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Complexities of Concrete Walrasian Systems*

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Mathematical form powerfully contributes to defining a philosophy of economic analysis whose major tenets include rigor, generality and simplicity. It commands the long search for the most direct routes from assumptions to conclusions. It directs its aesthetic code, and it imposes its terse language. Another tenet of that philosophy is recognition, and acceptance, of the limits of economic theory, which cannot achieve a grand unified explanation of economic phenomena.

Gerard Debreu[1]

1 Introduction

The rigorous elaboration of the Walras-Pareto 'theory of value' has often been hailed as 'one of the most notable intellectual achievements' of economic theory. The volume of research on the refinements of the Arrow-Debreu-McKenzie model, the axiomatic style of exposition, and the growing use of a variety of mathematical techniques have led to appraisals of the area by

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[1] See 'Random Walk and Life Philosophy', in *Eminent Economists*, p. 114.

methodologists, philosophers, historians of economic thought as well as by well-known economic theorists.[2] Some of the appraisals, not surprisingly, have been quite negative, characterizing the state of axiomatic general equilibrium theory as 'a wealth of mathematics but no proper object of study' (Jolink 1993, p. 1311). But more balanced assessments have also raised serious questions about the direction of future developments. To take a recent example, one might turn to Morishima's article (1991) on general equilibrium theory ('GET'). Morishima feels that 'unlike physics, economics has unfortunately developed in a direction far removed from its empirical source, and GET in particular, as the core of economic theory, has become a mathematical social philosophy'. Morishima emphasizes that the *price taking behaviour* that underlies the Walrasian model is applicable to only a small part of the modern industrialized economies. Moreover, the axioms of utility and profit maximization are inadequate to capture the motivations behind economic decisions, and may well be irrelevant in understanding the institutions of the highly productive economies of the Far East. On the whole, Morishima's article reflects unhappiness over the 'inadequate concern for actuality' reflected by the research efforts in GET.

Whether a model should be judged by the realism or empirical content of its *assumptions* has been a prominent issue in many a methodological debate. A somewhat extreme view (Friedman 1946) emphasized *predictive power* of the ultimate propositions as opposed to the realism of the assumptions as the key test. Others look for an *explanatory power* or more broadly, an 'understanding' of economic phenomena and institutions from a model, with due recognition that simplifications (in choosing assumptions) are inevitable for making significant progress in economic theory. A prominent theme in Samuelson's *Foundations* was the need to derive 'meaningful' (in principle refutable by empirical evidence)

[2] A survey of some of the appraisals is in Weintraub (1985). This book also contains a fascinating discussion of the developments in research leading to the landmarks of the early fifties. See also Radner (1991).

results on comparative statics and dynamics from formal mathematical models. In *Causality in Economics* (1979), Hicks mentioned five reasons for pursuing economic theory, but stressed: 'When theory is applied, it is being used as a means of explanation: we ask not merely what happened, but why it happened'.

If one recognizes that economic theory has its limits, and it 'cannot achieve a grand explanation of economic phenomena', one ought not to be surprised that a *particular* formal model is inadequate for explaining or understanding the forces behind the evolution of market prices in different economies and in different periods of history. It is better to interpret some of Morishima's criticisms as items on the agenda for future research. In this article, our focus is the difficulty of prediction in a Walrasian framework. In Section 1, we consider a Walrasian exchange economy. The *price-taking* agents have well behaved indifference curves, and maximize utility subject to the budget constraint. But, it turns out that maximizing behaviour and the standard assumptions on preferences (including strong convexity and monotonicity) do not impose any special restriction on the excess demand function of the economy. In the absence of special structures on the excess demand function, there is no bound on the number of Walrasian equilibria, and the scope of comparative statics or qualitative economics appears quite limited. Next, we go on to more 'concrete' models, namely, those with only two goods, and examine a Walrasian tatonnement cast in the form of a non-linear difference equation (following the tradition of 'period analysis' in economic dynamics). Even with two goods, one can construct examples of 'chaotic' tatonnement. The price path or trajectory over time may turn out to display *sensitive dependence on initial conditions:* trajectories emanating from nearby initial conditions exhibit remarkably different qualitative behaviour. Also, there may be an uncountable set of initial conditions from which the trajectories do not converge to an equilibrium or a periodic trajectory. At the same time, periodic or cyclical trajectories of *all* periods may be present (see Theorem 4.1 for a precise statement).

One way to look at the examples is to recognize the very limited possibility of predicting long run behaviour. From another perspective, since the initial condition crucially affects the dynamic behaviour, we are reminded of the importance of historical evolutions that presumably start the process from a particular initial condition. And, finally, we see that to 'explain' complicated dynamics, we do *not* have to invoke 'external shocks' (although these may be of importance in a particular context) or to develop complicated models with many variables (see May 1976).

2 THE BASIC MODEL

2.1 Notation

If $x = (x_k)$ is any vector in R^l, we say that x is *non-negative* $(x \geq 0)$ if $x_k \geq 0$ for all k; x is *positive* $(x > 0)$ if x is non-negative and $x_k > 0$ for some k, x is *strictly positive* $(x >> 0)$ if $x_k > 0$ for all k. For any two vectors x, y in R^l, we write $x \geq y$ (respectively, $x > y$; $x >> y$) if $x - y \geq 0$ (resp. $x - y \geq 0$, $x - y >> 0$). The set of all non-negative (resp. strictly positive) vectors in R^l is denoted by R^l_+ (resp. R^l_{++}).

By a preference preorder $\lesssim$ on R^l_+, we mean a binary relation on R^l_+ that is reflexive, transitive and complete (i.e., '$x \lesssim x$' for all x in R^l_+, '$x \lesssim y$, $y \lesssim z$' implies '$x \lesssim z$' for any x, y, z in R^l_+; for any pair x, y in R^l_+ either $x \lesssim y$ or $y \lesssim x$ (or both)). We write $x \sim y$ if $x \lesssim y$ *and* $y \lesssim x$; and $x < y$ if $x \lesssim y$ holds but $y \lesssim x$ does not hold. We interpret '$x \lesssim y$' as 'the commodity bundle x is no better than the commodity bundle y;' $x \sim y$ means that the consumer is indifferent between x and y, $x < y$ means that 'y is preferred to x.'

2.2 Excess Demand Functions

Consider first a Walrasian exchange economy with $l \geq 2$ goods, the prices of which are represented by strictly positive vectors with unit Euclidean norm, i.e., the set of prices is given by

$$S = \{p \in R^l : p >> 0, \; ||p|| = 1\} \tag{2.1}$$

where $||p|| = \left[\sum_{k=1}^{l} P_k^2 \right]^{1/2}$

We shall restrict our attention mostly to prices bounded away from zero: for a sufficiently small positive ε, let

$$S_\varepsilon = \{p \in S: p_k \geq \varepsilon > 0, k = 1, \ldots, l\} \tag{2.2}$$

A *consumer* is defined by a pair $(\lesssim, e)$ where $\lesssim$ represents his preferences and e is his endowment vector. We assume:

(C.1) *$\lesssim$ is a strictly convex, monotone, continuous preference preorder on R^l_+ and the endowment e is an element of R^l_+.*

Recall that $\lesssim$ is *strictly convex* if the convex combination with weights different from 0 and 1 of two distinct indifferent consumption vectors is strictly preferred to both.

We say that $\lesssim$ is *monotone* if for any two vectors x, y in R^l_+ with $x > y$, it is true that $y < x$. The preference preorder $\lesssim$ is *continuous* if for any x in R^l_+, both the sets:

$$\{y \in R^l_+ : y \lesssim x\} \text{ and } \{z \in R^l_+ : x \lesssim z\}$$

are closed in R^l_+.

Given our assumptions, it is known that '$\lesssim$' can be 'represented' by a *utility function* u; in other words, there is some $u: R^l_+ \rightarrow R$ such that:

$$u(x) \geq u(y) \text{ if and only if } y \lesssim x.$$

For the sake of completeness, we recall a simple method of constructing such a utility function. Define

$$M = \{x \in R^l_+ : x = (\lambda, \ldots, \lambda), \lambda \geq 0\}$$

When $l = 2$, M is just the 'forty-five degree line' from the origin. For any $x = (\lambda, \ldots, \lambda)$ in M, define $u(x) = \lambda$. Next, since '$\lesssim$' is continuous as well as monotone, corresponding to any $\hat{y}$ in R^l_+, there is a *unique* $\hat{x} = (\hat{\lambda}, \ldots, \hat{\lambda})$ in M such that $\hat{y} \sim \tilde{x}$. Hence, define $u(\hat{y}) = u(\hat{x}) = \hat{\lambda}$. Again, when $l = 2$, it is helpful to visualize the construction by drawing the indifference curve through $\hat{y}$ which intersects the forty-five degree line at a unique $\hat{x}$.

Now, given a price system p in S, a consumer i, represented by $(\lesssim_i, e_i)$, chooses the best element d_i for $\lesssim_i$ in his *budget set*

$$B_i = \{x \in R^l_+ : p\,x \leq p\,e_i\}.$$

The monotonicity assumption on $\lesssim_i$ ensures that d_i will be on the 'budget hyperplane', i.e., $pd_i = pe_i$ (the expenditure on d_i will be equal to the income pe_i). The excess demand of consumer i is the vector $z_i = d_i - e_i$. Thus, the *value* of excess demand of consumer i is necessarily zero, i.e., $pz_i = 0$. Hence, the aggregate excess demand of the economy with n agents defined as

$$z = z_1 + \ldots + z_n$$

also satisfies 'Walras' law':

$$pz = 0 \tag{2.3}$$

The motivation behind the next two definitions should now be clear.

A function $f: S \rightarrow R^l$ is the *individual excess demand function* of consumer $(\lesssim, e)$ if for every p in S, $e + f(p)$ is the best element for $\lesssim$ of $B = \{x \in R^l : p\,x \leq p\,e\}$, i.e., any x in B satisfies $x \lesssim (e + f(p))$.

A continuous function $f: S \rightarrow R^l$ is an *excess demand function* (for the economy) if for every p in S, $p\,f(p) = 0$.

The basic result of Debreu (1974) can now be stated. Recall that l is the number of commodities.

Theorem 2.1 Let f be an excess demand function. For every $\varepsilon > 0$, there are l consumers whose individual excess demand functions sum to f on S_ε.

Any continuous function satisfying Walras' law can be viewed as an excess demand function (on S_ε) of an appropriately constructed exchange economy with l consumers. Thus, the utility maximization hypothesis in the context of the Walrasian economy imposes no special structure on the excess demand function of the economy (besides Walras' law) *even when the preferences are required to be monotone, strictly convex and continuous*, i.e., the 'indifference curves' look like those appearing in Hicks' *Value and Capital*.

Debreu's theorem has been extended in many directions. In particular, there is no hope of getting any special property of excess demand functions by restricting the *dispersion* of endowments and income. Also of interest is the following result due to Kirman and Koch (1986):

Theorem 2.2 Let n be a positive integer greater than or equal to l, and $\upsilon_1, \ldots, \upsilon_n$ be distinct positive real numbers with $\upsilon_1 + \ldots + \upsilon_n = 1$. Let $f: S \to R^l$ be an excess demand function. Then for every $\varepsilon > 0$, there exists a continuous, monotone, strictly convex preference preorder $\lesssim$ on R^l_+ and an endowment vector e in R^l_+, such that the individual excess demand functions of agents $i (= 1, \ldots, n)$ represented by $(\lesssim, \upsilon_i e)$ sum up to f on S_ε.

We see that even when we impose the additional *restriction* that all agents have the *same* preference preordering and that their endowments are of the type $\upsilon_i e$ (so that the distribution of relative income is fixed and price independent) there is still no special restriction on the class of excess demand functions for the economy.

Let us spell out some implications of these results for qualitative economics. The set of Walrasian equilibrium prices is formally given by

$$W = \{p \in S: f(p) = 0\} \tag{2.4}$$

Clearly, for 'explaining' some observed prices perceived as equilibrium prices, or predicting equilibrium prices from appropriate information on excess demand functions, it is essential to have as detailed a knowledge of W as possible. The first task, of course, is to understand conditions on f such that W is *non-empty*. Here, in addition to Walras law, some continuity, desirability and boundedness conditions on f certainly suffice to assure that W is non-empty.[3] However, the existence of an equilibrium price system (mathematically, a 'fixed point' problem) does not imply that W consists of a *single* element: indeed, examples of non-uniqueness cannot be dismissed as pathological. A programme of research initiated by Debreu aimed at studying the properties of W for 'typical' or 'generic' models of exchange economies. The literature is understandably technical, and leads to the conclusion that a 'typical' or 'regular' economy has a *finite* number of equilibria (i.e., the set W is *discrete*). Furthermore, it was also proved that in a 'neighborhood' of a regular economy (i.e., for small 'local' variations around a typical model) the set of equilibrium prices will change continuously with variations in the parameters (preferences/endowments). The importance of such continuity was stressed by Debreu (1975) along these lines: [in the absence of such continuity] 'the slightest error of observation on the data of the economy might lead to an entirely different set of predicted equilibria. This consideration, which is common in the study of physical systems, applies with even greater force to the study of social systems'.

But it is *not* possible to put any upper bound on the number of elements of W, and our Theorem 2.1 indicates why it is difficult to make 'general' qualitative predictions on W.

There are two other routes that have been explored. The first possibility (pioneered by H. Scarf) is to rely on explicit computation of W by assuming particular forms of excess demand functions. The functional forms presumably have to be chosen on the basis of estimates from the relevant data (and practical considerations of computational feasibility will be relevant). 'Computable' general equilibrium models will bring the theory closer to the empirical world (as often urged) and will be used more effectively as researchers gain experience with more sophisticated machines that have a vast memory and exceptional speed.

[3] Formally, assume that f is continuous at any $p >> 0$, and for any sequence p^n in S converging to some p in the boundary of S, $||f(p^n)|| \to \infty$; also f is bounded below. See Arrow and Hahn (1971).

Yet another route has been to develop 'concrete' Walrasian models. Here, drastic assumptions on the number of goods and/or the nature of technology and preferences are usually made in order to derive insightful comparative static or dynamic results. The literature on 'pure' theory of international trade is probably the best example of successful efforts in this direction.

It is of interest to note that in assessing Paul Samuelson's contributions to economics, Lindbeck (1970) observed that Samuelson's 'most important contributions in general equilibrium theory is probably a "concretization" of the Walrasian system, implying a simplification of general equilibrium theory. This makes it possible to analyse concrete problems and to reach operationally meaningful theorems, rather than limiting the analysis to counting of equations and unknowns and saying that "everything depends on everything else".'

In what follows, we look at such a 'concrete' model with only two commodities. It turns out that even with such an extreme simplification, we face complex problems if we want to develop the dynamic processes that underlie comparative static exercises.

3 Walrasian Tatonnement

What can we say about the behavior of markets when the price system is *not* an equilibrium? Walras himself discussed a market-by-market adjustment process through which an equilibrium can be attained. He emphasized that the *direct pressure* of excess demand in a market on its prevailing (non-equilibrium) price will push it towards an equilibrium level at which the equality of demand and supply will prevail. He recognized that such a change in one market would disturb other markets, but these 'indirect influences, some in the direction of equality and others in the opposite direction' . . . 'up to a point cancelled each other out' (Walras, *Elements of Pure Economics*, 1954 Jaffee translation from the 1926 edition).

A mathematical formulation of the Walrasian tatonnement was presented by Samuelson in his *Foundations*. We know now that unless one has some special features (see Arrow and Hahn (1971) on sufficient conditions for local and global stability of tatonnement), a dynamic price process in which the price in one market increases (decreases) if there is positive (negative) excess demand need not converge to an equilibrium. In a model with two commodities, however, a particularly interesting result was obtained by Arrow and Hurwicz (1958), a result which indicates the possibility of using the tatonnement to approach an equilibrium. We shall first describe a similar situation somewhat informally.

3.1 Adjustments in a Two Commodity Model

We can conveniently represent a two commodity economy (given (W)) by a single *excess demand function*

$$\zeta_1(p) \equiv z_1(p_1, 1-p_1), 0 < p_1 < 1 \tag{3.1}$$

In this section we shall drop the subscript for the commodity in order to simplify notation.

Now suppose that we have a family of economies E_θ each described by an excess demand function of the type (3.1). Formally, the family of economies E_θ is described by a family of functions $\{\zeta_\theta(p)\}$ where the parameter θ belongs to some (non-empty) set C. Assume that *for each* θ in C, $\zeta_\theta(p)$ *is continuous on* $(0,1)$, *and has finitely many zeros* (i.e., $\zeta_\theta(p) = 0$ has finitely many solutions). Also, assume the following *boundary condition:*

'*For each* $\theta \in C$, $\zeta_\theta(p) > 0$ (resp. < 0) *for all p sufficiently close to* 0. (resp. 1).'

Consider any equilibrium $\bar{p}_\theta$ of the economy E_θ, i.e., $\zeta_\theta(\bar{p}_\theta) = 0$; next, we change the value of θ to some other θ'. For concreteness, assume that $\zeta_{\theta'}(\bar{p}_\theta) > 0$. In other words, we imagine that the economy E_θ is initially in equilibrium at price $\bar{p}_\theta$ and then there is a shift of the excess demand function generating positive excess demand at the price $\bar{p}_\theta$. Now, suppose that the price adjustment in disequilibrium is modelled as a *continuous time* tatonnement process

$$\frac{dp}{dt} = \alpha\, \zeta_{\theta'}(p(t)),\ \alpha > 0 \tag{3.2}$$

where $\alpha > 0$ is the speed of adjustment. In this case a theorem of Arrow and Hurwicz can be invoked to assert that *p(t)* starting from $\bar{p}_\theta$ will converge to *some* equilibrium of the economy $E_{\theta'}$. Since at $\bar{p}_\theta$, excess demand of the economy $E_{\theta'}$ is assumed to be positive (i.e., $\zeta_{\theta'}(\bar{p}_\theta) > 0$), and, by our boundary condition, $\zeta_{\theta'}(p) < 0$ when p is sufficiently close to one, we can use the intermediate value theorem and assert that there is *some* $\bar{p}$ in $(\bar{p}_\theta, 1)$ such that $\zeta_{\theta'}(\bar{p}) = 0$. Clearly, if we take the *smallest* such $\bar{p}$ in $(\bar{p}_\theta, 1)$, say $\bar{p}_{\theta'}$, the process (3.2) starting from $\bar{p}_\theta$ will increase to $\bar{p}_{\theta'}$. Hence, a shift from ζ_θ to $\zeta_{\theta'}$ that generates a positive [respectively, negative] excess demand at the initial equilibrium $\bar{p}_\theta$ leads to an increasing [resp. decreasing] *p(t)* that will converge to a new equilibrium $\bar{p}_{\theta'} > \bar{p}_\theta$. A formal proof of convergence is spelled out in Arrow and Hahn (1971), and convergence does *not* imply that 'markets will settle down to the new equilibrium $\bar{p}_{\theta'}$ in finite time', but the special structure of the model yields an unambiguous prediction as to the direction of change from the old equilibrium $\bar{p}_\theta$ to the new equilibrium $\bar{p}_{\theta'}$.

But economic theory has a tradition of 'period' analysis in which time is treated as a discrete variable. Whether a continuous time formulation is adequate or well-suited for depicting the evolution of an economic process has been discussed extensively (see Baumol 1971). When we think of the Walrasian auctioneer whose role is to announce prices and then to make the appropriate revisions in the light of responses from the agents in various markets, we feel that Saari's remarks (1985, p. 119) on period analysis are quite persuasive. 'It can be argued', he said, 'that the correct dynamical process associated with the tatonnement process is an iterative one. Just one supporting argument is that the differential dynamic process requires a continuum of information. At each instant of time the information must be updated; so a continuous mechanism is far beyond the capability of any "auctioneer".'

In the tradition of 'period analysis' an adjustment process may also be cast in terms of a discrete-time difference equation. For example, we might write

$$p_{t+1} = p_t + \alpha\, \zeta_{\theta'}(p_t) \text{ where } p_0 = \bar{p}_\theta$$

or simply

$$p_{t+1} \equiv G_{\theta'}(p_t) \tag{3.3}$$

The global behaviour of (3.3) may be in sharp contrast with that of (3.2). It is remarkable that even a simple 'one dimensional' system (3.3) may display extremely complicated behaviour. In the next section we elaborate on this point.

4 A Disgression on Chaotic Dynamics

4.1 Abundance of Cycles

It is useful to recall some mathematical definitions. Consider a first order difference equation

$$x_{t+1} = F(x_t) \tag{4.1}$$

where F is a continuous map from some interval X of real numbers into itself. The set X is the *state space* and F is the *law of motion* of the *dynamical system* defined as the pair (X, F). To set the notation, write $F^0(x) \equiv x$, and for each $j \geq 1, F^j(x) \equiv F\,[F^{j-1}(x)]$. Starting from an initial x in X, the rule (4.1) provides us with the *trajectory* $(F^j(x))_{j=0}^{\infty}$. Once the initial state x is specified, the law of motion unambiguously specifies x_t, the state of the system in period t. A point x is a *fixed point* of F if $F(x) = x$. A point x is *periodic* if there is some $k \geq 1$ such that $F^k(x) = x$; the smallest such k is the *period* of F. In particular, a fixed point of F is a periodic point of period one. If X is a (non-empty) compact interval and F

is continuous, then F has at least one fixed point. A fixed point x is *locally stable* if we can find an open interval V containing x such that for each $y \in V$, the trajectory $\{F^j(y)\}_{j=0}^{\infty}$ converges to x (approaches x).

Generations of undergraduate students are introduced to difference equations by the linear 'cobweb' models. Recall that if we have a first order, linear, homogeneous difference equation:

$$x_{t+1} = \lambda x_t \tag{4.L}$$

We know that the solution

$$x_t = \lambda^t x_0 \tag{4.LS}$$

displays 'oscillatory' behaviour when $\lambda < 0$. But if $|\lambda| > 1$, it is 'explosive'; and, if $|\lambda| < 1$, it is 'damped'. A *persistent* oscillation or cycle is produced when $\lambda = -1$, and in this case the solution is *periodic* with period 2. When $\lambda > 0$, the solution is monotone (increasing if $\lambda > 1$, constant if $\lambda = 1$, and decreasing to zero if $\lambda < 1$). Thus, the periodic behaviour is somewhat of an accidental feature, a knife-edge possibility in this linear world (4.L). Turning to the solution of a *non-linear* first order equation (4.1), Samuelson observed in his *Foundations:*

'It could no doubt be shown that it must do one of the following: (a) go off to infinity; (b) approach an equilibrium level; or (c) approach a periodic motion of some finite period.'

Of course, if we choose in (4.1) a function F from a bounded set X into X, the possibility (a) that Samuelson alluded to is ruled out. We are then left with possibilities (b) and (c). As far as we recall, in our student days no other systematic analysis of non-linear first-order systems (4.1) that was accessible to us challenged Samuelson's conjecture. But we are now in a position to appreciate the spectrum of possibilities much better.

To gain some insights into the possible complexities of trajectories of (4.1), consider the *tent map* $\hat{F}$ defined as:

$$\hat{F}(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2(1-x) & \text{for } x \in [1/2, 1] \end{cases} \tag{4.2}$$

$\hat{F}$ is 'piece-wise' linear. It has two fixed points, namely, $x = 0$ and $x = 2/3$. Neither, however, is locally stable (by drawing the graph of $\hat{F}$ and looking at the iterates of points close to the fixed points, this can be easily seen).

The map (4.2) is admittedly just a step away from the linear case (4.L) and is simple to describe; yet, it is quite useful for illustrating some general results and capturing some subtle arguments. Let us recall some well-known results from real analysis. In what follows by an *interval* we shall always mean a *non-degenerate* interval, and if X is an interval, a *subinterval*, X_1 of X is an interval X_1 contained in X (for example, $H \equiv [0, 1/2]$ and $T \equiv [1/2, 1]$ are both subintervals of the interval $X = [0, 1]$). An interval is *compact* if it is both closed and bounded (i.e., contains both the end points and does not stretch out into infinity in either direction!). If X is an interval and F is a continuous real valued function on X, $F(X)$ is an interval; moreover, if X is a compact interval, so is $F(X)$. Observe that $\hat{F}$ maps the subintervals H and T of $[0, 1]$ *onto* $[0, 1]$ i.e., $\hat{F}(H) = \hat{F}(T) = [0, 1]$. Thus, T is a subinterval of $\hat{F}(H)$ and H is a subinterval of $\hat{F}(T)$.

We shall characterize a family of dynamical systems (of which the dynamical system with state space $X = [0,1]$ and the law of motion $\hat{F}$ defined by the tent map (4.2) is a member) in which there is an abundance of periodic points. The main theorem, due to Li and Yorke (1975), is one of the most striking results in the literature on dynamical systems. On the way, we pick up some propositions that throw light on the difficulties of predicting the long run qualitative properties of dynamical systems. To avoid misunderstanding we shall state the results somewhat formally.

Lemma 4.1 Let G be a real-valued continuous function on an interval I. For any compact subinterval I_1, of $G(I)$, there is a compact subinterval Q of I such that $G(Q) = I_1$.

One can figure out the subinterval Q directly as follows. Let $I_1 = [G(p), G(q)]$ where p and q are in I. Assume that $p < q$. Let r be the last point of the interval $[p, q]$ such that $G(r) = G(p)$; let s be the first point after r such that $G(s) = G(q)$. Then the subinterval

$Q = [r, s]$ is mapped *onto* I_1 under G. The case $p > q$ is similarly dealt with.

The next lemma has some fairly deep implications regarding the complexity of a class of dynamical systems. Fortunately, the 'proof' based on an induction argument is short and entirely elementary.

Lemma 4.2 Let J be an interval and let $F: J \to J$ be continuous. Suppose that $(I_n)_{n=0}^{\infty}$ is a sequence of compact subintervals of J and, for all n,

$$I_{n+1} \subset F(I_n). \tag{4.3}$$

Then there is a sequence of compact subintervals (Q_n) of J such that, for all n,

$$Q_{n+1} \subset Q_n \subset I_0$$

and (4.4)

$$F^n(Q_n) = I_n$$

Hence, for any $x \varepsilon Q = \cap_n Q_n$, we have $F^n(x) \in I_n$.

The proof 'by induction' is constructed as follows: $Q_0 = I_o$. Then $F^0(Q_0) = I_0$ and $I_1 \subset F(I_0)$. If Q_{n-1} is defined as a compact subinterval such that $F^{n-1}(Q_{n-1}) = I_{n-1}$ then $I_n \subset F(I_{n-1}) = F^n(Q_{n-1})$. Apply our previous Lemma, 4.1 to the map $G \equiv F^n$ on Q_{n-1} to get a compact subinterval Q_n of Q_{n-1} such that $F^n(Q_n) = I_n$. This completes the induction argument.

A fundamental characterization of compactness implies that the intersection $Q \equiv \cap_n Q_n$ of the 'nested' compact intervals $\{Q_n\}$ must be non-empty. Hence, there is surely *some* x in Q, for which we have $F^n(x) \varepsilon I_n$ for all n.

Let us reflect on some implications of Lemma 4.2 using our tent map (4.2) for the sake of concreteness. Let Λ be the uncountable set of all sequences with two symbols $\{H, T\}$. Choose an *arbitrary* element s of Λ. By identifying H with the subinterval $[0, 1/2]$ and T with $[1/2, 1]$ as before, we see that the chosen s corresponds to an 'arrangement' or a sequence of compact intervals $(I_n)_{n=0}^{\infty}$ where each I_n is either H or T. Let us stress that the sequence s is *completely arbitrary:* it could even be viewed as a record of the outcomes of an infinite sequence of coin-tossing; the order in which H and T appears is not 'controlled' in any way whatsoever! Now, we have already noted that T is a subinterval of $\hat{F}(H)$ and H is a subinterval of $\hat{F}(T)$; thus, for this sequence $(I_n)_{n=0}^{\infty}$, it is certainly true that I_{n+1} is always a subinterval of $\hat{F}(I_n)$. The last statement of Lemma 4.2 now applies; there is *some* initial x in [0, 1] which generates a trajectory $\{\hat{F}^n(x)\}_{n=0}^{\infty}$ with the property that $\hat{F}^n(x)$ is in I_n. Thus, however 'randomly' we arrange H and T, the dynamical system $(X, \hat{F})$ where $X = [0, 1]$ and $\hat{F}$ is the tent map (4.2), is capable of generating a trajectory that will bounce from I_n into I_{n+1} over time according to this arrangement. Among other things, this means that we can think of 'repeating' or 'cyclical' arrangements like

$$(H\,H\,H \ldots)$$
$$(T\,T\,T \ldots)$$
$$(HT\,HT\,HT \ldots)$$
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No matter which 'cyclical' arrangement is contemplated, we can generate a trajectory that will provide an exact 'match'. Now, the set of all such cyclical arrangements (sequences with repeating finite 'blocks') is countable. But, as we said earlier, the set Λ is uncountable. Hence, there is an uncountable number of aperiodic arrangements. And, our simple dynamical system following the law of motion (4.2) on the state space [0, 1] is also capable of producing a trajectory matching *any* such aperiodic arrangement!

By using the two results discussed above, and the 'intermediate value theorem' we can prove the first part [*T.1*] of the following theorem of Li and Yorke (1975); [*T.1*] is also a special case of a deep theorem of Sarkowskii (see Devaney (1989)).

Theorem 4.1 Let J be an interval and $F: J \to J$ be continuous. Assume that there is some point a in J for which there are points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$ satisfying:

$$d \leq a < b < c \text{ (or, } d \geq a > b > c)$$

Then:

[*T.1*] For every positive integer $k = 1, 2, \ldots$, there is a periodic point of period k.

[*T.2*] (i) There is an uncountable set W containing no periodic points such that for all

$$x, y \in W, x \neq y$$

$$\limsup_{n \to \infty} |F^n(x) - F^n(y)| > 0; \liminf_{n \to \infty} |F^n(x) - F^n(y)| = 0$$

(ii) If x is any periodic point, then for all y in W,

$$\limsup_{n \to \infty} |F^n(x) - F^n(y)| > 0.$$

Of course, by considering (4.2) and the point $x = 1/4$, one notes that $\hat{F}(1/4) = 1/2$, $\hat{F}(1/2) = 1$ and $\hat{F}(1) = 0$. Hence, with [0, 1] as the state space and the tent map (4.2) as the law of motion $\hat{F}$ we get an example where Theorem 4.1 readily applies.

The fact that the existence of a periodic point of period three implies the existence of periodic points of *all* periods is surely bewildering. In particular, it means that by 'observing' a computer print out of a million terms of a trajectory it may not be possible to predict whether we have an aperiodic sequence of numbers, or the first million terms of a periodic orbit of, say, two million periods. But the second part [*T.2*] of the Li-Yorke theorem enables us to challenge Samuelson's conjecture that we quoted above. There is an uncountable set of initial points such that the emanating trajectories will *not* converge to *any* periodic orbit, and these trajectories will 'approach' and 'turn away' from one another along different subsequences of time periods. For brevity, any dynamical system satisfying (*T.1*) and (*T.2*) is often called 'chaotic' (there are other definitions). Taking $X = [0, 1]$, $(X, \hat{F})$ is chaotic; so is $X, F^*)$ where F^* is a 'quadratic map' defined by:

$$F^*(x) = 4x(1-x) \tag{4.5}$$

5 Excess Demand Functions Once Again

The long digression of the last section was intended to emphasize the point that there are easily verifiable conditions to identify 'chaotic' behavior. A second point is that chaos may be present in quite simple non-linear (even piecewise linear) dynamical systems. Our task now is to combine the Li-Yorke results (Lemma 4.2 and Theorem 4.1) with the earlier propositions on excess demand functions. What emerges is the striking conclusion that even in a 'concrete' economy of two commodities and (by Debreu's theorem) two agents, the Walrasian adjustment process cast in discrete time may display chaotic behaviour.[4] We shall proceed somewhat heuristically to keep the computational details at a minimal level.

Let us define a function $\hat{G}$ from $X = [0, 1]$ into itself by the formula:

$$\hat{G}(x) = \begin{cases} 1.95x & \text{when } 0 \le x \le 1/2 \\ 1.95(1-x) & \text{when } 1/2 \le x \le 1 \end{cases} \tag{5.1}$$

We can easily see that $\hat{G}$ is a piecewise linear continuous function that attains its maximum at $x = 1/2$. The maximum value $\hat{G}(1/2)$ is 0.975. We can verify that $\hat{G}$ maps the compact subinterval

[4] We should mention an alternative approach followed by Day and Pianigiani (1991). Consider an exchange economy with two goods and two agents. Both have the same utility functions $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. The endowments are specified as (1, 0) and (0, 1) respectively. Let $\zeta(p)$ be the excess demand vector for the economy at $p >> 0$. If we consider $p_{t+1} = p_t + \alpha\, \zeta(p_t)$, we can, for a 'high' value of the speed of adjustment parameter α, verify the Li-Yorke conditions.

$X_1 = [0.02, 0.98]$ into itself. Now, going back to (3.3), we can define a price adjustment process

$$p_{t+1} = \hat{G}(p_t) \tag{5.2}$$

starting from some initial price in X. We can compute that if we set the speed of adjustment parameter $\alpha = 1$, the map

$$\hat{\varsigma}(p) = \begin{cases} 0.95p & \text{when } 0 \leq p \leq 1/2 \\ 1.95 - 2.95p & \text{when } 1/2 \leq p \leq 1 \end{cases} \tag{5.3}$$

satisfies

$$\hat{G}(p) \equiv p + \hat{\varsigma}(p) \qquad \text{for } p \text{ in } [0, 1] \tag{5.4}$$

Note that $\hat{\varsigma}(p)$ is positive for all positive p less than 1/2; and, for all p (in X) greater than $\bar{p} \simeq 0.661$, $\hat{\varsigma}(p)$ is negative. There is a unique positive $\bar{p}$ at which $\hat{\varsigma}(\bar{p})$ equals zero. Now, using Debreu's theorem, we can assert that there is a 'well behaved' Walrasian economy, whose excess demand function agrees with our $\hat{\varsigma}(p)$ on, say, $X_1 = [0.02, 0.98]$. Thus, for this economy

$$p_{t+1} = \hat{G}(p_t)$$

or,

$$p_{t+1} = p_t + \hat{\varsigma}(p_t) \tag{5.5}$$

provides an example of a tatonnement.

We should point out that there *are* points (other than $\bar{p}$ itself!) from which we can get to the equilibrium $\bar{p}$ in a finite number of steps by following (5.5). For example, if the initial $p_0' = \bar{p}/0.95$ then we can get to $\bar{p}$ in just one period. Then, there are surely points from which we can arrive at p_0' in one period, so that the equilibrium $\bar{p}$ is attained in two periods, and so on. A look at the graph of $\hat{G}$ [defined by (5.1)] is useful to see how these points are generated. But all such points belong to a 'small' countable set (hence, to a set of Lebesgue measure zero).

For this dynamical system $(X_1, \hat{G})$, we can select

$$a = 0.246;$$

hence, $b = \hat{G}(a) = 0.4797;$

and, $c = \hat{G}(b) = 0.935415;$

and, $d = \hat{G}(c) = 0.1259407.$

We can apply Theorem 2.1, and assert that (5.5) gives an example of a chaotic tatonnement. It should be stressed that the example is by no means a 'rare' or knife-edge phenomenon. Saari (1985) and Bala and Majumdar (1992) have investigated in detail the question of 'robustness' of chaos, and have demonstrated that models of chaotic tatonnement are 'non-negligible' in a precise sense. Since the arguments are quite technical we do not pursue this issue here.

The 'tent map' we use leads to particularly simple calculations. But there are more complicated functional forms which are better suited to reveal other types of complexities in dynamics. Again, taking $X = [0, 1]$ consider the map $G: X \to X$ defined as:

$$G(p) = 7.86\,p - 23.31\,p^2 + 28.75\,p^3 - 13.30\,p^4 \qquad (5.6)$$

For this dynamical system (X, G), G has a unique, positive fixed point $\bar{p}^* = 0.72$ [and this $\bar{p}^*$ is also 'locally stable': $|G'(\bar{p}^*)| = 0.89 < 1$]. But G also has a (locally stable) periodic point of period two, the trajectory (0.3217, 0.93, 0.3217, 0.93, . . .) from 0.3217 [as well as the trajectory from 0.93] is periodic.

Let us contrast a comparative static approach with a dynamic one with the help of the map G in (5.6). First, consider *any* economy (with two goods and two agents) such that $p_0 = 0.3217$ is a unique equilibrium price of the first good. As before (subject to the 'ε-qualification' of Theorem 2.1), we can think of the map $G(p)$ of (5.6) as an excess demand function for the first good, and let us view it as a 'new' excess demand function. Of course, if we are *just* interested in comparing the old equilibrium with the 'new' equilibrium ($\bar{p}^* = 0.72$ where $G(\bar{p}^*) = 0$) we can say that the 'equilibrium price will change from p_0 to $\bar{p}^*$ as a result of the shift to the new excess demand function G.' However, if we consider a dynamic process

$$p_{t+1} = G(p_t) \tag{5.7}$$

with $p_0 = 0.3217$, (and G given by (5.6)), we note that the trajectory from p_0 will be periodic (with a two period cycle). There will be no 'movement' towards the 'new' equilibrium price.

6 Concluding Comments

'Concrete' Walrasian models are now an integral part of the basic box of tools that economic theorists draw upon. Using such models to derive insights has been, and hopefully will continue to be, an attractive direction of research. Whether 'simple' models can provide a firm foundation for advocating policy measures has been the subject of lively debates and, we are sure, will continue to be a controversial methodological issue. As Frank Hahn (1983) observed: 'the notion of *simple* is not simple' and 'sometimes the uselessness of the general model is simply a frank statement of ignorance'.

Our purpose has been to use such models to get a glimpse of the complex dynamics and to appreciate the programme of research that started in the thirties and has continued ever since, a programme that aims at explaining the nature of dynamic processes that drive a market which is not in equilibrium. It is now clear that even the simplest model with two goods and two agents can generate 'robust' chaotic behavior. This, of course, raises questions about the predictive power of economic models. But, to us, 'it seems satisfactory that we should have detailed empirical knowledge before we can go into the prediction business. This circumstance in no way reduces the importance of theory' Hahn (1983).

Appendix

Proof of T.1 in Theorem 4.1

In addition to Lemma 4.1–4.2, we need

Lemma 4.3 Let J be an interval and $G: J \rightarrow R$ be continuous. Let I be a compact subinterval of J. Assume $I \subset G(I)$. Then there is some p in I such that $G(p) = p$.

Proof: Let $I = [\beta_0, B_1]$. Choose $\alpha_i (i = 0, 1)$ in I such that $G(\alpha_i) = \beta_i$. It follows that $\alpha_0 - G(\alpha_0) \geq 0$ and $\alpha_1 - G(\alpha_1) \leq 0$. So continuity of G implies that $G(x) - x$ must be zero for some $x = p$ in I. To complete the proof of [*T.1*], assume that $d \leq a < b < c$ as in the theorem. The proof for the case $d \geq a > b > c$ is similar. Write $K = [a, b]$ and $L = [b, c]$.

Now, let k be any positive integer. For $k > 1$, let $\{I_n\}$ be a sequence of intervals where $I_n = L$ for $n = 0, \ldots, k-2$; $I_{n-1} = K$ and define I_n to be periodic inductively: $I_{n+k} = I_n$ for $n = 0, 1, 2, \ldots$. If $k = 1$, let $I_n = L$ for all n. Let Q_n be the sets in Lemma 4.2. Notice that $Q_k \subset Q_0$ and $F^k(Q_k) = Q_0$. Apply Lemma 4.3 to the map $G \equiv F^k$ to get a fixed point p_k of F^k in Q_k. It is clear that p_k cannot have period less than k for F. Otherwise, we need to have $F^{k-1}(p_k) = b$, contrary to $F^{k+1}(p_k) \in L$. The point p_k is thus a periodic point of period k for F.

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