present paper is most closely related to [2] and [3]. With Clemhout and Wan [2] it shares the restriction to a cake-eating game and the specification of the utility functions. Clemhout and Wan [2], however, use a continuous-time formulation and focus on issues different from those discussed in the present paper. In particular, they assume strictly positive discounting throughout their analysis. Dutta and Sundaram [3], on the other hand, study existence and properties of MPNE in discounted as well as undiscounted resource games. One important difference between their approach and ours is that they consider stochastic models and formulate the undiscounted game by means of long-run average utility functions, whereas we have a deterministic model and do not take any average, but look at the discounted sum of utilities for discount factors that are smaller than, equal to, or larger than 1. Despite the lack of discounting, the objective functionals of all players remain finite as a consequence of the boundedness of the non-renewable resource stock and the nature of the strategic interaction among the players, which forces the consumption rates to converge to 0 sufficiently fast.

The rest of the paper is organized as follows. In Sect. 2, we formulate the model and define Markov perfect Nash equilibria (MPNE). In Sect. 3, we derive necessary and sufficient conditions for a strategy profile consisting of linear strategies to qualify as a MPNE. Section 4 deals with symmetric MPNE in a game with identical players, whereas Sect. 5 presents results for games with heterogeneous players. Finally, Sect. 6 concludes.

2 Model Formulation

Consider a dynamic game in discrete time, in which $m \geq 2$ infinitely lived players have access to a non-renewable common property resource and derive utility solely from the consumption of this resource. The set of players is denoted by $\mathbf{M} = \{1, 2, \ldots, m\}$ and the time-domain is denoted by $\mathbb{N}_0 = \{0, 1, \ldots\}$. Denoting the resource stock at the start of period $t \in \mathbb{N}_0$ by $x_t \in \mathbb{R}_+$ and the consumption of player $i \in \mathbf{M}$ in period $t \in \mathbb{N}_0$ by $c_{i,t} \in \mathbb{R}_+$, it follows that

$$x_{t+1} = x_t - \sum_{i=1}^{m} c_{i,t} \tag{1}$$

holds for all $t \in \mathbb{N}_0$, where $x_0 > 0$ is the exogenously given initial stock of the resource. By choosing the consumption path $(c_{i,t})_{t=0}^{+\infty}$, player i derives utility

$$\sum_{t=0}^{+\infty} \rho_i^t \frac{c_{i,t}^{1-\alpha_i}}{1-\alpha_i}, \tag{2}$$

where $\rho_i > 0$ is the time-preference factor and $\alpha_i \in (0, 1)$ is the elasticity of the marginal utility of consumption.

A *stationary Markovian strategy* for player $i \in \mathbf{M}$ is a function $\sigma_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\sigma_i(x) \leq x$ for all $x \in \mathbb{R}_+$. We assume that players can only choose such strategies and we will drop the adjectives 'stationary' and 'Markovian' in what follows. At the outset of the game, each player $i \in \mathbf{M}$ chooses a strategy σ_i and determines consumption in period $t \in \mathbb{N}_0$ according to the rule $c_{i,t} = \sigma_i(x_t)$. A *strategy profile* is an ordered sequence of strategies, one for each player. The strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is *feasible* if

$$\sum_{i=1}^{m} \sigma_i(x) \le x \tag{3}$$

holds for all $x \in \mathbb{R}_+$.

Consider a feasible strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ and an arbitrary player $i \in \mathbf{M}$. Furthermore, suppose that player i's opponents $j \neq i$ play according to their strategies σ_j. In that case, the strategy σ_i is a best response for player i if choosing $c_{i,t} = \sigma_i(x_t)$ for all $t \in \mathbb{N}_0$ maximizes the objective functional in (2) subject to the constraint

$$x_{t+1} = x_t - \sum_{j \neq i} \sigma_j(x_t) - c_{i,t} \ge 0 \tag{4}$$

and the given initial stock x_0. A feasible strategy profile is a *Markov perfect Nash equilibrium* (MPNE) if every player's strategy is a best response to its opponents' strategies.[1] More formally, the feasible strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a MPNE if the following three conditions hold for all $x_0 > 0$:

(i) For all $i \in \mathbf{M}$ and given the strategies σ_j for all $j \neq i$, there exists an optimal solution $(\bar{x}_{i,t}, \bar{c}_{i,t})_{t=0}^{+\infty}$ of the dynamic optimization problem of maximizing (2) subject to (4) and the initial resource stock x_0.

(ii) The optimal solution of player i's optimization problem mentioned in condition (i) satisfies $\bar{c}_{i,t} = \sigma_i(\bar{x}_{i,t})$ for all $t \in \mathbb{N}_0$ and all $i \in \mathbf{M}$.

(iii) For all $(i, j) \in \mathbf{M}^2$ and all $t \in \mathbb{N}_0$, it holds that $\bar{x}_{i,t} = \bar{x}_{j,t}$, where $(\bar{x}_{i,t})_{t=0}^{+\infty}$ is the optimal state trajectory of player i's problem mentioned in conditions (i)–(ii) above.

3 Linear Strategies

In this section, we derive conditions under which a feasible strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ consisting of linear strategies of the form $\sigma_i = A_i x$ constitutes a MPNE. To formulate the conditions, we define for every $i \in \mathbf{M}$ the function $g_i : [0, 1] \mapsto \mathbb{R}$ by

$$g_i(z) = (1 - z)\left[1 - \rho_i^{1/\alpha_i}(1 - z)^{(1-\alpha_i)/\alpha_i}\right].$$

Proposition 1 *Let A_1, A_2,..., A_m be nonnegative real numbers and define a strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ by $\sigma_i(x) = A_i x$ for all $x \in \mathbb{R}_+$. This strategy profile is feasible and constitutes a MPNE if and only if the following conditions hold for all $i \in \mathbf{M}$:*

$$\sum_{j=1}^{m} A_j \le 1, \tag{5}$$

$$A_i = g_i\left(\sum_{j \neq i} A_j\right), \tag{6}$$

$$\rho_i \left(1 - \sum_{j=1}^{m} A_j\right)^{1-\alpha_i} < 1. \tag{7}$$

[1] Note that we define the property of being a MPNE only for feasible strategy profiles, i.e., for profiles satisfying condition (3). Sundaram [10] uses an ad hoc rule to determine the consumption rates for infeasible strategy profiles. The important point is that there is no way to place conditions on the primitives of the model that guarantee that a strategy profile will always be feasible.

Proof It is obvious that the proposed strategy profile is feasible if and only if the coefficients A_i, $i \in \mathbf{M}$, are nonnegative numbers satisfying condition (5). It remains to prove necessity and sufficiency of conditions (6)–(7). To this end, consider the optimization problem of player $i \in \mathbf{M}$ given the strategies σ_j of the opponents $j \neq i$. It consists of the maximization of the objective functional (2) subject to (4) and the given initial state x_0. Because of the assumed form of the strategies for players $j \neq i$, the state equation is

$$\bar{x}_{i,t+1} = \left(1 - \sum_{j \neq i} A_j\right) \bar{x}_{i,t} - \bar{c}_{i,t} \geq 0. \tag{8}$$

If $\sum_{j \neq i} A_j = 1$, then the only feasible consumption choice for player i (and, hence, the optimal one) is $\bar{c}_{i,t} = 0$ for all $t \in \mathbb{N}_0$. This optimal consumption path is generated by a strategy of the form $\sigma_i(x) = A_i x$ if and only if $A_i = 0$. Note that in the case where $\sum_{j \neq i} A_j = 1$ holds, the only nonnegative number A_i that satisfies conditions (6)–(7) is indeed $A_i = 0$.

Now consider the case where $\sum_{j \neq i} A_j < 1$ holds. Due to the infinite steepness of the utility function $c_{i,t}^{1-\alpha_i}/(1-\alpha_i)$ at $c_{i,t} = 0$, it cannot be optimal for player i to drive down the resource stock to 0. Hence, the optimal solution must be an interior one, i.e., it must hold for all $t \in \mathbb{N}_0$ that $\bar{x}_{i,t} > 0$. A sequence $(\bar{x}_{i,t}, \bar{c}_{i,t})_{t=0}^{+\infty}$ is an interior optimal solution of player i's optimization problem if and only if it satisfies $\bar{x}_{i,0} = x_0$, the state equation (8), the Euler equation

$$(\bar{c}_{i,t})^{-\alpha_i} = \rho_i \left(1 - \sum_{j \neq i} A_j\right) (\bar{c}_{i,t+1})^{-\alpha_i},$$

and the transversality condition

$$\lim_{t \to +\infty} \rho_i^t (\bar{c}_{i,t})^{-\alpha_i} \bar{x}_{i,t+1} = 0. \tag{9}$$

The Euler equation can be rewritten as

$$\bar{c}_{i,t+1} = \rho_i^{1/\alpha_i} \left(1 - \sum_{j \neq i} A_j\right)^{1/\alpha_i} \bar{c}_{i,t}. \tag{10}$$

Substituting the proposed strategy $\sigma_i(x) = A_i x$ into (8) and (10), we obtain

$$\bar{x}_{i,t+1} = \left(1 - \sum_{j=1}^{m} A_j\right) \bar{x}_{i,t} \tag{11}$$

and

$$\bar{x}_{i,t+1} = \rho_i^{1/\alpha_i} \left(1 - \sum_{j \neq i} A_j\right)^{1/\alpha_i} \bar{x}_{i,t},$$

respectively. Because of $\bar{x}_{i,0} = x_0 > 0$, these two equations can hold simultaneously if and only if

$$1 - \sum_{j=1}^{m} A_j = \rho_i^{1/\alpha_i} \left(1 - \sum_{j \neq i} A_j\right)^{1/\alpha_i},$$

which is easily seen to be equivalent to (6).

As for the transversality condition (9), we note that Eq. (11) holds along the path generated by the proposed equilibrium, which implies that

$$\bar{x}_{i,t} = \left(1 - \sum_{j=1}^{m} A_j\right)^t x_0.$$

Using this observation as well as $c_{i,t} = A_i \bar{x}_{i,t}$, we see that condition (9) holds if and only if condition (7) is satisfied. This completes the proof of the proposition. □

4 Identical Players

In this section, we consider the case of identical players, that is, we assume that there exist real numbers $\alpha \in (0, 1)$ and $\rho > 0$ such that $\alpha_i = \alpha$ and $\rho_i = \rho$ hold for all $i \in \mathbf{M}$. In such a case, it makes sense to study the existence of a *symmetric* equilibrium, i.e., of a MPNE $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ in which the strategies of all m players coincide. Let us denote the common strategy of the players by σ and let us continue to assume that this strategy takes the linear form $\sigma(x) = Ax$, where A is a nonnegative number. The necessary and sufficient equilibrium conditions of Proposition 1 are given by

$$A \leq 1/m, \tag{12}$$

$$A = g((m-1)A), \tag{13}$$

$$\rho(1 - mA)^{1-\alpha} < 1, \tag{14}$$

where $g : [0, 1] \mapsto \mathbb{R}$ is defined by

$$g(z) = (1 - z)\left[1 - \rho^{1/\alpha}(1 - z)^{(1-\alpha)/\alpha}\right]. \tag{15}$$

In what follows, we distinguish between the discounted case $\rho < 1$ and the case without discounting $\rho \geq 1$. In the latter case, we shall need to restrict the elasticity of marginal utility by $\alpha < (m - 1)/m$. It is therefore convenient to introduce the following assumption.

Assumption 1 It holds that $\rho < 1$ or $\alpha < (m - 1)/m$.

We shall maintain this assumption for the rest of this section. Moreover, we define

$$\bar{A} = \frac{(1-\alpha)m - 1}{(1-\alpha)m(m-1)} \tag{16}$$

and

$$\bar{\rho} = \frac{m\alpha^\alpha(1-\alpha)^{1-\alpha}}{(m-1)^\alpha}. \tag{17}$$

We start the analysis with a preliminary lemma.

Lemma 1 *If* $\alpha < (m - 1)/m$*, then it holds that* $0 < \bar{A} < 1/m$ *and* $1 < \bar{\rho} < m$.

Proof The statement about $\bar{A}$ follows immediately from the definition of $\bar{A}$ and from the assumption $0 < \alpha < (m-1)/m$. To prove the statement about $\bar{\rho}$, let us denote the right-hand side of (17) by $R(\alpha)$. The function R is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$ with derivative

$$R'(\alpha) = -\frac{m\alpha^\alpha(1-\alpha)^{1-\alpha}}{(m-1)^\alpha} \ln\left[\frac{(m-1)(1-\alpha)}{\alpha}\right].$$

The assumption $0 < \alpha < (m-1)/m$ implies $(m-1)(1-\alpha)/\alpha > 1$ and it follows therefore that $R'(\alpha) < 0$ holds for all $\alpha \in (0, (m-1)/m)$. Hence, we obtain for all $\alpha \in (0, (m-1)/m)$ that

$$1 = R((m-1)/m) < R(\alpha) = \bar{\rho} < R(0) = m.$$

This completes the proof of the lemma. □

We now turn to Eq. (13). This is a single equation for the unknown parameter A. From (12), we know that A must be contained in the interval $[0, 1/m]$. The following lemma presents a complete characterization of the solution set of Eq. (13) for all possible values of the discount factor ρ.

Lemma 2 *Let the parameters m, α, and ρ be given such that Assumption 1 holds.*

- **(a)** *If $0 < \rho < 1$, then there exists a unique solution of Eq. (13) in the interval $[0, 1/m]$. This solution satisfies $A \in (\bar{A}, 1/m)$.*
- **(b)** *If $\rho = 1$, then there exist two solutions of Eq. (13) in the interval $[0, 1/m]$. One of them is $A = 0$ and the other one satisfies $A \in (\bar{A}, 1/m)$.*
- **(c)** *If $1 < \rho < \bar{\rho}$, then there exist two solutions of Eq. (13) in the interval $[0, 1/m]$. One of them satisfies $A \in (0, \bar{A})$ and the other one satisfies $A \in (\bar{A}, 1/m)$.*
- **(d)** *If $\rho = \bar{\rho}$, then the unique solution of Eq. (13) in the interval $[0, 1/m]$ is $A = \bar{A}$.*
- **(e)** *If $\rho > \bar{\rho} > 1$, then there does not exist a solution of Eq. (13) in the interval $[0, 1/m]$.*

Proof We note that Eq. (13) can be written as $h(A) = 1/\rho$, where $h : [0, 1/m) \mapsto \mathbb{R}$ is defined by

$$h(A) = \frac{1-(m-1)A}{(1-mA)^{\alpha}}. \tag{18}$$

Note that $h(0) = 1$ and $\lim_{A \to (1/m)-} h(A) = +\infty$. Furthermore, h is continuously differentiable with derivative

$$h'(A) = \frac{1}{(1-mA)^{\alpha}} \left\{ \frac{\alpha m[1-(m-1)A]}{1-mA} + 1 - m \right\}.$$

It follows that

$$h'(A) \begin{cases} < 0 & \text{if } A < \bar{A}, \\ = 0 & \text{if } A = \bar{A}, \\ > 0 & \text{if } A > \bar{A}. \end{cases}$$

Consequently, the graph of h is U shaped on $[0, 1/m)$ and attains its unique minimum at $\bar{A}$.

Suppose first that $\rho < 1$. In this case, we have $h(0) < 1/\rho < \lim_{A \to (1/m)-} h(A)$ such that the equation $h(A) = 1/\rho$ must have at least one solution in the interval $[1, m]$ and this solution must be an interior one. Moreover, because h has only one minimum in $[0, 1/m]$, the solution must be unique.

Next, suppose that $\rho = 1$. In this case, we have $h(0) = 1/\rho$ such that $A = 0$ qualifies as a solution of the equation $h(A) = 1/\rho$. Moreover, it holds that $h'(0) = 1 - (1-\alpha)m < 0$ which, together with the U shape of the graph of h, implies that there must exist a second solution satisfying $A \in (\bar{A}, 1/m)$.

Finally, suppose that $\rho > 1$. We can see that the minimum of h on $[0, 1/m)$ is given by

$$h(\bar{A}) = \frac{(m-1)^{\alpha}}{m\alpha^{\alpha}(1-\alpha)^{1-\alpha}} = \frac{1}{\bar{\rho}}.$$

Together with $h(0) = 1 > 1/\rho$ and $\lim_{A\to(1/m)-} h(A) = +\infty$, it follows that statements (c)–(e) must hold. □

Having studied the possible solutions of Eq. (13), we need to check whether condition (14) is satisfied or not.

Lemma 3 *Let the parameters m, α, and ρ be given such that $0 < \rho \leq \bar{\rho}$ and Assumption 1 hold. Assume that $A \in [0, 1/m]$ is a solution of Eq. (13). If $\rho = 1$ and $A = 0$, then condition (14) is not satisfied. In all other cases mentioned in Lemma 2, condition (14) holds.*

Proof It is clear that (14) fails to hold when $\rho = 1$ and $A = 0$. Let us therefore assume that A is a strictly positive solution of (13). From the proof of lemma 2, it follows that A must satisfy

$$h(A) = \frac{1-(m-1)A}{(1-mA)^{\alpha}} = \frac{1}{\rho}.$$

Consequently, we have

$$\rho(1-mA)^{1-\alpha} = \frac{\rho}{(1-mA)^{\alpha}}(1-mA) = \frac{1-mA}{1-(m-1)A} < 1.$$

This completes the proof of the lemma. □

After these preliminary steps, we can state the main result of the present section.

Theorem 1 *Let the parameters m, α, and ρ be given such that Assumption 1 holds.*

(a) *If $0 < \rho \leq 1$ or $\rho = \bar{\rho} > 1$, then there exists a unique symmetric MPNE consisting of linear strategies of the form $\sigma(x) = Ax$.*
(b) *If $1 < \rho < \bar{\rho}$, then there exist two symmetric MPNE consisting of linear strategies of the form $\sigma(x) = Ax$.*
(c) *If $\rho > \bar{\rho}$, then there does not exist a symmetric MPNE consisting of linear strategies of the form $\sigma(x) = Ax$.*

Proof The theorem follows from Proposition 1 and Lemmas 2 and 3. □

The above theorem shows that there exist MPNE even in the case where the discount factor exceeds 1. Despite the lack of discounting, the objective functionals of all players remain finite because the boundedness of the non-renewable resource stock and the nature of the strategic interaction of the players imply that consumption rates converge to 0 sufficiently fast. This is in sharp contrast to the case of a single decision maker, in which optimal extraction paths fail to exist even in more general circumstances; see [4]. Theorem 1 therefore demonstrates that competition for the resource can somehow compensate for lack of time-preference.

Clemhout and Wan [2] study the existence of MPNE in a continuous-time cake-eating game with strictly positive time-preference rates (which would correspond to the assumption $\rho < 1$ in our formulation) and the very same class of utility functions (with constant elasticity of marginal utility) as in the present paper. It is interesting to note that in their setting, existence is ensured only if the elasticity of marginal utility α is sufficiently *high*, namely $\alpha > (m-1)/m$. In contrast, Theorem 1 does not restrict the parameter value α at all if ρ is smaller than 1. Only if we consider the case of $\rho \geq 1$, Assumption 1 bites and imposes the *upper* bound $\alpha < (m-1)/m$. Hence, the modeling of time seems to make a substantial difference.

Part (c) of Theorem 1 shows that there does not exist a symmetric MPNE consisting of linear strategies if $\rho > \bar{\rho}$. The following result generalizes this finding.

Proposition 2 *Let the parameters m, α, and ρ be given such that* $\rho > \bar{\rho} > 1$ *and Assumption 1 hold. There is no symmetric MPNE with an equilibrium strategy* $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *that is continuously differentiable and satisfies* $\sigma'(0) \neq 0$.

Proof Suppose to the contrary that such a MPNE exists. Feasibility of the strategy profile $(\sigma, \sigma, \ldots, \sigma)$ requires that $m\sigma(x) \leq x$ and therefore

$$\sigma'(0) = \lim_{x \to 0+} \frac{\sigma(x)}{x} \in (0, 1/m].$$

Furthermore, note that $c_{i,t} = \sigma(x_t)$ and $x_{t+1} = x_t - m\sigma(x_t)$ must hold for all $t \in \mathbb{N}_0$ and all $i \in \mathbf{M}$. From the Euler equation, it follows therefore that

$$\sigma(x)^{-\alpha} = \rho[1 - (m-1)\sigma'(x - m\sigma(x))]\sigma(x - m\sigma(x))^{-\alpha}.$$

We rewrite this equation as

$$\frac{\sigma(x - m\sigma(x))}{\sigma(x)} = \rho^{1/\alpha}[1 - (m-1)\sigma'(x - m\sigma(x))]^{1/\alpha}$$

and take the limit as x approaches 0 on both sides. Because of $\lim_{x\to 0+} \sigma(x) = 0$ and $\lim_{x\to 0+}[x - m\sigma(x)] = 0$, the limit on the right-hand side is $\rho^{1/\alpha}[1 - (m-1)\sigma'(0)]^{1/\alpha}$. As for the left-hand side, we can use de L'Hopital's rule and the assumption $\sigma'(0) \neq 0$ to obtain

$$\lim_{x \to 0+} \frac{\sigma(x - m\sigma(x))}{\sigma(x)} = 1 - m\sigma'(0).$$

Combining the above results and abbreviating $\sigma'(0)$ by A, it follows that $A \in (0, 1/m]$ and

$$1 - mA = \rho^{1/\alpha}[1 - (m-1)A]^{1/\alpha}. \tag{19}$$

The latter equation can also be written as $A = g((m-1)A)$, where g is defined in (15). Since we know from Lemma 2 that for $\rho > \bar{\rho}$ there does not exist a solution of Eq. (13) in the interval $[0, 1/m]$, the proof of the proposition is complete. □

In the rest of this section, we discuss the properties of the MPNE identified in Theorem 1. First of all, let us point out that for the special case $\rho = 1$, Assumption 1 is not only sufficient for the existence of a symmetric MPNE of the form $\sigma(x) = Ax$ but also necessary. As a matter of fact, if $\rho = 1$, then assumption 1 boils down to $\alpha < (m-1)/m$. It is easily seen from the proof of lemma 2 that $\alpha \geq (m-1)/m$ implies $h'(0) \geq 0$ and, therefore, that $A = 0$ is the unique solution of $h(A) = 1/\rho$. But since Lemma 3 shows that condition (14) fails to hold for that solution, the strategy $\sigma(x) = Ax$ with $A = 0$ cannot be an equilibrium strategy.

Next, we turn to the case $\rho \in (1, \bar{\rho})$, in which there exist two different MPNE. The following result compares these two equilibria in terms of the utility derived by the players.

Lemma 4 *Let the parameters m, α, and ρ be given such that* $1 < \rho < \bar{\rho}$ *and Assumption 1 hold. Consider the two symmetric MPNE with linear strategies that exist according to Theorem 1(b). The MPNE with the smaller coefficient A leads to higher utility for all players than the one with the larger coefficient A.*

Proof Along every symmetric equilibrium with strategies of the form $\sigma(x) = Ax$, it holds that

$$\sum_{t=0}^{+\infty} \rho^t \frac{c_{i,t}^{1-\alpha}}{1-\alpha} = \sum_{t=0}^{+\infty} \frac{\rho^t (Ax_t)^{1-\alpha}}{1-\alpha} = \frac{(Ax_0)^{1-\alpha}}{1-\alpha} \sum_{t=0}^{+\infty} [\rho(1 - mA)^{1-\alpha}]^t.$$

Because of condition (14), the infinite sum on the right-hand side of this equation has a finite value which is given by

$$\frac{x_0^{1-\alpha} W(A)}{1-\alpha},$$

where $W : [0, \hat{A}) \cup (\hat{A}, 1/m] \mapsto \mathbb{R}$ is defined by

$$W(A) = \frac{A^{1-\alpha}}{1 - \rho(1 - mA)^{1-\alpha}}$$

and where $\hat{A} \in (0, 1/m)$ is given by

$$\hat{A} = \frac{1 - \rho^{-1/(1-\alpha)}}{m}.$$

Thus, we have to compare the value $W(A)$ for the two solutions of Eq. (13) that exist according to Theorem 1(b). Note that the function W has a singularity at $A = \hat{A}$ and that it is strictly decreasing on the two intervals $(0, \hat{A})$ and $(\hat{A}, 1/m)$. Hence, it is sufficient to prove that both solutions of Eq. (13) are located on the same side of $\hat{A}$. Since we know from the proof of lemma 2 that Eq. (13) is equivalent to the equation $h(A) = 1/\rho$, where the function h is defined in (18), all we need to show is that $h(\hat{A}) > 1/\rho$. From the definition of $\hat{A}$ and from (18), it follows that

$$h(\hat{A}) = \rho^{\alpha/(1-\alpha)} \left[\frac{1 - \rho^{-1/(1-\alpha)}}{m} + \rho^{-1/(1-\alpha)} \right] = \frac{\rho^{\alpha/(1-\alpha)} - \rho^{-1}}{m} + \frac{1}{\rho} > \frac{1}{\rho},$$

where the last inequality follows from the assumptions $\alpha \in (0, 1)$ and $\rho > 1$. This completes the proof of the lemma. □

Finally, we study how the aggregate propensity to consume, mA, depends on the model parameters ρ and m. To this end, we restrict ourselves to the case where $\rho \in (0, 1]$, in which the MPNE in linear strategies is unique.

Lemma 5 *Let the parameters m, α, and ρ be given such that $0 < \rho \leq 1$ and Assumption 1 hold. In the MPNE described in Theorem 1, it holds that the total propensity to consume, mA, is decreasing with respect to ρ and increasing with respect to m.*

Proof From the proof of lemma 2, we know that the equilibrium value of A satisfies $A > \bar{A}$ and $h(A) = 1/\rho$, where the function h is given by (18). We know furthermore that $A > \bar{A}$ implies $h'(A) > 0$. These properties obviously demonstrate that A is a decreasing function of ρ. Hence, mA must also be decreasing with respect to ρ.

To study the dependence of mA on m, we rewrite the equation $h(A) = 1/\rho$ as $\tilde{h}(mA) = (m-1)/m$, where

$$\tilde{h}(z) = \frac{\rho - (1-z)^{\alpha}}{\rho z}.$$

The derivative of the function $\tilde{h}$ is given by

$$\tilde{h}'(z) = \frac{(1-z)^{-(1-\alpha)}(1 - z + \alpha z) - \rho}{\rho z^2}.$$

Whenever $\tilde{h}(z) = (m-1)/m$ holds, it must be the case that

$$(1-z)^\alpha = \frac{\rho[m-(m-1)z]}{m}$$

and we obtain

$$\tilde{h}'(z)\Big|_{\tilde{h}(z)=(m-1)/m} = \frac{1-(1-\alpha)(m+z-mz)}{mz(1-z)}.$$

Therefore, we have

$$\tilde{h}'(z)\Big|_{\tilde{h}(z)=(m-1)/m} \begin{cases} <0 & \text{if } z < m\bar{A}, \\ =0 & \text{if } z = m\bar{A}, \\ >0 & \text{if } z > m\bar{A}, \end{cases}$$

where $\bar{A}$ is given by (16). Because we know that $A > \bar{A}$ holds, it follows that $z = mA > m\bar{A}$ and, hence, $\tilde{h}'(z) > 0$. Since $(m-1)/m$ is increasing in m and $\tilde{h}(mA) = (m-1)/m$ must hold, it follows that mA must be increasing in m. □

5 Two Heterogeneous Players

In the present section, we drop the assumption of homogeneity of the players, but, for analytical convenience, we restrict the presentation to the case of $m = 2$ players. We continue to focus on linear strategies. Under these assumptions, the equilibrium condition stated as Eq. (5) can be written as

$$A_1 + A_2 \leq 1 \tag{20}$$

whereas condition (6) becomes

$$A_1 = g_1(A_2), \tag{21}$$

$$A_2 = g_2(A_1). \tag{22}$$

We start the analysis under the assumption that the time-preference factors ρ_1 and ρ_2 are strictly less than 1, in which case, condition (7) is automatically satisfied.[2] Let us define Δ by

$$\Delta = \{(A_1, A_2) \mid A_1 \geq 0, A_2 \geq 0, A_1 + A_2 \leq 1\}.$$

We can prove the following result.

Theorem 2 *Assume that there are $m = 2$ players and that the parameters α_1, α_2, ρ_1, and ρ_2 are real numbers contained in the interval $(0, 1)$. There exists a unique MPNE (σ_1, σ_2) consisting of linear strategies of the form $\sigma_i(x) = A_i x$. The pair of coefficients (A_1, A_2) is the unique solution of Eqs. (21)–(22) in the set Δ.*

Proof Under the present assumptions, the necessary and sufficient equilibrium conditions from Proposition 1 boil down to (20)–(22). It is therefore sufficient to prove that Eqs. (21)–(22) have a unique solution in Δ. To this end, just note that $g_i(0) = 1 - \rho_i^{1/\alpha_i} > 0$, $g_i(1) = 0$, $g_i(z) \in (0, 1)$, and

$$g_i'(z) = \frac{\rho_i^{1/\alpha_i}(1-z)^{(1-\alpha_i)/\alpha_i}}{\alpha_i} - 1$$

[2] Later in this section, we shall discuss the model without discounting.

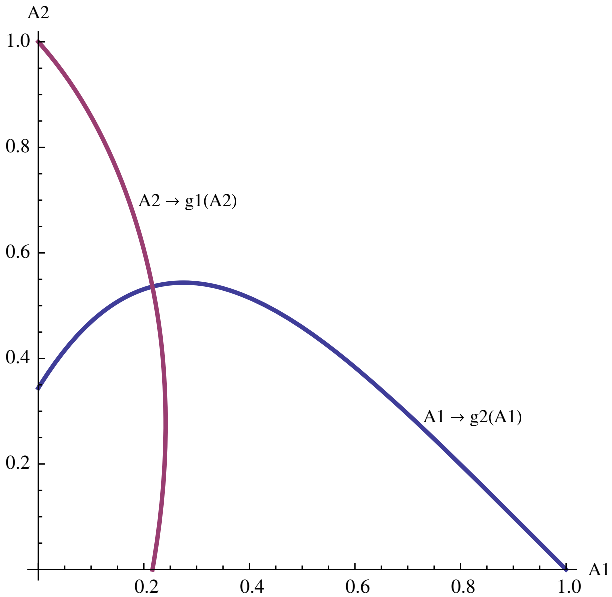

Fig. 1 Equations (21)–(22) for $\alpha_1 = 2/3$, $\alpha_2 = 1/4$, $\rho_1 = 0.85$, and $\rho_2 = 0.9$

hold for all $z \in [0, 1]$. The derivative $g_i'(z)$ is strictly decreasing with respect to z which implies that g_i is a strictly concave function. Moreover, it holds for all $z \in [0, 1]$ that $g_i'(z) \geq g_i'(1) = -1$. Drawing the graphs of the mappings $A_1 \mapsto g_2(A_1)$ and $A_2 \mapsto g_1(A_2)$ into a diagram with A_1 on the horizontal axis and A_2 on the vertical one (as shown in Fig. 1), it follows that the graph of $A_1 \mapsto g_2(A_1)$ is a strictly concave curve that starts in the point $(0, g_2(0))$ on the A_2-axis, ends at the point $(1, 0)$ on the A_1-axis, and is located inside the set Δ. Analogously, the graph of the mapping $A_2 \mapsto g_1(A_2)$ is a strictly concave curve (as seen from the A_2-axis) starting at $(g_1(0), 0)$ on the A_1-axis and ending at $(0, 1)$ on the A_2-axis. It follows from the continuity of the two curves that they have an intersection in the interior of Δ, and it follows from the curvature properties of the two curves that this intersection must be unique. This completes the proof of the theorem. □

Having established the existence of a unique MPNE (in linear strategies), we can now do a comparative static analysis of this equilibrium. This is greatly facilitated by the fact that (A_1, A_2) is the unique solution of the system (21)–(22).

Lemma 6 *Let the assumptions of Theorem 2 be satisfied and consider any player* $i \in \mathbf{M} = \{1, 2\}$. *Then, it holds that*

$$\frac{\partial A_i}{\partial \alpha_i} < 0 \;\; \textit{and} \;\; \frac{\partial A_i}{\partial \rho_i} < 0.$$

Proof Equation (21) can be written as

$$A_1 = 1 - A_2 - \rho_1^{1/\alpha_1}(1 - A_2)^{1/\alpha_1}.$$

Consider any fixed value of $A_2 \in (0, 1)$. Since ρ_1 and $1 - A_2$ are positive and smaller than 1, it follows that the right-hand side of the above equation is decreasing with respect to α_1. Hence, the graph of the mapping $A_2 \mapsto g_1(A_2)$ in Fig. 1 shifts to the left as α_1 increases.

Obviously, this implies that the value of A_1 at which the two curves intersect goes down. An analogous argument proves that A_1 is also decreasing with respect to ρ_1 such that the statement of the lemma for $i = 1$ is proven. The case $i = 2$ follows in an analogous way. □

We would like to note that the dependence of A_i on the opponent's parameters α_j and ρ_j is not clear-cut. For example, in the case that is shown in Fig. 1, the intersection of the two curves occurs at a point where the graph of $A_1 \mapsto g_2(A_1)$ is increasing, but the graph of $A_2 \mapsto g_1(A_2)$ is decreasing. This shows that a sufficiently small downward shift of the former curve increases A_1, whereas a shift to the left of the latter curve decreases A_2. We do get clear-cut result if we assume heterogeneity only with respect to one of the parameters (elasticity of marginal utility or time-preference factor) but not with respect to the other one. To see this, we first note that Eqs. (21)–(22) can also be written as

$$A_1 + A_2 = 1 - \rho_1^{1/\alpha_1}(1 - A_2)^{1/\alpha_1} = 1 - \rho_2^{1/\alpha_2}(1 - A_1)^{1/\alpha_2}, \tag{23}$$

which implies that

$$\rho_1^{\alpha_2}(1 - A_2)^{\alpha_2} = \rho_2^{\alpha_1}(1 - A_1)^{\alpha_1}. \tag{24}$$

We have the following result.

Lemma 7 *Let α and ρ be arbitrary real numbers in* $(0, 1)$.

(a) *If $0 < \alpha_1 < \alpha_2 < 1$ and $\rho_1 = \rho_2 = \rho$, then it follows that $A_1 > A_2$.*
(b) *If $0 < \rho_1 < \rho_2 < 1$ and $\alpha_1 = \alpha_2 = \alpha$, then it follows that $A_1 > A_2$.*

Proof **(a)** In this case, we obtain from Eq. (24) that

$$\rho^{\alpha_2}(1 - A_2)^{\alpha_2} = \rho^{\alpha_1}(1 - A_1)^{\alpha_1} > \rho^{\alpha_2}(1 - A_1)^{\alpha_2}$$

and, hence, $A_1 > A_2$.

(b) In this case, Eq. (24) implies that

$$\frac{1 - A_1}{\rho_1} = \frac{1 - A_2}{\rho_2}$$

and, hence, that $A_1 > A_2$. □

The interesting consequence of part (b) of the above lemma is that, unlike the well-known proverb, patience is not a virtue. In the equilibrium outcome, the more patient player gets to use less of the resource *at each date*, compared with the less patient player. From a conservationist's point of view, however, patience is beneficial as shown in the next lemma. This lemma explores the effect of impatience on the total propensity to consume, which we define as $A = A_1 + A_2$.

Lemma 8 *Assume that $\alpha_1 = \alpha_2 = \alpha \in (0, 1)$. Consider two games which differ from each other only in that the time-preference profile of the players in the first game is (ρ_1, ρ_2), whereas it is (ρ_1', ρ_2') in the second game. Assume that $(0, 0) < (\rho_1, \rho_2) \leq (\rho_1', \rho_2') < (1, 1)$ and $(\rho_1', \rho_2') \neq (\rho_1, \rho_2)$ hold. Denote by A and $(x_t)_{t=0}^{+\infty}$ the total propensity to consume and the state trajectory in the unique MPNE from Theorem 2 in the first game, and by A' and $(x_t')_{t=0}^{+\infty}$ the corresponding variables in the MPNE in the second game. Furthermore let $c_t = Ax_t$ and $c_t' = Ax_t'$, respectively, be total consumption of the resource in period t along the two MPNE. It holds that*

$$A' < A \tag{25}$$

and, for all $t \geq 1$,

$$x'_t > x_t \tag{26}$$

$$\sum_{s=0}^{t-1} c_s > \sum_{s=0}^{t-1} c'_s \quad \textit{and} \quad \sum_{s=0}^{+\infty} c_s = \sum_{s=0}^{+\infty} c'_s \tag{27}$$

Proof From (23), we obtain that

$$1 - A_2 = \frac{(1-A)^\alpha}{\rho_1} \quad \text{and} \quad 1 - A_1 = \frac{(1-A)^\alpha}{\rho_2}.$$

Adding these equations, it follows that

$$2 - A = (1-A)^\alpha \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right),$$

which yields the formula

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{q(A)}, \tag{28}$$

where $q : [0, 1] \mapsto \mathbb{R}$ is defined by

$$q(z) = \frac{(1-z)^\alpha}{2-z}.$$

It holds that $q(0) = 1/2$ and $q(1) = 0$. Further, for all $z \in [0, 1)$, we have

$$q'(z) = \frac{(1-z)^\alpha}{(2-z)^2} \left[1 - \frac{(2-z)\alpha}{1-z} \right].$$

Thus, defining $\bar{z} = (1-2\alpha)/(1-\alpha)$, we see that

$$q'(z) \begin{cases} > 0 & \text{if } z < \bar{z}, \\ = 0 & \text{if } z = \bar{z}, \\ < 0 & \text{if } z > \bar{z}. \end{cases}$$

Clearly, q is continuous on $[0, 1]$ and, therefore, attains a maximum at some point in $[0, 1]$. Because $q(1) < q(0)$, this point cannot be $z = 1$, and because $q'(0) > 0$, it cannot be $z = 0$. Thus, q has an interior maximum at which $q'(z) = 0$. This proves that the maximum is attained at $z = \bar{z}$. Using the fact that $q(0) = 1/2$ and that q is increasing on $[0, \bar{z}]$, it follows that $q(z) \geq 1/2$ holds for all $z \in [0, \bar{z}]$. Condition (28) together with the assumptions $\rho_1 < 1$ and $\rho_2 < 1$ implies that

$$\frac{1}{q(A)} = \frac{1}{\rho_1} + \frac{1}{\rho_2} > 2$$

such that we can conclude that $q(A) < 1/2$. Because we have seen that $q(z) \geq 1/2$ holds for all $z \in [0, \bar{z}]$, we finally obtain that $A \in (\bar{z}, 1)$ and therefore $q'(A) < 0$.

Because the change in time-preference factors from (ρ_1, ρ_2) to (ρ'_1, ρ'_2) makes the left-hand side of (28) smaller, the right-hand side must get smaller too, which implies that $q(A')$ must be larger than $q(A)$. Because of $q'(A) < 0$ this implies that $A' < A$.

Because $x_0 > 0$ is exogenously fixed and the resource dynamics (1) imply $x_{t+1} = (1-A)x_t$ and $x'_{t+1} = (1-A')x'_t$, it follows that

$$x'_t > x_t \tag{29}$$

holds for all $t \geq 1$. Since $(x_t)_{t=0}^{+\infty}$ is a decreasing sequence and $c_t = Ax_t$, it follows that $(c_t)_{t=0}^{+\infty}$ is decreasing as well. Similar remarks apply to $(x_t')_{t=0}^{+\infty}$ and $(c_t')_{t=0}^{+\infty}$. From the state dynamics (1), we obtain for all $T \geq 1$ that

$$\sum_{t=0}^{T-1} c_t = x_0 - x_T \quad \text{and} \quad \sum_{t=0}^{T-1} c_t' = x_0 - x_T'.$$

Combining this with (29), we observe that

$$\sum_{t=0}^{T-1} c_t > \sum_{t=0}^{T-1} c_t'.$$

Since the consumption sequences are decreasing over time, this inequality implies that the $T-1$ highest consumption rates added together must be less in the game with more patient players than in the game with more impatient players. By efficiency of the two paths (which follows from $\lim_{t\to+\infty} x_t = \lim_{t\to+\infty} x_t' = 0$), we also have

$$\sum_{t=0}^{+\infty} c_t = x_0 = \sum_{t=0}^{+\infty} c_t'.$$

This completes the proof of the lemma. □

Conditions (25) and (26) demonstrate that the resource stock is depleted more slowly in the game with more patient players, which can be seen as a social benefit from the conservationist's point of view. Condition (27) says that the path $(c_t')_{t=0}^{+\infty}$ Lorenz dominates $(c_t)_{t=0}^{+\infty}$.

The results stated above continue to hold without any change if one of the two players has the time-preference factor $\rho_i = 1$, whereas the other player has a time-preference factor $\rho_j < 1$. As a matter of fact, the situation depicted in Fig. 1 changes only to the extent that the curve $A_j \mapsto g_i(A_j)$ starts in the origin rather than on the positive A_i-axis. But it is still the case that there must exist a unique intersection of the two curves, that this intersection is located in the interior of Δ, and that the curves have the same curvature properties as shown in Fig. 1. Under the assumptions of the present section, condition (7) can be written as

$$\rho_1(1 - A_1 - A_2)^{1-\alpha_1} < 1, \tag{30}$$

$$\rho_2(1 - A_1 - A_2)^{1-\alpha_2} < 1. \tag{31}$$

Since (A_1, A_2) is in the interior of Δ and since ρ_1 and ρ_2 do not exceed 1, it is obvious that (30)–(31) are satisfied.

If both players have the common time-preference factor $\rho_1 = \rho_2 = 1$, then both curves from Fig. 1 start in the origin such that $(A_1, A_2) = (0, 0)$ is a solution of (21)–(22). But obviously this solution does not satisfy conditions (30)–(31) and so it does not correspond to a MPNE. Whether a second solution in the interior of Δ exists depends on the steepness of the two curves at the origin. If $1/g_1'(0) < g_2'(0)$ holds, then the two curves intersect in the interior of Δ, otherwise they do not. The condition $1/g_1'(0) < g_2'(0)$ is easily seen to be equivalent to $\alpha_1 + \alpha_2 < 1$. We can summarize these observations in the following lemma.

Lemma 9 *Assume that there are $m = 2$ players and that the parameters α_1, α_2, ρ_1, and ρ_2 satisfy $\rho_1 = \rho_2 = 1$, $\alpha_1 \in (0, 1)$, and $\alpha_2 \in (0, 1)$. There exists a MPNE (σ_1, σ_2) consisting of linear strategies of the form $\sigma_i(x) = A_i x$ if and only if $\alpha_1 + \alpha_2 < 1$. If this condition holds, then there is a unique such MPNE.*

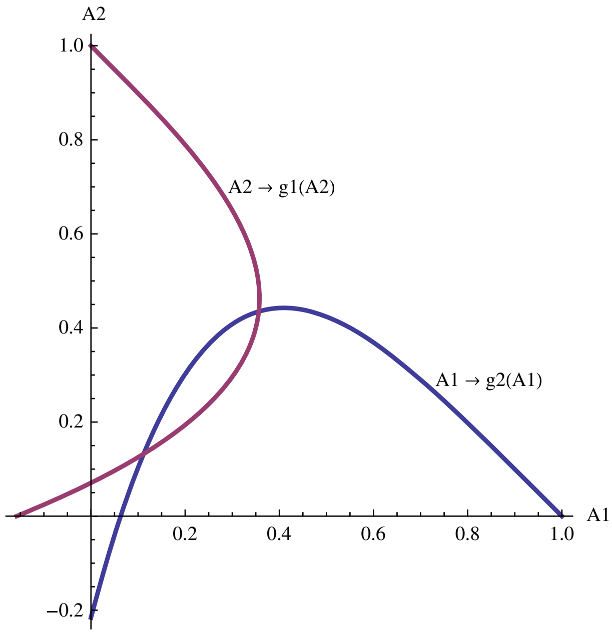

Fig. 2 Equations (21)–(22) for $\alpha_1 = 1/3$, $\alpha_2 = 1/4$, and $\rho_1 = \rho_2 = 1.05$

Finally, we may consider situations in which one or both discount factors exceed the value 1. It is clear from the graphical arguments used before, that the curves in Fig. 1 may still intersect in Δ. If they do, there must generically be two intersections. Instead of deriving the exact conditions under which this happens, we just provide a numerical example; see Fig. 2. It illustrates the case where $\alpha_1 = 1/3$, $\alpha_2 = 1/4$, $\rho_1 = \rho_2 = 1.05$. Note that compared to the situation depicted in Fig. 1, we have not only increased the time-preference factors above 1, but have also changed the elasticity of marginal utility of player 1 from $\alpha_1 = 2/3$ to $\alpha_1 = 1/3$. This was done in order to satisfy the condition $\alpha_1 + \alpha_2 < 1$ that was already stated in lemma 9.[3] Obviously, there are two intersections in Fig. 2 and it is easy to verify that conditions (30)–(31) are satisfied at both of them. Hence, there exist two different MPNE consisting of linear strategies.

6 Concluding Remarks

We have shown that Markov perfect Nash equilibria exist in a dynamic game describing the joint noncooperative exploitation of a non-renewable resource even if there is no discounting at all. Our model is admittedly a very simple one and, we have made explicit assumptions about the form of the utility functions in order to obtain analytical solutions. Nevertheless, we believe that the main point of the paper, namely that such equilibria can exist even in the absence of discounting may have wider applicability. This would mean that the discussion about pressing environmental problems need not revolve around the question of what the

[3] For the parameter set $(\alpha_1, \alpha_2, \rho_1, \rho_2) = (2/3, 1/4, 1.05, 1.05)$ there does not exist a MPNE consisting of linear strategies.

appropriate discount factor is, but one could abolish discounting altogether and treat all generations alike.

There are quite a few directions into which one could explore this issue further. One obvious alternative assumption would be that the resource is renewable as in [6] or [8]. Another direction would be to allow for market interactions between the players. In such a model, the utility of any given player would not only depend on its own extraction rate but—via a market demand function—also on the opponents' extraction rates; see, e.g., [9].

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