JOURNAL OF ECONOMIC THEORY 53, 12–50 (1991)

The Economics of Orchards: An Exercise in Point-Input, Flow-Output Capital Theory*

TAPAN MITRA

Department of Economics, Cornell University, Ithaca, New York 14853

AND

DEBRAJ RAY AND RAHUL ROY

Indian Statistical Institute, New Delhi 110016, India

Received May 27, 1989; revised March 7, 1990

This paper is concerned with the qualitative properties of optimal intertemporal programs in a model of point-input flow-output capital theory, when future utilities are discounted. Under a mild condition on the flow-output vector, we establish that optimal programs for every discount factor and every initial state (other than a unique stationary optimal state) will exhibit *non-convergence*. Furthermore, we provide a necessary and sufficient condition on the flow-output vector for which a *neighborhood turnpike theorem* holds; that is, long-run fluctuations on an optimal program are "small" when the discount factor is "close" to unity. *Journal of Economic Literature* Classification Numbers: 111, 131.

1. INTRODUCTION

Consider the following framework of forest management, which we shall refer to as the *orchard problem*. There is a plot of land of unit size. On this land are planted trees of various ages. This initial "forest" is inherited by the "planner" or the "manager." The manager can choose to clear some or all of the land under trees of any age. This cleared land is replaced with seedlings (trees of age zero). This act of clearing and replacement moves the system to a *new* forest, where the untouched trees are now one year

* The authors thank Aloisio Araujo, R. B. Bapat, Doug Bernheim, Henry Wan, and an anonymous referee for useful comments and suggestions. D. Ray and R. Roy express their gratitude for a pleasant stay at the Instituto de Matematica Pura e Aplicada, Rio de Janeiro, Brazil, where a substantial part of this article was written. T. Mitra's research was partially supported by a National Science Foundation Grant.

0022-0531/91 $3.00

older. At the next date, another process of clearing and replacement takes place, and this continues into the indefinite future.

Each tree has a *growth pattern*, defined by the fruit it yields in each year of its life. This growth pattern is identical for each tree. The yield of the forest at any date is obtained by multiplying the yield of age "s" trees by the number of such trees, and adding up over all the ages present in the forest. The manager possesses a one-period *utility function* defined on this yield. The objective is to choose a policy of clearing and replacement to maximize an infinite discounted sum of one-period utilities.

The orchards model, we feel, is a canonical example of what might be termed *point-input, flow-output capital theory*. Trees may be viewed as capital goods that yield a flow of output net of costs over their lifetimes. Capital goods age, and finally die out. The orchards model may then be viewed as a framework for studying the optimal replacement policy and age composition of capital goods in a planned organization (firm, economy). For the interested reader we place our model in a broader context in Section 6.

A related model of forest management, which might be referred to as the *timber model*, has been much studied in the literature. In this model, a tree only yields an output (its "timber content") when it is cut down. This corresponds to "point-input, *point*-output" capital theory, and the formal framework is actually very different from the one studied here. For studies of the timber model, the reader is referred to Faustmann [7], Schreuder [17], Samuelson [16], Mitra and Wan [11, 12], and Mitra and Ray [13].

Why do we wish to study the orchards model? First, as we have already pointed out, it corresponds to an important class of capital-theory models which has not been well studied in the literature. Second, it turns out to be an excellent device for illustrating certain aspects of global asymptotic stability in models of optimal economic growth; in particular, for understanding and evaluating certain "asymptotic turnpike theorems" that have been extensively studied in the literature. We shall presently expand on this theme.

We are interested in asking two types of questions. First: can one characterize the class of *optimal stationary forests*? These are forests which have the property that once you *start* with such a forest, it is optimal to carry out a policy which replicates the age composition of that forest period after period. They correspond to optimal stationary stocks in models of multisectoral economic growth.

Second: if one starts from an arbitrary initial forest, is it optimal to carry out a policy that causes the age composition to converge over time to that of an optimal stationary forest? Such a result would be analogous to *turnpike theorems* obtained in models of economic growth.

We find the following answers. First, it turns out that the set of optimal stationary forests is *invariant* with respect to the utility function, as long as the latter is increasing, twice continuously differentiable, and concave. (This property, for the "timber" model, was noted by Mitra and Wan [11].) We find the result of some interest, because our model generally corresponds to a multisectoral framework of intertemporal accumulation, where such an invariance property is not usually obtainable. In any case, the invariance result permits us to easily compute the set of optimal stationary forests by simply looking at the growth pattern of an individual tree and the discount factor. This entire exercise is carried out in stages in Propositions 3.1, 4.2, and 5.1. It might be of some interest to note that in the *proofs* of these propositions, we do not use any duality arguments. An elementary primal approach is employed throughout.

At a more down-to-earth level, the invariance result is of interest, because it requires limited information. In particular, apart from knowledge of the discount factor, no knowledge of the utility function is required. The optimal stationary composition can be easily computed and used in practical applications of forest management.

Now we turn to the second question. We consider two cases: one, where the utility function is linear, and the other, where the utility function is *strictly* concave. In the first case, if the initial forest is not an optimal stationary forest, then convergence to an optimal stationary forest is generally not obtained. All trees are *always* cut down at one of two possible ages, which can be exactly characterized. From the point of view of optimal growth theory, this result (the aspect dealing with nonconvergence) is hardly surprising. With linear utility functions, one does not generally expect turnpike properties to be present.

By far the more interesting case is where the utility function is strictly concave. The analogous model is that of optimal capital accumulation with a strictly concave utility function. In such a framework the following results are now fairly well known:

(i) With no restrictions on the magnitude of the discount factor, one does not usually obtain convergence to an optimal stationary program. Instead, optimal programs can "cycle" (for two early examples, see Sutherland [19] and Kurz [8]; for an example in the timber model of forestry, see Mitra and Wan [11]).

(ii) However, there exists, under some "standard" assumptions, a critical discount factor strictly less than unity, such that if the actual discount factor exceeds this critical value, all optimal programs converge to the stationary optimal program. Various versions of this result appear in Brock and Scheinkman [5], Cass and Shell [6], Rockafellar [14], Scheinkman [18], Araujo and Scheinkman [1], Bewley [4], McKenzie [9], and others.

The "standard" assumptions in (ii) can, however, be violated for some fairly interesting models of capital theory. Furthermore, in such models, it can be shown that the result stated in (ii) above fails to hold. Samuelson [15] reports an example due to Weitzman, which involves a pure aging process (wine is produced from grape juice), where for *every positive discount factor less than one*, optimal programs cycle around a unique non-trivial stationary optimal program. (For further discussion of this example, see Scheinkman [18], McKenzie [10], and Benhabib and Nishimura [2].)

In the timber model of forestry, Wan [20] has presented an example in which there is a critical discount factor strictly less than one, such that for all discount factors which are less than one and which exceed this critical value, all optimal programs cycle around a unique non-trivial stationary optimal program. [For further discussion of this example, see Wan [21]].

It turns out that the feature discussed by Wan [20] can be considerably generalized and extended to the orchards model (where the "standard" assumptions used to obtain the result in (ii) above are, interestingly enough, *never* satisfied). We describe an entire class of situations (not just an example) where for *every* discount factor, and for *every* initial forest which is not an optimal stationary forest, optimal programs fail to converge. This is Proposition 5.2 below. We feel that this result is indicative of the interesting idiosyncracies of the forestry model. Formally it *is* a special case of the general framework of intertemporal accumulation. But the assumptions that appear "natural" in that framework, and are in fact employed, are just *not* natural here and cannot be invoked.

Nevertheless, under a mild condition on growth patterns, we do obtain a weaker turnpike property; one that has been notably emphasized by McKenzie [9] in the general framework of intertemporal accumulation. This is the *neighborhood turnpike theorem*, stated in Proposition 5.3. It is this: given any $\varepsilon > 0$, however small, there exists a critical discount factor $\delta(\varepsilon) \in (0, 1)$ such that for any discount factor $\delta \geqslant \delta(\varepsilon)$ and every initial forest, an optimal program ultimately finds itself in the ε-neighborhood of the optimal stationary forest, and *remains* in that neighborhood from a certain time onwards. To sum up, optimal programs may not converge for *any* discount factor (viz. Proposition 5.2), but the limiting oscillations go to zero as the discount factor converges to unity.

We should remark that even the assumptions needed to guarantee the neighborhood turnpike theorem in the general framework are *not* satisfied here. Consequently, we use a technique of proof, which, while inspired by McKenzie [9], is substantially modified. This modified use of Lyapunov functions may be of some technical interest and applicability. Finally, it should be noted that unlike the remainder of the propositions, the proof of this result relies on a duality argument for the large part.

2. Orchards

Consider a unit plot of land (say, the unit square of $\mathfrak{R}^2$) with trees of various ages planted on it. Each tree lives for periods 0, 1, ..., T, after which it is incapable of bearing fruit. The tree yields an amount $R(s)$ in period s, $s = 0, ..., T$. We assume that there exist integers P, Q, with $0 < P \leqslant Q \leqslant T$ such that

$$R(0) \leqslant \cdots \leqslant R(P) = \cdots = R(Q) \geqslant \cdots \geqslant R(T) \tag{2.1}$$

with at least one strict inequality between 0 and P and with $R(s) > R(s+1)$ if $Q \leqslant s < T$.

We also normalize returns so that $R(s) \geqslant 0$ for all s.

Our wider interest is, of course, in vintage capital models and not orchards. It is easily seen that our assumptions make perfect sense in this context. In particular, the assumption that the maximum yield is reached at a *positive* age is easily seen by netting out setup costs, and indeed, purchase costs from returns in the period of installment of the new machine.

The manager of the orchard inherits, at time zero, a *forest*, which we identify with a vector $\alpha \in A$, where A is the nonnegative unit simplex of $\mathfrak{R}^{T+1}$. The interpretation is that $\alpha(s)$, $s = 0, ..., T$, is the fraction of land devoted to trees of age s.

A forest α yields a *harvest* or *consumption* $c(\alpha)$, which is given by

$$c(\alpha) = \sum_{s=0}^{T} \alpha(s) R(s). \tag{2.2}$$

Now, given a forest α_t at time t, $t \geqslant 0$, it is possible to move to a new forest at time $t+1$. The set of possible new forests depends, of course, on the existing forest α_t, and this point-set mapping will be denoted by ϕ. It is given by the following:

$$\begin{aligned}\phi(\alpha) = \{\alpha' \in A \mid & \text{there exist nonnegative reals } \varepsilon(0), ..., \varepsilon(T) \text{ with} \\ & \alpha'(s+1) = \alpha(s) - \varepsilon(s),\ s = 0, ..., T-1,\ \varepsilon(T) = \alpha(T), \\ & \text{and } \alpha'(0) = \textstyle\sum_{s=0}^{T} \varepsilon(s)\}.\end{aligned} \tag{2.3}$$

The interpretation of $\phi(\alpha)$ is that between any two periods, trees of age s grow to the age of $s+1$. Of course, all trees of age T are felled, while some, none, or all of the other trees may be felled. The number $\varepsilon(s)$ denotes the land originally devoted to age s trees which has now been cleared. All the cleared land is then planted with age zero trees.

Given an initial forest α, a *(feasible) program* $\langle \alpha_t \rangle_0^\infty$ from α is a sequence such that

$$\alpha_0 = \alpha \qquad \text{and} \qquad \alpha_{t+1} \in \phi(\alpha_t), \qquad t \geqslant 0 \tag{2.4}$$

The corresponding *consumption program* is the sequence $\langle c_t \rangle_0^\infty$, where

$$c_t = c(\alpha_t) \qquad \text{for all } t \geqslant 0 \tag{2.5}$$

Now we describe preferences. We assume that there is a one-period utility function u defined on current consumption, and that the future is discounted by some discount factor $\delta \in (0, 1)$. We will assume further that u is increasing, concave, continuous on $\Re_+$, and twice continuously differentiable on $\Re_{++}$.

The *problem of forest management* is: given an initial forest α, to choose a feasible program $\langle \alpha_t \rangle_0^\infty$ to solve

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t). \tag{2.6}$$

Given an initial forest α, a feasible program $\langle \alpha_t \rangle_0^\infty$ is *optimal* if it solves (2.6).

A program $\langle \alpha_t \rangle_0^\infty$ is *stationary* if $\alpha_{t+1} = \alpha_t$ for all t. It is easy to see that the set of forests that can be attained as outcomes on stationary programs is completely characterized by the condition $\alpha \in \phi(\alpha)$, or equivalently by the set

$$S = \{\alpha \in A \mid \alpha(s) \geqslant \alpha(s+1) \text{ for all } s = 0, ..., T-1\}. \tag{2.7}$$

A *stationary optimal program* is a stationary program from some initial forest $\alpha \in S$ which is also optimal. A *stationary optimal forest* is a forest that can prevail along some stationary optimal program.

In this paper, we explore some answers to the following two questions:

(1) Is it possible to provide an exact characterization of the set of stationary optimal forests?

The answer to (1) will provide some insight into the dynamics of forest management, but is lacking in the following sense. The *initial* forest is historically given, and there is no reason to suppose that it will be a stationary optimal forest. Generally, then, the forests along an optimal program will vary with time. However, in a manner perfectly analogous to turnpike theory in optimal growth models, one can ask:

(2) From any initial forest α_0, does an optimal program $\langle \alpha_t \rangle$ exhibit a sequence of forests that "converge" to a stationary optimal forest, as time goes to infinity?

The remaining sections are devoted to these issues.

We end this section with some remarks on the orchards model, which amplify some points raised in the introduction to this paper.

It should be clear that the orchards model serves as a canonical example of frameworks involving "point inputs" and "flow outputs." A tree can certainly be equated to a machine, which yields a flow of net outputs. The questions here then translate into issues dealing with the optimal age composition of machines. One caveat should be noted, however, in partial equilibrium applications. Our model assumes that there is no "capital market," so that intertemporal fluctuations in consumption cannot always be fully smoothed out. If there is a capital market, however, the reader will easily note that this corresponds to the case of a linear utility function, which is fully analyzed and solved in Section 4.

Note, however, that our exercise keeps fixed the *total* stock of machines. We focus on the intertemporal behavior of the *composition* of vintages. For a brief sketch of a generalization, see Section 6. It should also be clear that the orchards model can be placed in the general framework of optimal growth theory (e.g., that used in McKenzie [9]). The problem is that the standard assumptions which are made in that general framework are simply not satisfied here. Therefore, the results from that framework cannot be directly applied, and the particular nuances of this model need to be exploited.

3. The Single Tree Problem: A Prelude to the Main Exercise

Consider the optimal management of a single tree, with respect to a sequence of cutting times, when the utility function is linear. That is, we take as our objective the maximization of

$$\sum_{t=0}^{\infty} \delta^t c_t$$

with respect to a sequence of cutting times $(X_1, X_2, ...)$, where $X_i \geqslant 1$ denotes the number of periods the ith installed tree is allowed to exist.

For the later development of our model, it is essential to characterize the optimal solution to this exercise. This is done in

Proposition 3.1. *Let the initial tree be of age τ, where $0 \leqslant \tau \leqslant T$. There exist two integers N_1 and N_2 with $N_1, N_2 \leqslant T$ and $Q \leqslant N_1 \leqslant N_2 \leqslant N_1 + 1$, such that the set of optimal cutting sequences is given precisely by those sequences which allow tree i, $i \geqslant 2$, to exist for N_1 or N_2 periods, and tree 1 to exist for* $\max(0, N-\tau)$ *periods, where $N = N_1$ or N_2.*

Moreover, N_1 and N_2 form the set of solutions to

$$\max_{0 \leqslant N \leqslant T} \frac{\sum_{s=0}^{N} R(s)\, \delta^s}{1-\delta^{N+1}} \tag{3.1}$$

The solutions N_1 and N_2 will play a critical role in the rest of the paper.

4. Forest Management: Linear Utility Functions

The following proposition completely characterizes the set of optimal programs from any initial forest α.

Proposition 4.1. *If u is linear then given an initial forest α, a program $\langle \alpha_t \rangle$ is optimal if and only if*

(i) $\alpha_t(s) = 0$ *for all* $s > N_2$, *and each* $t \geqslant 1$

(ii) *for all* $t \geqslant 0$, α_{t+1} *is attained from* α_t *only by cutting down some or all trees of age* N_1, *and all trees of age* N_2 *or more.*

Proposition 4.1 characterizes what might be called the *Faustmann solution* to the forest management problem (Faustmann [7]). Initially, all trees of age N_2 or greater are cut down (and perhaps some or all of age N_1 as well). Thereafter, trees are never permitted to grow beyond the age of N_2, and are only cut at the age of N_1 or N_2. Given the linearity of the utility function and *given* Proposition 3.1, the reader should not find this result surprising at all.

A program satisfying (i) and (ii) above will be referred to as a *Faustmann program.* Such programs will be of use in proving some of the results stated below. Note that if $N_1 = N_2$, then for any $\alpha \in A$, there is a unique Faustmann program from α.

Proposition 4.1 allows us to completely characterize (with only a little more work) the set of stationary optimal forests, when the utility function is linear. Define the stationary forest $\alpha^*(\beta, \gamma)$, for any β, $\gamma \geqslant 0$ and $\beta + \gamma = 1$; as

$$\alpha^*(\beta, \gamma) = \left(\frac{\beta}{N_1+1} + \frac{\gamma}{N_2+1}, ..., \frac{\beta}{N_1+1} + \frac{\gamma}{N_2+1}; \frac{\gamma}{N_2+1}; 0, ..., 0\right), \tag{4.1}$$

where the first $N_1 + 1$ entries in the above vector involve both β and γ, while the $(N_2 - N_1)$th entries (if any) involve only γ, and the remaining $T - N_2$ entries (if any) involve zeros. In the light of (2.7), it should be obvious that $\alpha^*(\beta, \gamma)$ is a stationary forest. We may now state:

PROPOSITION 4.2. *If utility is linear, then the set of stationary optimal forests is given by* $\{\alpha^*(\beta, \gamma) : (\beta, \gamma) \geqslant 0$ *and* $\beta + \gamma = 1\}$.

As a corollary: if $N_1 = N_2 \equiv N$, so that the solution to the one-tree problem has a unique cutting time N, then there is a unique stationary optimal forest, given by

$$\alpha^* = \Big(\underbrace{\frac{1}{N+1}, \frac{1}{N+1}, ..., \frac{1}{N+1}}_{N+1 \text{ times}}; 0, ..., 0\Big). \tag{4.2}$$

Proposition 4.2 completely characterizes the set of stationary optimal forests, and therefore fully answers the first of our two questions in the linear case. Proposition 4.1 provides the answer to our second question. *In the linear case, there is no tendency for optimal forests to converge to a stationary optimal forest if the initial forest is not a stationary optimal forest.* For example, if $N_1 = N_2 = N$, then optimal programs from any initial forest which is not a stationary optimal forest will exhibit, from time 1 onwards, a periodicity of length $N + 1$.

5. FOREST MANAGEMENT: STRICTLY CONCAVE UTILITY FUNCTIONS

5.1. *Introductory Remarks*

This section is the heart of our paper. Throughout, we assume that the utility function is *strictly* concave, with $u''(c) < 0$ for all $c > 0$. In Section 5.2, we characterize the set of stationary optimal forests. In Section 5.3, we take up the question of convergence. We recall that the issue of *existence* of an optimal program is not serious here; existence can be readily established using standard compactness–continuity arguments.

5.2. *Stationary Optimal Forests*

It is of some interest that the set of stationary optimal forests is *invariant* with respect to the utility function, as long as it is concave. Of course, this property is not generally true of multisectoral growth models. Specifically, the main result of this section is

PROPOSITION 5.1. *If u is concave and twice continuously differentiable, then the set of stationary optimal forests is identical to that in the linear case; namely, it is*

$$\{\alpha^*(\beta, \gamma) : (\beta, \gamma) \geqslant 0, \beta + \gamma = 1\}.$$

To compute the set of stationary optimal forests, then, the exact form of the utility function is quite irrelevant. All one needs are the growth characteristics of a single tree and the discount factor, and these may be used to solve problem (3.1). The solutions N_1 and N_2 may then be used to generate the set of stationary optimal forests, according to the formula (4.1).

The set of stationary optimal forests is therefore either uncountably infinite, or it is a singleton. Here is an example of a model with uncountably many stationary optimal forests.

EXAMPLE 5.1. $T=4$. $R(0)=0$, $R(1)=12$, $R(2)=20$, $R(3)=32$, $R(4)=8$. The discount factor $\delta=\frac{1}{2}$. The solution to (3.1) yields $N_1=3$, $N_2=4$. So, using Proposition 4.2, there are an uncountable number of stationary optimal forests.

However, the reader can easily verify the truth of the following assertion, a proof of which is omitted:

Given the growth pattern R, there are only a finite number of discount factors for which N_1 and N_2 are distinct. Consequently, for all but a finite number of discount factors, we have a unique stationary optimal forest.

5.3. *Convergence of Optimal Forests to a Stationary Optimal Forest*

Suppose that an initial forest which is *not* a stationary optimal forest is exogeneously given. Consider the sequence of forests generated along an optimal program. Does this sequence converge to a stationary optimal forest?

Throughout, we shall concentrate on the case where there is a *unique* stationary optimal forest. An assumption to formally guarantee this will be made shortly.

We first note that, in line with multisectoral capital theory models, such a result cannot be expected to hold in general for the *discounted* case. For an example in the context of optimal growth theory, see, for example, Sutherland [19]. The forest management problem is no different in this regard, as we shall presently see.

In the literature an optimal growth, it has therefore been customary to study asymptotic turnpike properties; that is, turnpike properties when the discount factor is "close to" unity. In the present context, does there exist a discount factor $\delta^* \in (0, 1)$, such that if $1>\delta \geqslant \delta^*$, then every sequence of optimal forests converges to the stationary optimal forest? Under some "standard" assumptions in optimal growth theory, the answer is "yes" (see the references given in Section 1).

These "standard" assumptions concern themselves with certain smoothness and strict concavity properties of the utility function. It turns out that in the present model, these assumptions are not satisfied. This by itself, of course, does *not* imply that the model lacks asymptotic turnpike properties, but only that the *proofs* followed in the literature are not directly applicable. However, for the "timber" model, Wan [20] has produced an ingenious example where there is a critical discount factor, $\delta^* \in (0, 1)$, such that for *every* discount factor $1 > \delta \geqslant \delta^*$, the model lacks turnpike properties.

We first argue that this feature of nonconvergence holds good for the "orchards" model. Not only that, we provide a fairly general subclass of cases where there is *never* any convergence to the stationary optimal forest, *regardless* of the discount factor. This subclass of cases is obtained by making the additional assumption on the growth pattern of an individual tree that $Q = T$ (recall (2.1)), or in other words,

$$R(0) \leqslant R(1) \leqslant \cdots \leqslant R(T), \tag{5.1}$$

with at least one strict inequality.

Now define a forest $\tilde{\alpha} \in A$ by

$$\tilde{\alpha}(s) = \frac{1}{T+1}, \qquad \text{for} \quad s = 0, ..., T. \tag{5.2}$$

It is easy to check that $N_1(\delta) = N_2(\delta) = T$ whenever (5.1) holds, for all δ, so that:

> *Under* (5.1), $\tilde{\alpha}$ *is the unique optimal stationary forest for every* $\delta \in (0, 1)$.

We can now state an important non-convergence result:

PROPOSITION 5.2. *Under* (5.1), *there do not exist any* $\alpha \neq \tilde{\alpha}$ *and any* $\delta \in (0, 1)$ *such that if* $\langle \alpha_t \rangle$ *is optimal from* α (*under* δ), *then* $\alpha_t \rightarrow \tilde{\alpha}$ *as* $t \rightarrow \infty$.

We reiterate that this result stands in striking contrast to the asymptotic stability theorems that have been obtained for optimal growth models. Nonconvergence to $\tilde{\alpha}$ is obtained for *every* discount factor (strictly less than unity) and for *every* initial forest (not equal to the stationary optimal forest). The reader can easily verify, using a continuity argument, that Proposition 5.2 implies that *optimal programs from* $\alpha \neq \tilde{\alpha}$ *do not converge at all to any* $\alpha' \in A$.

However, while optimal programs fail to converge, this does not rule out the possibility that the "amplitude" of their oscillations may tend to "dampen" as the discount factor goes to unity. This is what we turn to next.

The best result that one can hope to obtain in this regard is a "neighborhood turnpike theorem" [McKenzie [9]]. Such a result would state that for any preassigned ε-neighborhood of the stationary optimal forest, there exists a discount factor $\delta(\varepsilon) \in (0, 1)$ such that for any initial forest α and any discount factor $\delta \in [\delta(\varepsilon), 1)$, the optimal program eventually finds its way into the ε-neighborhood of the stationary optimal forest, *and never leaves it thereafter.* It is, of course, obvious that a neighborhood turnpike theorem of this sort is obtainable under somewhat weaker conditions than the asymptotic results discussed earlier. Unfortunately, even the weakest known conditions for the general model of capital accumulation (see McKenzie [9]) are not satisfied in the present model.

The main result of this section is the presentation of a necessary and sufficient condition on the parameters of the model such that a neighborhood turnpike theorem will hold. The condition is *necessary* in the sense that when it fails, one can find some initial forest and an $\varepsilon > 0$ such that for no discount factor close enough to 1 does the optimal program permanently enter the ε-neighborhood of the stationary optimal forest. And when the condition *is* satisfied, we should reiterate that it is *not* implied by the assumptions made for the general capital accumulation model. For example, the assumption of uniform strict concavity of the utility function in the "general" case is not satisfied here. So, while our technique of proof is inspired by the arguments of Bewley [4] and McKenzie [9], a substantially different line of reasoning is involved.

We proceed in steps. First, we make an assumption that will guarantee that a stationary optimal forest is unique, even in the limiting "undiscounted" case.

(A) There is a unique integer N that solves

$$\max_{M \in \{0, ..., T\}} \frac{\sum_{s=0}^{M+1} R(s)}{M+1}.$$

The following lemma is a useful preparatory step.

LEMMA 5.1. *There is* $\underline{\delta} \in (0, 1)$ *such that if* $\delta \in [\underline{\delta}, 1)$, *then* $N_1 = N_2 \equiv N(\delta)$, *say. Moreover,* $N(\delta) = N$, *where N is defined by* (A).

Lemma 5.1 states that if the discount factor is sufficiently close to unity, then the stationary optimal forest is unique *and* invariant to the discount factor. Let

$$\alpha^* \equiv \Big(\underbrace{\frac{1}{N+1}, ..., \frac{1}{N+1}}_{N+1 \text{ times}}; 0, ..., 0\Big). \tag{5.3}$$

Then, given Lemma 5.1, for all $\delta \geq \underline{\delta}$, α^* is the unique stationary optimal forest. Because we are concerned with limiting behaviour when δ is sufficiently close to unity, we will simply suppose, without loss of generality, that $\delta \geqslant \underline{\delta}$, and consequently study "neighborhood convergence" to α^*.

Now define an $(N+1)\times(N+1)$ matrix in the following way: let $y(i) \equiv R(i+1) - R(i)$, for $i = 0, ..., N-1$, and $y(N) \equiv R(0) - R(N)$. Define the "circulant matrix"

$$Y \equiv \begin{bmatrix} y(0) & y(1) & \cdots & y(N) \\ y(1) & y(2) & \cdots & y(0) \\ & \vdots & & \\ y(N) & y(0) & \cdots & y(N-1) \end{bmatrix}. \tag{5.4}$$

The basic turnpike result is the following:

PROPOSITION 5.3. *Suppose that Y is of rank N. Then for each $\varepsilon > 0$, there is $\delta_\varepsilon \in (0, 1)$ such that for every initial forest α and each $\delta \geqslant \delta_\varepsilon$, if $\langle \alpha_t \rangle$ is an optimal program, then*

$$\limsup_{t \to \infty} \|\alpha^* - \alpha_t\| \leqslant \varepsilon. \tag{5.5}$$

On the other hand, if Y is of rank less than N, then there exist $\varepsilon > 0$ and an initial forest α such that for all δ close enough to 1, if $\langle \alpha_t \rangle$ is an optimal program, then

$$\liminf_{t \to \infty} \|\alpha^* - \alpha_t\| > \varepsilon. \tag{5.6}$$

Observe that while Y is an $(N+1)\times(N+1)$ matrix, its rank can never be equal to $N+1$, because its row sums are zero. However, its possession of rank N is certainly generic in the space of all growth patterns $\langle R(s) \rangle_{s=0}^{T}$. For example, a sufficient condition for Y to have rank N is the non-singularity of the circulant matrix

$$R = \begin{bmatrix} R(0) & R(1) & \cdots & R(N) \\ R(1) & R(2) & \cdots & R(0) \\ & \vdots & & \\ R(N) & R(0) & \cdots & R(N-1) \end{bmatrix}.$$

We conclude, therefore, that a neighborhood turnpike result is typical of the point-input, flow-output capital theory framework. However, no convergence result can be expected for any discount factor δ strictly less than unity. This is in sharp contrast to the standard frameworks of optimal growth theory where "asymptotic turnpike" results have been obtained.

6. Remarks on the Vintage Capital Interpretation[1]

At a number of points in the exposition, we have alerted the reader to the obvious links between our exercise and a vintage capital framework. An unsatisfactory component of the exercise is the fixity of the land stock, which has no comfortable interpretation in the context of vintage capital theory, except perhaps as an inflexible "space" constraint. Nevertheless, it served as a useful device for focusing on issues of composition, and points the way towards a natural generalization.

Consider an extension of the model, where, at a cost to current consumption, the number of new trees installed is a *choice* variable which is not limited by the available free land from "felling." In such a model, one might focus on *two* state variables. The first is the vector of *age composition*, α, just as it was present in our exercise. The second is the *total stock*, a new variable. To capture the effect of varying returns to scale, the return from each machine at each date must now be viewed as *functions* of the stock.

At this stage, all we can offer is speculation about the intertemporal behaviour of such a model. But it is a reasonable conjecture that if returns to scale are decreasing, the stock variable would converge to a steady state (even in the discounted model, because there is a single output), while the age composition would continue to fluctuate in the manner we have described.

7. Proofs

Proof of Proposition 3.1. First consider the case where $\tau = 0$.

Define N^* as a solution to the following maximization problem:

$$\max_{0 \leqslant N \leqslant T} [1 - \delta^{N+1}]^{-1} \left[\sum_{s=0}^{N} R(s)\, \delta^s \right].$$

Clearly N^* exists, though not necessarily uniquely.

If $N^* > 0$, we obtain

$$[1 - \delta^{N^*}]^{-1} \left[\sum_{s=0}^{N^*-1} R(s)\, \delta^s \right] \leqslant [1 - \delta^{N^*-1}]^{-1} \left[\sum_{s=0}^{N^*} R(s)\, \delta^s \right], \quad (7.1)$$

[1] These suggestions for an extension owe much to discussions with Aloisio Araujo and to comments by an anonymous referee. In this connection, we would like to draw the reader's attention to the recent work on the choice of optimal capital vintages by Benhabib and Rustichini [3]. We became aware of their work after our paper was largely completed. While there are obviously several points of overlap between their paper and ours, a complete comparison of the models and results of the two papers will not be attempted here.

which yields, after some manipulation,

$$\sum_{s=0}^{N^*-1} [R(N^*)-R(s)]\,\delta^s \geqslant 0. \tag{7.2}$$

Also, if $N^* < T$, we get

$$[1-\delta^{N^*+2}]^{-1}\left[\sum_{s=0}^{N^*+1} R(s)\,\delta^s\right] \leqslant [1-\delta^{N^*+1}]^{-1}\left[\sum_{s=0}^{N^*} R(s)\,\delta^s\right] \tag{7.3}$$

which yields, after some manipulation,

$$\sum_{s=0}^{N^*} [R(N^*+1)-R(s)]\,\delta^s \leqslant 0. \tag{7.4}$$

First we show that $N^* \geqslant Q$. If not, $R(Q) \geqslant R(N^*+1) \geqslant R(s)$ for all $s=0, \ldots, N^*$, with strict inequality for some s. This yields $\sum_0^{N^*}[R(N^*+1)-R(s)]\,\delta^s > 0$, contradicting (7.4). Now we show that N^* can take on at most two adjoining values. Suppose not. Then there are N' and N'' with $N'' > N'+1$ and (7.2) and (7.4) satisfied, respectively, for $N^* = N'$ and $N^* = N''$. We have

$$\begin{aligned}
&\sum_{s=0}^{N''-1} [R(N'')-R(s)]\,\delta^s \\
&\quad= \sum_{s=0}^{N''-2} [R(N''-1)-R(s)]\,\delta^s + \sum_{s=0}^{N''-1} [R(N'')-R(N''-1)]\,\delta^s \\
&\quad= \sum_{s=0}^{N''-2} [R(N''-1)-R(s)]\,\delta^s + \frac{1-\delta^{N''}}{1-\delta}[R(N'')-R(N''-1)] \\
&\quad= \sum_{s=0}^{N'} [R(N'+1)-R(s)]\,\delta^s + \frac{1-\delta^{N'+2}}{1-\delta}[R(N'+2)-R(N'+1)] \\
&\qquad+ \cdots + \frac{1-\delta^{N''}}{1-\delta}[R(N'')-R(N''-1)].
\end{aligned} \tag{7.5}$$

Note that $N' \geqslant Q$, by the same argument made for N^*. So for all $M > M' \geqslant N'$, $R(M) < R(M')$, and therefore

$$\frac{1-\delta^M}{1-\delta}[R(M)-R(M-1)] < 0. \tag{7.6}$$

But the left-hand side of (7.5) is non-negative and so together with (7.6) we get

$$\sum_{0}^{N'} [R(N'+1) - R(s)]\, \delta^s > 0,$$

contradicting (7.4).

This proves that N^* can take at most two values, N_1 and N_2, with $Q \leqslant N_1 \leqslant N_2 \leqslant N_1 + 1$.

Next, we prove that any cutting sequence $\langle X_t \rangle$ with $X_t = N_1$ or N_2 has the same value and any other cutting sequence has strictly smaller value.

For any $\langle X_t \rangle$, the total value obtained, $V(\langle X_t \rangle)$, is given by

$$\begin{aligned} V(\langle X_t \rangle) &= \sum_{s=0}^{X_1} R(s)\, \delta^s + \delta^{X_1+1} \left[\sum_{s=0}^{X_2} R(s)\, \delta^s \right] \\ &\quad + \delta^{X_1+X_2+2} \left[\sum_{s=0}^{X_3} R(s)\, \delta^s \right] + \cdots \\ &\leqslant [1-\delta^{N^*+1}]^{-1} \left[\sum_{0}^{N^*} R(s)\, \delta^s \right] \{(1-\delta^{X_1+1}) \\ &\quad + \delta^{X_1+1}(1-\delta^{X_2+1}) + \delta^{X_1+X_2+2}(1-\delta^{X_3+1}) + \cdots \} \end{aligned}$$

(with strict inequality if any $X_i \neq N_1$ or N_2, and with equality if all $X_i = N_1$ or N_2).

But the term in the curly brackets equals unity because $\delta^n \to 0$ as $n \to \infty$. Hence,

$$V(\langle X_t \rangle) \leqslant [1-\delta^{N^*+1}]^{-1} \left[\sum_{0}^{N^*} R(s)\, \delta^s \right] \equiv V^* \qquad (7.7)$$

with equality if and only if $X_i = N_1$ or N_2 for all i. This completes the proof in the case $\tau = 0$.

Finally consider the general case where $0 \leqslant \tau \leqslant T$. We divide this into two cases:

(A) $\tau \leqslant N_2$: The proposition here immediately follows from the Principle of Optimality and the previous argument.

(B) $\tau > N_2$: Here we need only show that $X_1 = 1$ (that $X_i = N_1$ or N_2, $i \geqslant 2$, follows from the arguments above). Define a sequence $\mathbf{X}^k$ by $X_1 = k$, and $X_i = N_1$ or N_2, $i \geqslant 2$. We are done if we show that $V(\mathbf{X}^k) > V(\mathbf{X}^{k+1})$. We have

$$V(\mathbf{X}^k)=\sum_{s=0}^{k-1}\delta^s R(\tau+s)+\delta^k V^* \tag{7.8}$$

$$V(\mathbf{X}^{k+1})=\sum_{s=0}^{k}\delta^s R(\tau+s)+\delta^{k+1} V^*, \tag{7.9}$$

where V^* is defined in (7.7) above. Subtracting (7.8) from (7.9) yields

$$V(\mathbf{X}^{k+1})-V(\mathbf{X}^k)=\delta^k R(\tau+k)-\delta^k(1-\delta)\,V^*,$$

so

$$\begin{aligned}
&\delta^{-k}[V(\mathbf{X}^{k+1})-V(\mathbf{X}^k)](1-\delta^{N_2+1})\\
&=(1-\delta^{N_2+1})\,R(\tau+k)-(1-\delta)\sum_{s=0}^{N_2} R(s)\,\delta^s\\
&=(1-\delta)\sum_{s=0}^{N_2}[R(\tau+k)-R(s)]\,\delta^s\\
&<(1-\delta)\sum_{s=0}^{N_2}[R(N_2+1)-R(s)]\,\delta^s\\
&\leqslant 0.
\end{aligned}$$

This completes the proof. Q.E.D.

Proof of Proposition 4.1. If utility is linear, we can take it to be of the form $u(c)=c$. Consider any program $\langle\alpha_t\rangle$ from some initial forest α. Then

$$\sum_{t=0}^{\infty}\delta^t u(c(\alpha_t))=\sum_{t=0}^{\infty}\delta^t c(\alpha_t)=\sum_{t=0}^{\infty}\delta^t\left(\sum_{s=0}^{T}\alpha_t(s)\,R(s)\right).$$

Let J denote the unit plot of land, and let $E_t(s)$ be the (measurable) subset of J with trees of age s at time t. Clearly, $\alpha_t(s)=\mu(E_t(s))$, where μ is Lebesgue measure. Let χ_E be the characteristic function of E. Now

$$\begin{aligned}
\sum_{t=0}^{\infty}\delta^t\left[\sum_{s=0}^{T}\alpha_t(s)\,R(s)\right]&=\sum_{t=0}^{\infty}\delta^t\left[\sum_{s=0}^{T}\int_J\chi_{E_t(s)}R(s)\,d\mu\right]\\
&=\sum_{t=0}^{\infty}\delta^t\left[\int_J\left\{\sum_{s=0}^{T}\chi_{E_t(s)}R(s)\right\}d\mu\right]\\
&=\int_J\left\{\sum_{t=0}^{\infty}\delta^t\left[\sum_{s=0}^{T}\chi_{E_t(s)}R(s)\right]\right\}d\mu\\
&\leqslant\int_J V^*_{\tau(\omega)}\,d\mu(\omega),
\end{aligned} \tag{7.10}$$

where $V^*_{\tau(\omega)}$ is the maximum return obtainable from the one-tree problem with initial age τ, and where $\tau(\omega)$ is the age of a tree "at point ω" at time zero.

Using Proposition 3.1, it is now easy to check that the last equality in (7.10) can hold with equality if and only if $\langle \alpha_t \rangle$ satisfies the conditions of Proposition 4.1. Q.E.D.

Proof of Proposition 4.2. It is immediate from Proposition 4.1 that all stationary forests of the form $\alpha^*(\beta, \gamma)$, where $\beta, \gamma \geqslant 0$ and $\beta + \gamma = 1$, constitute stationary optimal programs.

Now we will show that this is *exactly* the set of stationary optimal forests. Pick any α; suppose that it is a stationary optimal forest. Then, by Proposition 4.1,

$$\begin{aligned} \alpha(s) &= \alpha(s+1) && \text{for all } s = 0, ..., N_1 - 1 \\ \alpha(N_1+1) &\leqslant \alpha(N_1) && \\ \alpha(s) &= \alpha(s+1) && \text{for all } s = N_1 + 1, ..., N_2 - 1 \\ \alpha(s) &= 0 && \text{for all } s > N_2. \end{aligned} \tag{7.11}$$

Using (7.11) and the fact that $\sum_{s=0}^{T} \alpha(s) = 1$, we have, defining $\sigma \equiv [\alpha(N_1) - \alpha(N_1+1)]/\alpha(N_1)$, that

$$\alpha(0)[N_1 + 1] + (1 - \sigma)\, \alpha(0)(N_2 - N_1) = 1$$

or

$$\alpha(0) = \frac{1}{(N_1+1) + (1-\sigma)(N_2 - N_1)}. \tag{7.12}$$

Consequently, using (7.12) and the fact that $0 \leqslant \sigma \leqslant 1$, we have

$$\frac{1}{N_1+1} \geqslant \alpha(0) \geqslant \frac{1}{N_2+1}$$

so that there exist $(\beta, \gamma) \geqslant 0$ with $\beta + \gamma = 1$ such that

$$\alpha(0) = \alpha(1) = \cdots = \alpha(N_1) = \frac{\beta}{N_1+1} + \frac{\gamma}{N_2+1}.$$

It is now easy to check that $\alpha(s) = \gamma/(N_2+1)$ for all $s = N_1 + 1, ..., N_2$, and this completes the proof. Q.E.D.

Proof of Proposition 5.1. This proof is along the lines of Mitra and Wan [11]. First we show that if α^* is a stationary optimal forest under a linear utility function, then it is a stationary optimal forest under a concave utility function, say u. Let $c(\alpha^*)=c^*$. For any feasible program $\langle \alpha_t \rangle$ from α^*, we have

$$\begin{aligned}\sum_{t=0}^{\infty} \delta^t[u(c(\alpha_t))-u(c^*)] &\leqslant \sum_{t=0}^{\infty} \delta^t u'(c^*)[c(\alpha_t)-c^*] \\ &= u'(c^*) \sum_{t=0}^{\infty} \delta^t[c(\alpha_t)-c^*] \\ &\leqslant 0,\end{aligned}$$

where the first inequality uses the concavity of u and the last inequality exploits the fact that α^* is an optimal stationary forest for a linear utility function.

Now we show that if α^* is an optimal stationary forest under a concave C^2 utility function, then it is so for a linear utility function. Suppose this is not true. Then there are a concave C^2 utility function u, a discount factor $\delta \in (0, 1)$, and some α^* which is a stationary optimal forest under u, but at the same time there is a program $\langle \alpha^t \rangle$ from α^* with

$$\sum_{t=0}^{\infty} \delta^t c(\alpha_t) > \frac{c^*}{1-\delta}. \tag{7.13}$$

On the other hand, because α^* is an optimal stationary forest under u,

$$\sum_{t=0}^{\infty} \delta^t u(c(\alpha_t)) \leqslant \frac{u(c^*)}{1-\delta}. \tag{7.14}$$

Pick $\hat{\lambda} \in (0, 1)$. For any $0 < \lambda \leqslant \hat{\lambda}$, note that the program $\langle \hat{\alpha}_t \rangle$ given by $\hat{\alpha}_t = \lambda \alpha_t + (1-\lambda)\alpha^*$, $t \geqslant 0$, is a feasible program from α^*. Let $c(\alpha_t)=c_t$, $c(\hat{\alpha}_t)=\hat{c}_t$, for $t \geqslant 0$. Now

$$\begin{aligned}&\sum_{t=0}^{\infty} \delta^t[u(\hat{c}_t)-u(c^*)] \\ &\quad = \sum_{t=0}^{\infty} \delta^t \left\{u'(c^*)[\hat{c}_t - c^*] + u''(\xi_t)\frac{(\hat{c}_t - c^*)^2}{2}\right\} \\ &\quad = \lambda \left[u'(c^*) \sum_{t=0}^{\infty} \delta^t (c_t - c^*) + \lambda \sum_{t=0}^{\infty} \delta^t u''(\xi_t) \frac{(c_t - c^*)^2}{2}\right]\end{aligned} \tag{7.15}$$

for some ξ_t lying between $\hat{c}_t$ and c^*. Note that $\hat{c}_t = \lambda c_t + (1-\lambda)c^* \geqslant (1-\hat{\lambda})c^*$, so that for all t, $\xi_t \geqslant \min\{\hat{c}_t, c^*\} \geqslant (1-\hat{\lambda})c^* > 0$. Using the fact that $u''(\cdot)$ is continuous on $\Re_{++}$ and that $(1-\hat{\lambda})\, c^* \leqslant \xi_t \leqslant \max_{1 \leqslant i \leqslant t} R(i)$, we know that there exists $K > -\infty$ such that

$$\sum_{t=0}^{\infty} \delta^t u''(\xi_t) \frac{(c_t - c^*)^2}{2} \geqslant K, \tag{7.16}$$

where the inequality in (7.16) holds *uniformly* in $\lambda \in (0, \hat{\lambda}]$. Consequently, using (7.13), (7.15), and (7.16), we see that for λ small enough but positive,

$$\sum_{t=0}^{\infty} \delta^t [u(\hat{c}_t) - u(c^*)] > 0,$$

which contradicts the fact that α^* is a stationary optimal forest under u. Q.E.D.

Proof of Proposition 5.2. We will use the following result.

CLAIM. *Suppose that $\alpha \in A$ is of the form $\alpha(s) = 1$ for some $0 \leqslant s < T$. Let $\langle \hat{\alpha}_t \rangle$ be the Faustmann program from α, and $\langle \alpha_t \rangle$ be any other program with $\alpha_1(0) = 1$. Then there is $\Theta(\delta) \in (0, 1)$ such that*

$$\Theta(\delta) \sum_{t=0}^{\infty} \delta^t c(\hat{\alpha}_t) > \sum_{t=0}^{\infty} \delta^t c(\alpha_t). \tag{7.17}$$

Proof. Fix any $s \in \{0, ..., T-1\}$, and $\alpha \in A$ accordingly as given by the claim. Let $\langle \alpha'_t \rangle$ be the program from α such that (i) $\alpha'_1(0) = 1$ and (ii) $\langle \alpha'_t \rangle_{t \geqslant 1}$ is a Faustmann program from α'_1. Then, just as in Proposition 4.1,

$$\sum_{t=0}^{\infty} \delta^t c(\alpha'_t) \geqslant \sum_{t=0}^{\infty} \delta^t c(\alpha_t), \tag{7.18}$$

where $\langle \alpha_t \rangle$ is any program from α satisfying $\alpha_1(0) = 1$. Moreover, by Proposition 4.1 and using the fact that $N_1(\delta) = N_2(\delta) = T$, we have, for the Faustmann program,

$$\sum_{t=0}^{\infty} \delta^t c(\hat{\alpha}_t) > \sum_{t=0}^{\infty} \delta^t c(\alpha'_t). \tag{7.19}$$

Because there are only a finite number of programs of the form $\langle \alpha'_t \rangle$ (one for each $s \in \{0, ..., T-1\}$), there exists $\Theta(\delta) \in (0, 1)$ (possibly depending on δ) such that

$$\Theta(\delta) \sum_{t=0}^{\infty} \delta^t c(\hat{\alpha}_t) > \sum_{t=0}^{\infty} \delta^t c(\alpha'_t). \tag{7.20}$$

Combining (7.18) and (7.20), we have established the claim.

Now we turn to the main argument. Suppose the proposition is false. Then there are $\delta \in (0, 1)$ and some $\bar{\alpha} \neq \alpha^*$ such that for some optimal program $\langle \alpha_t \rangle$ from $\bar{\alpha}$, $\alpha_t \to \alpha^*$ as $t \to \infty$.

Observe that there exists $\varepsilon > 0$ with the following properties:

$$\text{(i)} \quad \|\bar{\alpha} - \alpha^*\| > \varepsilon \tag{7.21}$$

(ii) there exist $\underline{c}$, $\bar{c} \geqslant 0$, and $\eta > 0$, with $\underline{c} - \eta > 0$, such that if $\|\alpha - \alpha^*\| \leqslant \varepsilon$, then

$$\underline{c} \leqslant c(\alpha) \leqslant \bar{c} \tag{7.22}$$

$$u'(\bar{c} + \eta) > u'(\underline{c} - \eta)\, \Theta(\delta), \tag{7.23}$$

where $\Theta(\delta)$ is given by (7.17) above.

Now note that if $\alpha_t \to \alpha^*$ as $t \to \infty$, there must exist some date $M \geqslant 0$ such that

$$\alpha_{M+1}(\hat{s} + 1) < \alpha_M(\hat{s}) \qquad \text{for some } \hat{s} \neq T \tag{7.24}$$

$$\|\alpha_t - \alpha^*\| < \varepsilon \qquad \text{for all } t \geqslant M. \tag{7.25}$$

To see this, it suffices to look at the first date t when $\langle \alpha_t \rangle$ permanently "enters" the ε-neighborhood of α^*. By (7.21), $t \geqslant 1$. For $M \equiv t - 1$, (7.24) and (7.25) must hold.

Without loss of generality, (by the Principle of Optimality) take $M = 0$. So, by (7.25), $\underline{c} \leqslant c(\alpha_t) \leqslant \bar{c}$ for all $t \geqslant 0$. Now we can find $\lambda > 0$ such that

$$\lambda < \alpha_0(\hat{s}) - \alpha_1(\hat{s} + 1) \tag{7.26}$$

and

$$\underline{c} - \eta \leqslant c(\alpha_t) - \lambda R(T) \leqslant c(\alpha_t) + \lambda R(T) \leqslant \bar{c} + \eta \qquad \text{for all } t \geqslant 0 \tag{7.27}$$

Now, it can be easily verified that because of our choice of λ in (7.26), there exist two initial stocks $\alpha' \in A$ and $\alpha'' \in A$ and two programs $\langle \alpha'_t \rangle$, $\langle \alpha''_t \rangle$ from α' and α'', respectively, such that

$$\alpha_t = (1 - \lambda)\, \alpha'_t + \lambda \alpha''_t, \qquad t \geqslant 0 \tag{7.28}$$

$$\alpha''(\hat{s}) = 1, \qquad \alpha''(s) = 0 \qquad \text{for} \quad s \neq \hat{s} \tag{7.29}$$

$$\alpha''_1(0) = 1, \qquad \alpha''_1(s) = 0 \qquad \text{for} \quad s \neq 0. \tag{7.30}$$

These follow from the fact that λ can be identified with the measure of a subplot which only possessed age $\hat{s}$ trees that were all cut down at date 0 (see (7.24) and (7.26)). On this subplot, the initial forest (α'') consisted of only age $\hat{s}$ trees. The forest α' may be identified with the initial forest on the remainder of the land.

Let $\langle \hat{\alpha}_t \rangle$ be the Faustmann program from α''. Define a program $\langle \alpha_t^* \rangle$ from $\bar{\alpha}$ by

$$\alpha_t^* = (1-\lambda)\,\alpha_t' + \lambda\hat{\alpha}_t, \qquad t \geqslant 0. \tag{7.31}$$

Let $c_t^* = c(\alpha_t^*)$, $c_t = c(\alpha_t)$, and $c_t' = c(\alpha_t')$, $t \geqslant 0$. Note that by virtue of (7.27),

$$\underline{c} - \eta \leqslant (1-\lambda)\, c(\alpha_t') \leqslant \bar{c} + \eta \tag{7.32}$$

$$\underline{c} - \eta \leqslant c(\alpha_t^*) \leqslant \bar{c} + \eta. \tag{7.33}$$

Now, for $t \geqslant 1$,

$$\begin{aligned} u(c_t^*) - u((1-\lambda)\, c_t') &\geqslant u'(c_t^*)\, \lambda c(\hat{\alpha}_t) \\ &\geqslant u'(\bar{c} + \eta)\, \lambda c(\hat{\alpha}_t) \\ &> u'(\bar{c} - \eta)\, \Theta(\delta)\, \lambda c(\hat{\alpha}_t), \end{aligned} \tag{7.34}$$

using (7.31), (7.33), and (7.23). At the same time, for $t \geqslant 1$,

$$\begin{aligned} u(c_t) - u((1-\lambda)\, c_t') &\leqslant u'((1-\lambda)\, c_t')\, \lambda c(\alpha_t'') \\ &\leqslant u'(\bar{c} - \eta)\, \lambda c(\alpha_t''), \end{aligned} \tag{7.35}$$

using (7.28) and (7.32).

Combining (7.34) and (7.35), we have, for $t \geqslant 1$,

$$u(c_t^*) - u(c_t) > u'(\bar{c} - \eta)\, \lambda \{\Theta(\delta)\, c(\hat{\alpha}_t) - c(\alpha_t'')\}, \tag{7.36}$$

while $c_0^* = c_0$. Consequently,

$$\begin{aligned} \sum_{t=0}^{\infty} \delta^t \{u(c_t^*) - u(c_t)\} &> u'(\bar{c} - \eta)\, \lambda \left[\Theta(\delta) \sum_{t=0}^{\infty} \delta^t c(\hat{\alpha}_t) - \sum_{t=0}^{\infty} \delta^t c(\alpha_t'') \right] \\ &> 0, \end{aligned}$$

using (7.17). This contradicts the supposition that $\langle \alpha_t \rangle$ was optimal.

Q.E.D.

Proof of Lemma 5.1. Consider the maximization problem (3.1), normalized by the discount factor and augmented to include the case $\delta = 1$,

$$\max_{0 \leqslant M \leqslant T} S(\delta, M), \tag{7.37}$$

where

$$S(\delta, M)=\frac{1-\delta}{1-\delta^{M+1}}\sum_{s=0}^{M} R(s)\,\delta^{s} \qquad \text{for} \quad \delta\in[0,1)$$
$$S(\delta, M)=\frac{\sum_{s=0}^{M} R(s)}{M+1} \qquad \text{for} \quad \delta=1. \tag{7.38}$$

By assumption (A), there is $\varepsilon>0$, such that for all $0\leqslant M\leqslant T$, $M\neq N$, $S(1,M)\leqslant S(1,N)-3\varepsilon$. Now, $S(\delta, M)$ is continuous in δ on $[0, 1]$ for each $0\leqslant M\leqslant T$. Consequently, for each $0\leqslant M\leqslant T$, there is $0<\delta_M<1$ such that $\delta_M\leqslant\delta<1$ implies $|S(\delta,M)-S(1,M)|<\varepsilon$. Define $\underline{\delta}=\max\{\delta_0, ..., \delta_T\}$; then $0<\underline{\delta}<1$. Furthermore, for $\underline{\delta}\leqslant\delta<1$, we have $S(\delta,M)\leqslant S(1,M)+\varepsilon\leqslant S(1,N)-2\varepsilon\leqslant S(\delta,N)-\varepsilon$. Thus, for all $\delta\in[\underline{\delta},1]$, N is the unique solution of (7.37). Q.E.D.

Proof of Proposition 5.3. We will break up the long proof of this proposition into a series of steps and lemmas. First, let $\underline{\delta}$ be given by Lemma 5.1.

Throughout this proof, unless otherwise stated, δ *will be taken to be in the interval* $[\underline{\delta}, 1)$.

Define $c^*\equiv c(\alpha^*)$ and

$$W(\alpha,\delta)\equiv \max_{\langle\alpha_t\rangle;\alpha_0=\alpha}\sum_{t=0}^{\infty}\delta^t[u(c(\alpha_t))-u(c^*)]. \tag{7.39}$$

LEMMA 7.1. *There exists a real-valued function* $B(\alpha)\geqslant B>-\infty$ *for all* $\alpha\in A$, *such that* $B(\alpha)\to 0$ *as* $\alpha\to\alpha^*$, *with*

$$W(\alpha,\delta)\geqslant B(\alpha) \qquad \textit{for all } \delta. \tag{7.40}$$

Proof. Fix $\alpha\in A$. Define

$$\underline{\alpha}\equiv \min_{s\in\{0,...,N\}}\alpha(s). \tag{7.41}$$

Clearly, $\underline{\alpha}\leqslant 1/(N+1)$, with equality holding iff $\alpha=\alpha^*$. Let

$$\lambda=\underline{\alpha}(N+1)\leqslant 1. \tag{7.42}$$

Then there exists $\alpha'\in A$ such that

$$\alpha=\lambda\alpha^*+(1-\lambda)\,\alpha'. \tag{7.43}$$

Define a program from α' in the following way:

$$\alpha'_0 = \alpha';$$

$$\text{for} \quad t = 1, ..., N, \qquad \alpha'_t = \Bigl(1 - \frac{t-1}{N+1}, \underbrace{\frac{1}{N+1}, ..., \frac{1}{N+1}}_{t-1 \text{ times}}; 0, ..., 0\Bigr); \tag{7.44}$$

$$\alpha'_t = \alpha^*, \qquad t \geqslant N+1.$$

Clearly, $\langle \alpha'_t \rangle$ is feasible from α'. Now define a program $\langle \alpha_t \rangle$ from α by

$$\alpha_t = \lambda\alpha^* + (1-\lambda)\,\alpha'_t, \qquad t \geqslant 0. \tag{7.45}$$

Note that $c(\alpha_t) = c^*$, for $t \geqslant N+1$. And for $t = 0, ..., N$,

$$\begin{aligned} c(\alpha_t) &= \lambda c^* + (1-\lambda)\, c(\alpha'_t) \\ &\geqslant \underline{\alpha}(N+1)\, c^*. \end{aligned} \tag{7.46}$$

Therefore,

$$\begin{aligned} W(\alpha, \delta) &\geqslant \sum_{t=0}^{\infty} \delta^t [u(c(\alpha_t)) - u(c^*)] \\ &\geqslant \sum_{t=0}^{\infty} \delta^t [u(\underline{\alpha}(N+1)\, c^*) - u(c^*)]. \end{aligned} \tag{7.47}$$

Define

$$B(\alpha) \equiv \sum_{t=0}^{N} \delta^t [u(\underline{\alpha}(N+1)\, c^*) - u(c^*)].$$

Noting that $\underline{\alpha} \to 1/(N+1)$ as $\alpha \to \alpha^*$, it is easy to see that $B(\alpha)$ has all the required properties. Q.E.D.

We start the main argument by constructing a suitable Lyapunov function. This will be done by first establishing a "price support property" of stationary optimal forests. To this end, we define a "price vector" $p_\delta \in \Re^T_{++}$ in the following way: first, define $q_\delta \in \Re^T_{++}$, for $\delta \in [\underline{\delta}, 1)$, by

$$\begin{aligned} q_\delta(0) &\equiv 1 \\ q_\delta(s) &= 1 + \frac{1-\delta^s}{\delta^s[1-\delta^{N+1}]} \sum_{\tau=0}^{N} \delta^\tau R(\tau) - \frac{1}{\delta^s} \sum_{\tau=0}^{s-1} \delta^\tau R(\tau) \\ &\qquad \text{for} \quad s = 1, ..., N \\ q_\delta(N+1) &= 1 - \varepsilon \qquad \text{for some} \quad 1 > \varepsilon > 0 \\ q_\delta(s) &= 1 \qquad \text{for} \quad s > N+1. \end{aligned} \tag{7.48}$$

For $\delta = 1$, define

$$q_1(0) \equiv 1$$

$$q_1(s) \equiv 1 + \frac{s}{N+1} \sum_{\tau=0}^{N} R(\tau) - \sum_{\tau=0}^{s-1} R(\tau)$$

$$q_1(N+1) \equiv 1 - \varepsilon$$

$$q_1(s) = 1 \qquad \text{for} \quad s > N+1.$$

Now define for $\delta \in [\underline{\delta}, 1]$

$$p_\delta(s) = u'(c^*)\, q_\delta(s) \qquad \text{for} \quad s = 0, ..., T \tag{7.49}$$

and

$$w(\delta) \equiv \begin{cases} \dfrac{1-\delta}{1-\delta^{N+1}} \displaystyle\sum_{\tau=0}^{N} \delta^\tau R(\tau) - (1-\delta) & \text{if} \quad \delta \in [\underline{\delta}, 1) \\ \dfrac{\sum_{\tau=0}^{N} R(\tau)}{N+1} & \text{if} \quad \delta = 1. \end{cases} \tag{7.50}$$

Lemma 7.2. *The vector q_δ has the following properties for ε small enough but positive*:

(i) $R(N) - q_\delta(N) + \delta q_\delta(0) = R(s) - q_\delta(s) + \delta q_\delta(s+1)$

$$= w(\delta), \qquad s = 0, ..., N-1 \tag{7.51}$$

(ii) $q_\delta(0) > q_\delta(N+1)$ (7.52)

(iii) $q_\delta(s) > q_\delta(0)$ *for* $s = 1, ..., N$ (7.53)

(iv) $R(s) - q_\delta(s) + \delta \max\{q_\delta(0), q_\delta(s+1)\} < w(\delta)$

for $s = N+1, ..., T-1$ (7.54)

$R(T) - q_\delta(T) + \delta q_\delta(0) < w(\delta)$.

The value of ε can be chosen independently of the discount factor.

Proof. We show (i), (ii), and (iii) for $\delta < 1$. The details for these cases at $\delta = 1$ are easily worked out. Subsequently, we prove (iv).

If $N > 1$, pick any $s \in \{1, ..., N-1\}$. Then,

$$\begin{aligned}
&R(s)-q_\delta(s)+\delta q_\delta(s+1)\\
&=R(s)-\left[1+\frac{1-\delta^s}{\delta^s[1-\delta^{N+1}]}\sum_{\tau=0}^{N}\delta^\tau R(\tau)-\frac{1}{\delta^s}\sum_{\tau=0}^{s-1}\delta^\tau R(\tau)\right]\\
&\quad+\left[\delta+\frac{1-\delta^{s+1}}{\delta^s[1-\delta^{N+1}]}\sum_{\tau=0}^{N}\delta^\tau R(\tau)-\frac{1}{\delta^s}\sum_{\tau=0}^{s}\delta^\tau R(\tau)\right]\\
&=-(1-\delta)+\frac{1-\delta}{1-\delta^{N+1}}\sum_{\tau=0}^{N}\delta^\tau R(\tau)\equiv w(\delta).
\end{aligned} \tag{7.55}$$

For $s=0$, we have

$$\begin{aligned}
R(0)-q_\delta(0)+\delta q_\delta(1)&=R(0)-1+\left[\delta+\frac{1-\delta}{[1-\delta^{N+1}]}\sum_{\tau=0}^{N}\delta^\tau R(\tau)-R(0)\right]\\
&=w(\delta).
\end{aligned}$$

To complete the proof of (i), note finally that

$$\begin{aligned}
&R(N)-q_\delta(N)+\delta q_\delta(0)\\
&=R(N)-\left[1+\frac{1-\delta^N}{\delta^N[1-\delta^{N+1}]}\sum_{\tau=0}^{N}\delta^\tau R(\tau)-\frac{1}{\delta^N}\sum_{\tau=0}^{N-1}\delta^\tau R(\tau)\right]+\delta=w(\delta)
\end{aligned}$$

Part (ii) is obvious from the fact that $\varepsilon>0$. To establish (iii), note that by (A),

$$\frac{\sum_{\tau=0}^{N}\delta^\tau R(\tau)}{1-\delta^{N+1}}>\frac{\sum_{\tau=0}^{s-1}\delta^\tau R(\tau)}{1-\delta^s}\qquad\text{for all } 1\leqslant s\leqslant N. \tag{7.56}$$

Using (7.56) in (7.48) completes the argument.

Finally, we establish (iv). To do so, we first show that we can obtain $1>\hat{\varepsilon}>0$ such that

$$R(s)\leqslant\hat{\varepsilon}[w(\delta)+(1-\delta)] \tag{7.57}$$

for all $s=N+1, ..., T$ and all $\delta\in[\underline{\delta}, 1]$. By Proposition 3.1, it suffices to show this for $s=N+1$, because $N\geqslant Q$ and so $R(N+1)>R(s)$ for all $s>N+1$. Now, using Lemma 5.1, we have for $\delta\in(\underline{\delta}, 1)$,

$$\frac{\sum_{s=0}^{N+1}\delta^s R(s)}{1-\delta^{N+2}}<\frac{\sum_{s=0}^{N}\delta^s R(s)}{1-\delta^{N+1}}$$

or

$$\begin{aligned}
\frac{\delta^{N+1}R(N+1)}{1-\delta^{N+2}}&<\sum_{s=0}^{N}\delta^s R(s)\left[\frac{1}{1-\delta^{N+1}}-\frac{1}{1-\delta^{N+2}}\right]\\
&=\left[\sum_{s=0}^{N}\delta^s R(s)\right]\frac{\delta^{N+1}[1-\delta]}{(1-\delta^{N+1})(1-\delta^{N+2})}.
\end{aligned}$$

Therefore,

$$R(N+1) < \frac{1-\delta}{1-\delta^{N+1}} \sum_{s=0}^{N} \delta^s R(s).$$

Note also that at $\delta = 1$, by assumption (A),

$$R(N+1) < \frac{\sum_{s=0}^{N} R(s)}{N+1}.$$

Combining these two inequalities with the definition of $w(\delta)$, we have

$$R(N+1) < w(\delta) + (1-\delta) \qquad \text{for all } \delta \in [\underline{\delta}, 1]. \tag{7.58}$$

Since $w(\delta)$, defined for $\delta \in [\underline{\delta}, 1]$, is continuous in its domain, we can find $1 > \hat{\varepsilon} > 0$ such that (7.57) holds for $s = N+1$ and for *all* $\delta \in [\underline{\delta}, 1]$.

Now pick $\varepsilon = \hat{\varepsilon}/2$. We claim that for such ε, (iv) holds (independently of the value of δ). To check this, first note that, for all $\delta \in [\underline{\delta}, 1]$, if $s = N+2, ..., T-1$,

$$R(s) - q_\delta(s) + \delta \max(q_\delta(0), q_\delta(s+1)) = R(s) - 1 + \delta < w(\delta),$$

using (7.57). Also, if $T > N+1$, (7.57) yields

$$R(T) - q_\delta(T) + \delta q_\delta(0) = R(T) - 1 + \delta < w(\delta)$$

by the same argument; and finally (if $T > N+1$),

$$\begin{aligned} &R(N+1) - q_\delta(N+1) + \delta \max\{q_\delta(0), q_\delta(N+2)\} \\ &\quad = R(N+1) - 1 + \varepsilon + \delta \\ &\quad < w(\delta), \end{aligned}$$

using (7.57). (If $T = N+1$, use a similar argument.)

This completes the proof. Q.E.D.

LEMMA 7.3. *For all $\alpha, \alpha' \in A$ such that $\alpha' \in \phi(\alpha)$,*

$$u(c^*) - p_\delta \alpha^* + \delta p_\delta \alpha^* \geqslant u(c(\alpha)) - p_\delta \alpha + \delta p_\delta \alpha' \tag{7.59}$$

with strict inequality holding in (7.59) whenever at least one of the following holds:

$$\begin{aligned} &\text{(a)} \quad \alpha'(s+1) < \alpha(s) \qquad \textit{for some} \quad s \neq N \\ &\text{(b)} \quad \alpha'(N+1) > 0 \\ &\text{(c)} \quad \alpha(s) > 0 \qquad \textit{for some} \quad s > N \\ &\text{(d)} \quad c^* \neq c(\alpha). \end{aligned} \tag{7.60}$$

Proof. Pick any $\alpha, \alpha' \in A$, with $\alpha' \in \phi(\alpha)$. Note that

$$\begin{aligned}
&c(\alpha) - q_\delta \alpha + \delta q_\delta \alpha' \\
&\quad = \sum_{s=0}^{T} \alpha(s)\, R(s) - \sum_{s=0}^{T} q_\delta(s)\, \alpha(s) + \delta \sum_{s=0}^{T} q_\delta(s)\, \alpha'(s) \\
&\quad = \sum_{s=0}^{N-1} \alpha(s)[R(s) - q_\delta(s) + \delta q_\delta(s+1)] \\
&\qquad + \alpha(N)[R(N) - q_\delta(N) + \delta q_\delta(0)] \\
&\qquad + \delta \sum_{s=1}^{N} q_\delta(s)[\alpha'(s) - \alpha(s-1)] + \alpha\delta q_\delta(0)[\alpha'(0) - \alpha(N)] \\
&\qquad + \sum_{s=N+1}^{T-1} \alpha(s)[R(s) - q_\delta(s) + \delta \max\{q_\delta(s+1), q_\delta(0)\}] \\
&\qquad + \alpha(T)[R(T) - q_\delta(T) + \delta q_\delta(0)] + \sum_{s=N+1}^{T} \alpha'(s)\, \delta q_\delta(s) \\
&\qquad - \sum_{s=N+1}^{T-1} \alpha(s)\, \delta \max\{q_\delta(s+1), q_\delta(0)\} - \alpha(T)\, \delta q_\delta(0) \\
&\quad \leqslant w(\delta) + \delta \Bigg[\alpha'(0) + \sum_{s=1}^{N} (\alpha'(s) - \alpha(s-1)) \\
&\qquad + \sum_{s=N+2}^{T} (\alpha'(s) - \alpha(s-1)) - \alpha(N) + \alpha'(N+1) - \alpha(T)\Bigg],
\end{aligned} \tag{7.61}$$

where the inequality above is derived from the following: Lemma 7.2, parts (i), (ii), (iii), and (iv), and the fact that $q_\delta(N+1) = 1 - \varepsilon$, $q_\delta(0) = 1$, $q_\delta(s) = 1$ for $s \geqslant N+2$. The reader is invited to check, using the strict inequalities of Lemma 7.2, that the above weak inequality will hold strictly whenever part (a), (b), or (c) of the current lemma holds.

But now note that the term in the square brackets of (7.61) exactly equals zero. So

$$c(\alpha) - q_\delta \alpha + \delta q_\delta \alpha' \leqslant w(\delta) \text{ for any } \alpha, \alpha' \in A \text{ such that } \alpha' \in \phi(\alpha).$$

Furthermore, it is easy to check using Lemma 7.2, part (i), that

$$c^* - q_\delta \alpha^* + \delta q_\delta \alpha^* = w(\delta).$$

So

$$c(\alpha) - q_\delta \alpha + \delta q_\delta(\alpha') \leqslant c^* - q_\delta \alpha^* + \delta q_\delta \alpha^*$$

or

$$c(\alpha) - c^* \leqslant q_\delta \alpha - \delta q_\delta \alpha' - q_\delta \alpha^* + \delta q_\delta \alpha^*. \tag{7.62}$$

Now observe that by strict concavity of u,

$$u(c(\alpha)) - u(c^*) \leqslant u'(c^*)(c(\alpha) - c^*), \tag{7.63}$$

with strict inequality holding if part (d) of the current lemma holds.

Combining (7.62) and (7.63), and using the definition of p_δ given in (7.49), we are done. Q.E.D.

Now define, for any $\alpha, \alpha' \in A$ with $\alpha' \in \phi(\alpha)$, and for $\delta \in [\underline{\delta}, 1]$,

$$l(\alpha, \alpha', \delta) \equiv [u(c^*) - p_\delta \alpha^* + \delta p_\delta \alpha^*] - [u(c(\alpha)) - p_\delta \alpha + \delta p_\delta \alpha']. \tag{7.64}$$

Note that $l(\alpha, \alpha', \delta) \geqslant 0$, with strict inequality holding whenever (α, α') satisfy at least one of conditions (a), (b), (c), or (d) of Lemma 7.3.

For any program $\langle \alpha_t \rangle$ from α_0, define, for each $t \geqslant 0$ and $\delta \in [\underline{\delta}, 1)$,

$$L_s(\langle \alpha_t \rangle, \delta) \equiv \sum_{t=s}^{\infty} \delta^{t-s} l(\alpha_t, \alpha_{t+1}, \delta). \tag{7.65}$$

LEMMA 7.4. *For every $\alpha \in A$, every discount factor $\delta \in [\underline{\delta}, 1)$, and each optimal program $\langle \alpha_t \rangle$ (under δ) from α,*

$$L_s(\langle \alpha_t \rangle, \delta) \leqslant F < \infty, \qquad s \geqslant 0. \tag{7.66}$$

Moreover, there exists a function $f\colon R_+ \to R_+$ such that $f(\varepsilon) \to 0$ as $\varepsilon \to 0$, with the property that

$$L_s(\langle \alpha_t \rangle, \delta) \leqslant f(\varepsilon), \qquad s \geqslant 0, \tag{7.67}$$

whenever $\langle \alpha_t \rangle$ is an optimal program under δ from some $\alpha \in A$ and $\|\alpha_s - \alpha^*\| \leqslant \varepsilon$.

Proof. It clearly suffices to establish these results for L_0. For any $\alpha \in A$, and $\delta \in [\underline{\delta}, 1)$, and any optimal program $\langle \alpha_t \rangle$ from α, it is easy to see that

$$\begin{aligned} L_0(\langle \alpha_t \rangle, \delta) &\equiv \sum_{t=0}^{\infty} \delta^t [u(c^*) - u(c(\alpha_t))] - p_\delta \alpha^* + p_\delta \alpha \\ &\leqslant -B(\alpha) + p_\delta(\alpha - \alpha^*) \\ &\leqslant -B(\alpha) + \|p_\delta\| \, \|\alpha - \alpha^*\|. \end{aligned} \tag{7.68}$$

Note that $\sup_{\delta \in [\underline{\delta}, 1]} \|p_\delta\| < \infty$, and so, using Lemma 7.1, (7.66) is true. Now defining

$$f(\varepsilon) \equiv \sup_{\|\alpha - \alpha^*\| \leqslant \varepsilon} \left[-B(\alpha) + \left\{ \sup_{\delta \in [\underline{\delta}, 1]} \|p_\delta\| \right\} \|\alpha - \alpha^*\| \right] \tag{7.69}$$

and again using Lemma 7.1, it is easy to check that $f(\varepsilon)$ has all the required properties and that (7.67) is true. Q.E.D.

LEMMA 7.5. *Suppose that Y is of rank N. Then there exists a real valued function* $g: \mathfrak{R}_{++} \to \mathfrak{R}_{++}$ *such that if* $\|\alpha - \alpha^*\| \geqslant \varepsilon$, *for some* $\varepsilon > 0$, *then for every discount factor* $\delta \in [\underline{\delta}, 1)$ *and every program* $\langle \alpha_t \rangle$ *from* α,

$$\sum_{t=0}^{T-1} l(\alpha_t, \alpha_{t+1}, \delta) \geqslant g(\varepsilon) > 0. \tag{7.70}$$

Proof. It suffices to show that for every α such that $\alpha \neq \alpha^*$, and for every program $\langle \alpha_t \rangle$ from α, there exists $t \in \{0, 1, ..., T-1\}$ such that

$$l(\alpha_t, \alpha_{t+1}, \delta) > 0 \tag{7.71}$$

Let us first prove this claim. Suppose (7.71) is true but not (7.70). Then for some $\varepsilon > 0$, there exist a sequence δ_k, a sequence α^k such that $\|\alpha^k - \alpha^*\| \geqslant \varepsilon$, and a sequence of programs $\langle \alpha_t^k \rangle$ from α^k such that

$$\lim_{k \to \infty} \sum_{t=0}^{T-1} l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) = 0. \tag{7.72}$$

Without loss of generality we can suppose that $\delta_k \to \delta \in [\underline{\delta}, 1]$ and that α^k, $\langle \alpha_t^k \rangle$, converge pointwise to some initial forest and program, α, $\langle \alpha_t \rangle$. Note that $\langle \alpha_t \rangle$ is feasible from α. The reader can also check, using the definition of q for $\delta < 1$ *and* for $\delta = 1$, and the definition of l, that $l(\alpha, \alpha', \delta)$ is a continuous function on $A \times A \times [\underline{\delta}, 1]$. So, passing to the limit and using (7.72),

$$\sum_{t=0}^{T-1} l(\alpha_t, \alpha_{t+1}, \delta) = 0$$

which means that

$$l(\alpha_t, \alpha_{t+1}, \delta) = 0 \qquad \text{for all } t \in \{0, ..., T-1\}. \tag{7.73}$$

But (7.73) contradicts (7.71), and this proves the claim.

So we now establish (7.71). By Lemma 7.3, it suffices to prove that if $\alpha \neq \alpha^*$ and $\langle \alpha_t \rangle$ is feasible from α, then there exists $t \in \{0, ..., T-1\}$ such that one of the following obtains:

$$\begin{aligned}
&\text{(a)}\quad \alpha_{t+1}(s+1)<\alpha(s) \qquad \text{for some} \quad s\neq N\\
&\text{(b)}\quad \alpha_{t+1}(N+1)>0\\
&\text{(c)}\quad \alpha_t(s)>0 \qquad \text{for some} \quad s>N\\
&\text{(d)}\quad c(\alpha_t)\neq c^*.
\end{aligned} \tag{7.74}$$

Suppose this is not true. Then, it can be checked that there exists a feasible program $\langle \alpha_t \rangle$ from some $\alpha \neq \alpha^*$ such that

$$c(\alpha_t)=c^* \qquad \text{for all } t\in\{0,...,T\} \tag{7.75}$$

$$\alpha_t(s)=0 \qquad \text{for all } s>N, \text{ for } t\in\{1,...,T\} \tag{7.76}$$

$$\left.\begin{aligned}&\alpha_{t+1}(s+1)=\alpha_t(s) \qquad \text{for} \quad s=0,...,N-1\\ &\alpha_{t+1}(0)=\alpha_t(N)\end{aligned}\right\} \quad t\in\{0,...,T\}. \tag{7.77}$$

This implies that there exists $\alpha \neq \alpha^*$ such that

$$\begin{aligned}
\alpha(0)\,R(0)+\alpha(1)\,R(1)+\cdots+\alpha(M)\,R(M)&=c^*\\
\alpha(M)\,R(0)+\alpha(0)\,R(1)+\cdots+\alpha(M-1)\,R(M)&=c^*\\
&\vdots\\
\alpha(1)\,R(0)+\alpha(2)\,R(1)+\cdots+\alpha(0)\,R(M)&=c^*.
\end{aligned} \tag{7.78}$$

In turn, (7.78) implies that

$$Y\alpha=0, \tag{7.79}$$

where Y is defined in (5.4). The set of all α's satisfying (7.79) is, of course, the null space $n(Y)$, and it is well known that

$$\dim n(Y)+\operatorname{rank}(Y)=N+1. \tag{7.80}$$

If $\operatorname{rank}(Y)=N$, it follows that

$$\dim n(Y)=1. \tag{7.81}$$

Therefore all α's satisfying (7.79) are scalar multiples of each other, and given the additional restriction $\alpha \in A$, it follows that there is a *unique* solution to (7.79). But we know that α^* is *a* solution to (7.79). This contradicts the supposition that $\alpha \neq \alpha^*$, and proves the lemma. Q.E.D.

The next two lemmas are much weaker variants of what we finally want to establish. The first (Lemma 7.6) essentially states a "maximum theorem" when the discount factor parametrically converges to one; its proof is *not* covered by any standard maximum theorem in the literature. The second

(Lemma 7.7) is a version of the well-known "visit lemma" (see Scheinkman [18]) which states that for discount factors close enough to unity, an optimal program must "visit," at some finite date, an ε-neighborhood of the stationary optimal forest.

LEMMA 7.6. *Suppose that* rank $Y=N$. *Let* α^k *be a sequence in* A *such that* $\alpha^k \to \alpha^*$ *as* $k \to \infty$. *Let* δ^k *be a sequence of discount factors in* $[\underline{\delta}, 1)$ *such that* $\delta^k \to 1$ *as* $k \to \infty$, *and let* $\langle \alpha_t^k \rangle$ *be a sequence of optimal programs under* δ^k, *from* α^k. *Then, for each* $t \geqslant 0$,

$$\alpha_t^k \to \alpha^* \qquad as \quad k \to \infty. \tag{7.82}$$

Proof. Suppose not. Then for some sequence $(\alpha^k, \delta_k) \to (\alpha^*, 1)$, with $\delta_k < 1$, and for the corresponding sequence of optimal programs $\langle \alpha_t^k \rangle$, we can suppose, without loss of generality, that

$$\alpha_t^k \to \alpha_t, \qquad t \geqslant 0,$$

where, for some $t \geqslant 0$,

$$\alpha_t \neq \alpha^*.$$

Let s be the first date such that $\alpha_{s+1} \neq \alpha^*$. Clearly, because $\alpha_0^k = \alpha^k$ for all k, and because $\alpha^k \to \alpha^*$, we have $s \geqslant 0$. It is easily seen that without loss of generality, we can regard s as equal to 0.

Recall p_δ as defined by (7.48) and (7.49). Write $p_{\delta_k} \equiv p_k$ and $c_t^k \equiv c(\alpha_t^k)$ for all k and $t \geqslant 0$. Using (7.64), we have, for all $t \geqslant 0$ and for all k,

$$l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) = [u(c^*) - p_k \alpha^* + \delta_k p_k \alpha^*] - [u(c_t^k) - p_k \alpha_t^k + \delta_k p_k \alpha_{t+1}^k].$$

Multiplying both sides by δ_k^t, and taking the infinite sum over all t,

$$\begin{aligned} \sum_{t=0}^{\infty} \delta_k^t l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) &= \sum_{t=0}^{\infty} \delta_k^t [u(c^*) - u(c_t^k)] - p_k(\alpha^* - \alpha_0^k) \\ &\leqslant -B(\alpha_0^k) - p_k(\alpha^* - \alpha_0^k), \end{aligned} \tag{7.83}$$

where $B(\cdot)$ is given by Lemma 7.1.

Now, by Lemma 7.1, $B(\alpha_0^k) \to 0$ as $k \to \infty$, because $\alpha_0^k \to \alpha^*$. Also, p_k is bounded in k, so that

$$\limsup_{k \to \infty} \sum_{t=0}^{\infty} \delta_k^t l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) \leqslant 0. \tag{7.84}$$

However, by our construction, there is $\varepsilon > 0$ such that for all k sufficiently large,

$$\|\alpha_1^k - \alpha^*\| > \varepsilon.$$

Using Lemma 7.5, we have

$$\sum_{t=1}^{T} l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) \geqslant g(\varepsilon) > 0. \tag{7.85}$$

Therefore, using (7.85) and the fact that $\delta \geqslant \underline{\delta}$ for all t and that $l \geqslant 0$, we have, for all k,

$$\begin{aligned}\sum_{t=0}^{\infty} \delta_k^t l(\alpha_t^k, \alpha_{t+1}^k, \delta_k) &\geqslant \sum_{t=1}^{T} \delta_k^t l(\alpha_t^k, \alpha_{t+1}^k, \delta_k)\\ &\geqslant \underline{\delta}^T \sum_{t=1}^{T} l(\alpha_t^k, \alpha_{t+1}^k, \delta_k)\\ &\geqslant \underline{\delta}^T g(\varepsilon). \end{aligned} \tag{7.86}$$

But (7.86) contradicts (7.84). Q.E.D.

LEMMA 7.7. *Suppose that rank* $Y = N$. *For each* $\varepsilon > 0$, *there exists* $\hat{\delta} \in [\underline{\delta}, 1)$ *such that if* $\delta \in [\hat{\delta}, 1)$, *then for each initial forest* $\alpha \in A$, *and optimal program* $\langle \alpha_t \rangle$ *(under* δ*) from* α, *there exists* $T < \infty$ *such that*

$$\|\alpha_T - \alpha^*\| < \varepsilon. \tag{7.87}$$

Proof. Let $\varepsilon > 0$ be given. Pick $\hat{\delta}$ such that

$$\frac{g(\varepsilon)}{1 - \hat{\delta}^T} > F, \tag{7.88}$$

where $g(\cdot)$ is given by Lemma 7.5 and F is given by Lemma 7.4. Now suppose, on the contrary, that for some $\delta \geqslant \hat{\delta}$, there are $\alpha \in A$ and an optimal program $\langle \alpha_t \rangle$ (under δ) such that $\|\alpha_t - \alpha^*\| \geqslant \varepsilon$ for all $t \geqslant 0$. Then, using Lemma 7.5,

$$L_0(\langle \alpha_t \rangle, \delta) = \sum_{t=0}^{\infty} \delta^t l(\alpha_t, \alpha_{t+1}, \delta) \geqslant \frac{g(\varepsilon)}{1 - \delta^T} \geqslant \frac{g(\varepsilon)}{1 - \hat{\delta}^T}. \tag{7.89}$$

But (7.89) and (7.88) together contradict (7.66). Q.E.D.

Now we prove the more important part of Proposition 5.3.

Proof of (5.5) *in Proposition* 5.3. We are given some $\varepsilon > 0$. First, choose $\bar{\varepsilon} > 0$ and $\delta \in [\underline{\delta}, 1)$ such that

$$\alpha \in A, \qquad \|\alpha - \alpha^*\| \leqslant \bar{\varepsilon}, \qquad \text{and} \qquad \langle \alpha_t \rangle \text{ optimal for } \delta \geqslant \delta$$

mply that

$$\|\alpha_t - \alpha^*\| < \varepsilon \qquad \text{for} \quad t = 0, 1, ..., T. \tag{7.90}$$

This is possible by Lemma 7.6.

Now choose $\delta^0 \in [\bar{\delta}, 1)$ such that

$$\left(\frac{1}{\delta} - 1\right) F(T-1) < g(\bar{\varepsilon}) \qquad \text{for} \quad \delta \geqslant \delta^0. \tag{7.91}$$

Next, pick $\varepsilon' \in (0, \bar{\varepsilon})$ such that

$$\frac{f(\varepsilon')}{\delta} + \left(\frac{1}{\delta} - 1\right) F(T-1) < g(\bar{\varepsilon}) \qquad \text{for} \quad \delta \geqslant \delta^0. \tag{7.92}$$

Now, pick $\varepsilon'' \in (0, \bar{\varepsilon})$ and $\delta_1 \in [\delta^0, 1)$ such that

(i) $\alpha \in A$, $\langle \alpha_t \rangle$ optimal for $\delta \geqslant \delta_1$ implies that there exists T with

$$\|\alpha_T - \alpha^*\| < \varepsilon''; \tag{7.93}$$

(ii) $\alpha \in A$, $\|\alpha - \alpha^*\| < \varepsilon''$, $\langle \alpha_t \rangle$ optimal for $\delta \geqslant \delta_1$ implies that

$$\|\alpha_1 - \alpha^*\| < \varepsilon'. \tag{7.94}$$

The statement (7.93) follows from Lemma 7.7, while (7.94) follows from Lemma 7.6.

Finally, pick $\delta^* \in [\delta_1, 1)$ such that

$$\left(\frac{1}{\delta^*} - 1\right) FT < g(\varepsilon''). \tag{7.95}$$

We will prove that for $\underline{\delta} \geqslant \delta^*$ and for any α and optimal program $\langle \alpha_t \rangle$, under δ,

$$\limsup_{t \to \infty} \|\alpha_t - \alpha^*\| < \varepsilon. \tag{7.96}$$

By virtue of (7.93), we may presume without loss of generality that the initial forest α satisfies $\|\alpha - \alpha^*\| < \varepsilon''$.

Define, for any feasible program $\langle \alpha_t \rangle$, and for any s, k with $k \geqslant s$,

$$\alpha(s, k) \equiv (\alpha_s, \alpha_{s+1}, ..., \alpha_k). \tag{7.97}$$

Next, define

$$d(\alpha(s, k), \alpha^*) \equiv \min_{s \leqslant t \leqslant k} \|\alpha_t - \alpha^*\|. \tag{7.98}$$

By virtue of (7.90), it suffices to prove that

$$d(\alpha(t, t+T), \alpha^*) \leqslant \bar{\varepsilon} \qquad \text{for all } t \geqslant 0. \tag{7.99}$$

We now prove this claim, to complete the proof. Suppose, on the contrary, that (7.99) is false. Then there are some $\alpha \in A$ with $\|\alpha - \alpha^*\| < \varepsilon$, some $\delta \geqslant \delta^*$, and a *first* integer S such that

$$d(\alpha(S, S+T), \alpha^*) > \bar{\varepsilon}$$

for the program $\langle \alpha_t \rangle$ optimal from α under δ.

In other words, $\|\alpha_t - \alpha^*\| > \bar{\varepsilon}$ for all $t = S, S+1, ..., S+T$. Because $\|\alpha_0 - \alpha^*\| = \|\alpha - \alpha^*\| < \varepsilon'' < \bar{\varepsilon}$, we have $S \geqslant 1$.

Write $l_t \equiv l(\alpha_t, \alpha_{t+1}, \delta)$ and $L_t = L_t(\langle \alpha_t \rangle, \delta)$ for the program and discount factor in question. An easy computation reveals that

$$\delta L_{t+1} - L_t = -l_t, \qquad \text{for all } t \geqslant 0$$

so that

$$L_{t+T} - L_t \leqslant \left(\frac{1}{\delta} - 1\right) \sum_{s=0}^{T-1} L_{t+s} - \sum_{s=0}^{T-1} l_{t+s}. \tag{7.100}$$

Let K be the greatest time period not exceeding S such that $\|\alpha_K - \alpha^*\| < \varepsilon''$. Then, using (7.94) and the "principle of optimality," $\|\alpha_{K+1} - \alpha^*\| < \varepsilon' < \bar{\varepsilon}$. So, if we define k to be the first integer such that

$$(K+1) + kT \in \{S, S+1, ..., S+T\},$$

we must have $k \geqslant 1$. Now, for each $i = 0, ..., k-1$, we have

$$\begin{aligned} &L_{(K+1)+(i+1)T} - L_{(K+1)+iT} \\ &\quad \leqslant \left(\frac{1}{\delta} - 1\right) \sum_{s=0}^{T-1} L_{(K+1)+iT+s} - \sum_{s=0}^{T-1} l_{(K+1)+iT+s} \\ &\quad \leqslant \left(\frac{1}{\delta^*} - 1\right) FT - g(\varepsilon'') < 0. \end{aligned} \tag{7.101}$$

The second inequality in (7.101) follows from Lemma 7.4, and the fact that $\|\alpha_s - \alpha^*\| \geqslant \varepsilon''$ for all $s \in \{K+1, ..., S+T\}$, so that Lemma 7.5 may be applied. The last inequality follows from (7.95).

Note also that because $\|\alpha_{K+1} - \alpha^*\| < \varepsilon'$, we have, by Lemma 7.4,

$$L_{K+1} < f(\varepsilon'). \tag{7.102}$$

Combining (7.101) and (7.102), we get

$$L_{(K+1)+kT} < f(\varepsilon'). \tag{7.103}$$

Now,

$$L_{(K+1)+(k+1)T} - L_{(K+1)+kT} \leqslant \left(\frac{1}{\delta_0} - 1\right) \sum_{s=0}^{T-1} L_{(K+1)+kT+s} - \sum_{s=0}^{T-1} l_{(K+1)+kT+s}$$

so that

$$\begin{aligned} L_{(K+1)+(k+1)T} &\leqslant \frac{1}{\delta_0} L_{(K+1)+kT} + \left(\frac{1}{\delta_0} - 1\right) F(T-1) - \sum_{s=0}^{T-1} l_{(K+1)+kT+s} \\ &< \frac{f(\varepsilon')}{\delta_0} + \left(\frac{1}{\delta_0} - 1\right) F(T-1) - g(\bar{\varepsilon}) \\ &< 0, \end{aligned} \tag{7.104}$$

where the second inequality uses (7.103), Lemma 7.4, and the fact that $\|\alpha_{(K+1)+kT} - \alpha^*\| > \bar{\varepsilon}$ by definition of k, so that Lemma 7.5 may be applied.

But (7.104) yields a contradiction, because by construction, $L_t \geqslant 0$ for all $t \geqslant 0$. Q.E.D.

To complete the proof of Proposition 5.3 in the case where rank $Y < N$, we need a final lemma.

LEMMA 7.8. *Suppose that* rank $Y < N$. *Then there exists* $\alpha \in A$, *with* $\alpha \neq \alpha^*$, *such that*

$$\begin{aligned} R(0)\,\alpha(0) + R(1)\,\alpha(1) + \cdots + R(N)\,\alpha(N) &= c^* \\ R(N)\,\alpha(0) + R(0)\,\alpha(1) + \cdots + R(N-1)\,\alpha(N) &= c^* \\ &\vdots \\ R(1)\,\alpha(0) + R(2)\,\alpha(1) + \cdots + R(0)\,\alpha(N) &= c^*. \end{aligned} \tag{7.105}$$

Proof. Suppose rank $Y < N$. Then, recalling the identity

$$\dim n(Y) + \text{rank } Y = N + 1$$

(see (7.80)), we have

$$\dim n(Y) \geqslant 2. \tag{7.106}$$

We know already that $\alpha^* \in n(Y)$. Because of (7.106), there is $\beta \in n(Y)$ such that α^* and β are linearly independent. Moreover, β can be chosen so that

$$\sum_{i=0}^{N} \beta(i) = 1. \tag{7.107}$$

Because $\alpha^* \gg 0$, the reader can easily check that there exists $\lambda \in (0, 1)$ such that

$$\alpha \equiv [\lambda\alpha^* + (1-\lambda)\beta] \in A. \tag{7.108}$$

Note that $\alpha \neq \alpha^*$. Moreover, because α^*, $\beta \in n(Y)$, we have

$$Y\alpha = 0. \tag{7.109}$$

More explicitly,

$$\begin{aligned}
&[R(1)-R(0)]\,\alpha(0)+[R(2)-R(1)]\,\alpha(1)+\cdots \\
&\qquad +[R(N)-R(N-1)]\,\alpha(N-1)+[R(0)-R(N)]\,\alpha(N)=0 \\
&[R(0)-R(N)]\,\alpha(0)+[R(1)-R(0)]\,\alpha(1)+\cdots \\
&\qquad\qquad +[R(N)-R(N-1)]\,\alpha(N)=0 \\
&\qquad \vdots \\
&[R(2)-R(1)]\,\alpha(0)+[R(3)-R(2)]\,\alpha(1)+\cdots+[R(1)-R(0)]\,\alpha(N)=0.
\end{aligned} \tag{7.110}$$

Put $\alpha(0)\,R(0)+\alpha(1)\,R(1)+\cdots+\alpha(N)\,R(N)=K$.
Then, using (7.110), it can be easily seen that

$$\begin{aligned}
&R(0)\,\alpha(0)+R(1)\,\alpha(1)+\cdots+R(N)\,\alpha(N)=K \\
&R(N)\,\alpha(0)+R(0)\,\alpha(1)+\cdots+R(N-1)\,\alpha(N)=K \\
&\qquad \vdots \\
&R(1)\,\alpha(0)+R(2)\,\alpha(1)+\cdots+R(0)\,\alpha(N)=K.
\end{aligned} \tag{7.111}$$

It remains to prove that $K=c^*$. This is done simply by adding up the left-hand sides and the right-hand sides of all the equations in (7.111), and noting that $\sum_{s=0}^{N} R(s)=(N+1)\,c^*$. Q.E.D.

We may now complete the proof of Proposition 5.3.

Proof of (5.6) *in Proposition* 5.3. Suppose that rank $Y<N$. Pick $\alpha \neq \alpha^*$ as given by Lemma 7.8. Define $\langle \alpha_t \rangle$ from α as follows:

$$\begin{aligned} \alpha_0 &= \alpha \\ \alpha_{t+1}(s+1) &= \alpha_t(s) \qquad \text{for all } s \neq N, \text{ for all } t \geqslant 0 \\ \alpha_{t+1}(0) &= \alpha_T(N) \qquad \text{for all } t \geqslant 0. \end{aligned} \tag{7.112}$$

Then it is easy to see that

$$c(\alpha_t) = c^* \qquad \text{for all } t \geqslant 0. \tag{7.113}$$

We know that there exists $\underline{\delta} \in (0, 1)$ such that for all $\delta \geqslant \underline{\delta}$, α^* is a stationary optimal forest. We claim now that for all $\delta \geqslant \underline{\delta}$, the program $\langle \alpha_t \rangle$ is optimal from $\alpha \in A$. For any other program $\langle \alpha'_t \rangle$ from α and any discount factor $\delta \geqslant \underline{\delta}$, we have

$$\begin{aligned} \sum_{t=0}^{\infty} \delta^t [u(c(\alpha'_t)) - u(c(\alpha_t))] &= \sum_{t=0}^{\infty} \delta^t [u(c(\alpha'_t)) - u(c^*)] \\ &\leqslant u'(c^*) \sum_{t=0}^{\infty} \delta^t [c(\alpha'_t) - c(\alpha_t)] \\ &\leqslant 0, \end{aligned}$$

where the last inequality follows by noting that $\langle \alpha_t \rangle$ satisfies conditions (i) and (ii) of Proposition 4.1.

It is now clear that the program $\langle \alpha_t \rangle$ satisfies (5.6) for every $\delta \in (\underline{\delta}, 1)$. This completes the proof. Q.E.D.

References

1. A. Araujo and J. Scheinkman, Smoothness, comparative dynamics and the turnpike property, *Econometrica* **45** (1977), 601–620.
2. J. Benhabib and K. Nishimura, Competitive equilibrium cycles, *J. Econ. Theory* **35** (1985), 284–306.
3. J. Benhabib and A. Rustichini, "A Vintage Capital Model of Investment and Growth: Theory and Evidence," Economic Research Report #89-26, C.V. Starr Center for Applied Economics, New York University, 1989.
4. T. Bewley, An integration of equilibrium theory and turnpike theory, *J. Math. Econ.* **10** (1982), 233–268.
5. W. A. Brock and J. Scheinkman, Global asymptotic stability of optimal control systems with applications to the theory of economic growth, *J. Econ. Theory* **12** (1976), 164–190.
6. D. Cass and K. Shell, The structure and stability of competitive dynamical systems, *J. Econ. Theory* **12** (1976), 31–70.
7. M. Faustmann, Calculation of the value which forestry land and immature stands possess for forestry, *in* "Martin Faustmann and the Evolution of Discounted Cash Flow" (M. Gane, Ed.), pp. 27–55, Institute Paper No. 42 (1849), Commonwealth Forestry Institute, University of Oxford, 1968.
8. M. Kurz, Optimal economic growth and wealth effects, *Int. Econ. Rev.* **9** (1968), 155–174.

9. L. McKenzie, A primal route to the turnpike and Liapounov stability, *J. Econ. Theory* **27** (1982), 194–209.
10. L. W. McKenzie, Turnpike theory, discounted utility, and the von Neumann facet, *J. Econ. Theory* **30** (1983), 330–352.
11. T. Mitra and H. Wan, Some theoretical results on the economics of forestry, *Rev. Econ. Stud.* **52** (1985), 263–282.
12. T. Mitra and H. Wan, On the Faustmann solution to the forest management problem, *J. Econ. Theory* **40** (1986), 229–249.
13. T. Mitra and D. Ray, A neighborhood turnpike theorem for the timber model of forest management, mimeo.
14. R. T. Rockafellar, Saddle points of Hamiltonian systems in convex Lagrange problems having a non-zero discount rate, *J. Econ. Theory* **12** (1976), 71–113.
15. P. A. Samuelson, Optimality of profit-including prices under ideal planning, *Proc. Nat. Acad. Sci. U.S.A.* **70** (1973), 2109–2111.
16. P. A. Samuelson, Economics of forestry in an evolving society, *Econ. Inquiry* **14** (1976), 66–492.
17. G. F. Schreuder, Optimal forest investment decisions through dynamic programming, *Yale Univ. School Forestry Bull.* **72** (1968).
18. J. Scheinkman, On optimal steady states of n-sector growth models when utility is discounted, *J. Econ. Theory* **12** (1976), 11–30.
19. W. Sutherland, On optimal development in a multi-sectoral economy: The discounted case, *Rev. Econ. Stud.* **37** (1970), 585–589.
20. H. Y. Wan, Optimal evolution of tree-age distribution for a tree farm, *in* "Mathematical Approaches to Problems in Resource Management and Epidemiology," (Carlos Castillo-Chavez *et al.*, Eds.), Lecture Notes in Biomathematics, Springer-Verlag, Berlin/New York, 1989.
21. H. Y. Wan, "Durable Assets and Replacement Cycles," Cornell University Working Paper #419, 1989.