Some Results on the Transfer Problem in an Exchange Economy*

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1. Introduction

In this chapter we review some results on the 'transfer problem' in the context of a general equilibrium exchange model with m agents and n commodities. Consider, first, an 'initial' Walrasian equilibrium of such an economy given a particular distribution of endowments. Now let one agent (the 'donor') give away ('transfer') some of its endowment to another agent (the 'recipient'), or to some/all of the other agents, and look at a post-transfer equilibrium. Two questions arise naturally:

(a) How will the post-transfer equilibrium price compare to the pre-transfer one?
(b) How will the post-transfer equilibrium welfares of the agents compare to the pre-transfer levels?

The effect of a transfer studied in question (a) is called a 'positive effect'; that in question (b) a 'welfare effect'.

Both the above questions were posed by Samuelson (1952, 1954). Since then, a considerable body of literature has developed on the transfer problem. Our primary interest is in the 'welfare effect'. Our exposition is organized as follows. We begin with a brief summary of the literature. A Walrasian model of an exchange economy with many goods and agents is outlined in Section 2 [a masterly presentation of this model, along with results on the existence of a Walrasian equilibrium is in Nikaido (1956)]. Section 3 focuses on a two-agent

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economy. The main result asserts that the statement 'for every feasible transfer the donor is no better-off in the post-transfer equilibrium' is equivalent to the statement 'for every distribution of endowment, there is a unique equilibrium'. In Section 4, two examples are discussed in detail: the first demonstrates that the result in Section 2 cannot be generalized to a many-agent economy. The second, due to Leonard and Manning (1983), is used as an introduction to some of the recent literature. In Section 5 we turn to a many-agent economy. The main result [taken from Majumdar and Mitra (1985)] indicates that the donor is worse-off after the transfer if (a) all goods are 'net substitutes' for the donor, (b) all goods are 'gross substitutes' for all other agents and (c) all goods are 'normal' for all agents.

The comparative static results in Section 3 and 5 are 'global'; these are derived *without* any differentiability assumptions. This presentation is in sharp contrast with that in much of the literature. It has often been stressed [among others, by Morishima (1964, p. 3)] that attempts should be made to get 'global' comparative static results, since an 'infinitesimally small' change of a parameter is *not* what one has in mind when one thinks of a transfer problem. An informal account of the Samuelsonian comparative static analysis is sketched in Section 6.

The Transfer Problem of International Trade

A large part of the literature focuses on the two-country, two-commodity model of international trade theory. The main proposition that emerges from this model is that if the transfer is 'small', the donor cannot be better-off (and so the recipient cannot be worse-off) after the transfer, provided the Walrasian (World) equilibrium is 'locally' stable under the usual 'tatonnement' adjustment process. [See, for example, Section 6 and, for a more extended treatment, some standard texts in international trade theory, such as Caves and Jones (1981), or Dixit and Norman (1980)].

More recently, the literature has focused on the three-country, two-commodity model, as it is felt that this framework leads to significantly different results, compared to the two-country counterpart. In particular, Brecher and Bhagwati (1981) and Yano (1981) have noted, in the context of a three-country, two-commodity model, that it is possible for the donor to be better-off after the transfer, even under local stability of the Walrasian equilibrium.

Gale (1974) noted this phenomenon with a concrete algebraic example, and established that the Walrasian equilibrium in his example is unique. (It turns out in this two-good example that the equilibrium is also locally stable as the reader can easily verify.) Gales's example involved L-shaped indifference curves for the three agents; an algebraic example with the same content as Gale's, but involving smooth preferences for the three agents, was constructed by Aumann and Peleg (1974). This example, incidentally, shows a close relationship between the welfare effect of the transfer problem, and the welfare effect of the problem of 'immiserizing growth', which had been studied earlier quite extensively in international trade theory by Bhagwati (1958), Johnson (1967) and Bhagwati (1968). In a recent paper, Bhagwati, Brecher and Hatta (1982) have undertaken a thorough analysis of the three-country, two-commodity case integrating the transfer problem, the problem of immiserizing growth, and the theory of 'distortions'. (A rather complete history and bibliography of contributions to the transfer problem, immiserizing growth and distortions is contained in this paper, so we have deliberately avoided going into all of the details.) The Leonard-Manning example of Section 4 provides a useful introduction for the interested reader. Yet another review of the transfer and other 'paradoxes' in trade theory with an emphasis on the role of income effects is in Jones (1985).

2. The Nikaido Model of an Exchange Economy

We consider an exchange economy in which there are n *commodities*, indexed $j = 1, \ldots, n$, and m *agents*, indexed $i = 1, \ldots, m$. The *total endowment* of economy is given, and denoted by e (where e is in R^n_{++}).

Each agent has an endowment vector, the sum of all such endowment vectors being the total endowment of the economy. In our notation, the endowment of the ith agent is e^i in R^n_{++}, and

$$\sum_{i=1}^{m} e^i = e$$

The *distribution of endowments* is then denoted by $E = (e^1, e^2, \ldots, e^m)$.

The *preferences* of agent i are represented by a continuous *utility function*, $u^i: R^n_+ \to R$. For each i, we define $D^i = \{x \text{ in } R^n_+ : u^i(x) > u^i(0)\}$. For each i, the following assumptions on u^i are maintained:

(A.1) (monotonocity)

(a) $x \geq \bar{x} \geq 0$ implies $u^i(x) \geq u^i(\bar{x})$

(b) $x >> \bar{x} \geq 0$ implies $u^i(x) > u^i(\bar{x})$

(A.2) (concavity)

$\bar{x}$ in D^i, $x \neq \bar{x}$, $u^i(x) > u^i(\bar{x})$ implies
$u^i[\theta x + (1 - \theta)\bar{x}] > u^i(\bar{x})$ for $0 < \theta < 1$

An *allocation* is denoted by $X \equiv (x^1, \ldots, x^m)$ with x^i in R^n_+ for each i, and

(feasibility) $\sum_{i=1}^{m} x^i \leq e$

An *equilibrium* (given E) is a pair (X, p) such that

X is an allocation (2.1)

p is in R^n_+, $pe = 1$ (2.2)

For each i, x^i is an element of the set

$B^i = \{x \text{ in } R^n_+ : px \leq pe^i\}$ (2.3)

and $u^i(x^i) \geq u^i(x)$ for all x in B^i.

A few remarks on the definition of equilibrium are in order. Given the assumptions on u^i, condition (2.3) implies $px^i = pe^i$, so

$$p[e - \sum_{i=1}^{m} x^i] = 0 \tag{2.4}$$

Furthermore, by condition (2.3), p is in R^n_{++}. Finally, since X is an allocation, and p is in R^n_{++}, so using (2.4), one gets

$$\sum_{i=1}^{m} x^i = e \tag{2.5}$$

We are interested in examining how an equilibrium changes when one agent 'transfers' some of its endowment to another (or to several of the other agents). This exercise clearly belongs to the more general class of comparative static problems concerned with the effect of a change in the distribution of endowments on the set of equilibrium allocations. [See Mukherji (1990, Chapter 2.8).]

In describing 'transfers', we can suppose, without loss of generality, that agent 1 *pays the transfer* (i.e., it is the 'donor'), and some or all of the other agents ($i = 2, \ldots, m$) *receive the transfer*. Keeping this in mind, we can define a set (writing ($\bar{E} = (\bar{e}^1, \ldots, \bar{e}^m)$) :

$$T = \{(E, \bar{E}): E \text{ and } \bar{E} \text{ are } \textit{distributions} \text{ of endowments, } \bar{e}^1 < e^1, \bar{e}^i \geq e^i \text{ for } i = 2, \ldots, m\} \tag{2.6}$$

Thus, when we write $(E, \bar{E})$ is in T we mean that E is the distribution of endowments *before* the transfer, and $\bar{E}$ *after* the transfer.

Now, let $(E, \bar{E})$ be in T, (X, p) be an equilibrium given E, $(\bar{X}, \bar{p})$ be an equilibrium given $\bar{E}$. The problem we are concerned with is the following. How do the post-transfer welfare levels $[u^1(\bar{x}), u^2(\bar{x}), \ldots, u^m(\bar{x})]$ compare to the pre-transfer welfare levels $[u^1(x), u^2(x), \ldots, u^m(x)]$?

3. The Two-Agent Case

Some insight into the transfer problem can be gained by examining the case in which the economy consists of precisely two agents. In this case agent one is the 'donor' and agent two the 'recipient' of the transfer. Indeed, much of the 'international trade' literature on the transfer problem relates precisely to this case. Unlike this literature (which also restricts the number of commodities to two), we allow an arbitrary (finite) number of commodities.

We will demonstrate in this section that the statement, 'For every feasible transfer the donor country is no better-off after the transfer' is equivalent to the statement, 'For every distribution of endowments, there is at most one competitive equilibrium'.

Thus, whether or not the donor of a transfer is worse-off after the transfer depends crucially on the uniqueness of competitive equilibria in the economy. So, before proceeding to the results, we define formally the concept of uniqueness that is relevant for the purpose.

Condition U: Given E, if (X, p) and $(\bar{X}, \bar{p})$ are equilibria, then $X = \bar{X}$.

Proposition 3.1: Let $(E, \bar{E}) \in T$; let (X, p) be an equilibrium given E, and $(\bar{X}, \bar{p})$ be an equilibrium given $\bar{E}$. If Condition U holds, then

$$u^1(x^1) \geq u^1(\bar{x}^1) \text{ and } u^2(x^2) \leq u^2(\bar{x}^2) \tag{3.1}$$

Proof: Since, in our framework, an equilibrium is necessarily a Pareto-optimum, to prove (3.1), it suffices to show that $u^1(x^1) \geq u^1(\bar{x}^1)$. Suppose, on the contrary, that

$$u^1(\bar{x}^1) > u^1(x^1) \tag{3.2}$$

It follows from (3.2) that:

$$px^1 < p\bar{x}^1 \tag{3.3}$$

Note also that:

$$px^1 = pe^1 > p\bar{e}^1 \tag{3.4}$$

Then (3.3) and (3.4) imply that there is $0 < \lambda < 1$, such that

$$p^1 = p[\lambda\, \bar{x}^1 + (1 - \lambda)\bar{e}^1] \tag{3.5}$$

Denote $[\lambda\, \bar{x}^1 + (1 - \lambda)\bar{e}^1]$ by $\tilde{x}^1$; define $\tilde{x}^2 = e - \tilde{x}^1$. Clearly $0 << \tilde{x}^1 << e$, so $0 << \tilde{x}^2 << e$, and $\tilde{x}^1 + \tilde{x}^2 = e$. Define $\tilde{E} = (\tilde{x}^1, \tilde{x}^2)$. Then $\tilde{E}$ is a distribution of endowments.

Now, $\overline{p}\tilde{x}^1 = \lambda\overline{p}\overline{x}^1 + (1 - \lambda)\overline{p}\overline{e}^1 = p\tilde{e}^1$; also $p\tilde{x}^2 = \overline{pe} - \overline{px}^1 = \overline{pe}^2$. Thus, $(\overline{X}, \overline{p})$ is an equilibrium given $\tilde{E}$.

Also, by (3.5), $p\tilde{x}^1 = px^1 = pe^1$; and $p\tilde{x}^2 = pe - p\tilde{x}^1 = pe - pe^1 = pe^2$. Thus, (X, p) is an equilibrium given $\tilde{E}$. Since $X \neq \overline{X}$ by (3.3), so Condition U is violated, a contradiction. This proves that $u^1(x^1) \geq u^1(\bar{x}^1)$.

The idea of the proof of Proposition 3.1 is simple. It can be depicted (for the case of two commodities) in the familiar Edgeworth box diagram (see Figure 1).

In Figure 1, the endowment pattern at the point $\tilde{E}$ is associated with two *distinct* equilibrium allocations X and $\overline{X}$. This violates (U). Observe that the move from E and $\overline{E}$ reflects a transfer from agent 1 to agent 2. Since E can be conceived of as an endowment pattern consistent with the 'pre-transfer' equilibrium allocation X (at prices given by the line connecting E to X), and $\overline{E}$ can be viewed as the 'post-transfer' endowment pattern leading to $\overline{X}$, the 'paradoxical' result appears.

Proposition 3.2: Suppose Condition U is violated; then there are $(E, \overline{E}) \in T$, an equilibrium (X, p) given E, an equilibrium $(\overline{X}, \overline{p})$ given $\overline{E}$, such that

$$u^1(x^1) < u^1(\bar{x}^1) \text{ and } u^2(x^2) > u^2(\bar{x}^2) \tag{3.6}$$

Proof: If Condition U is violated, then there is $\tilde{E}$, with two equilibria $(\hat{X}, \hat{p})$ and $(\overline{\overline{X}}, \overline{\overline{p}})$, given $\tilde{E}$, such that $\hat{X} \neq \overline{\overline{X}}$. Then $\hat{x}^1 \neq \overline{\overline{x}}^1$ and $\hat{x}^2 \neq \overline{\overline{x}}^2$. Clearly $u^1(\hat{x}^1) \neq u^1(\overline{\overline{x}}^1)$. For, if $u^1(\hat{x}^1) = u^1(\overline{\overline{x}}^1)$, then $u^2(\hat{x}^2) = u^2(\overline{\overline{x}}^2)$, since in our framework equilibria are Pareto-optimal. So

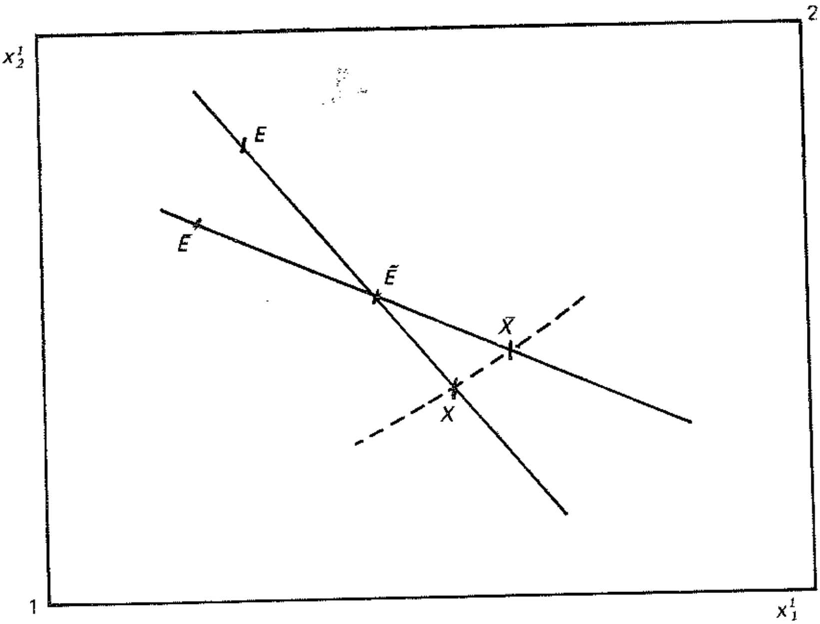

Figure 1

$$u^1\left(\frac{1}{2}\hat{x}^1 + \frac{1}{2}\bar{\bar{x}}^1\right) > u^1(\hat{x}^1) \text{ and}$$

$$u^2\left(\frac{1}{2}\hat{x}^2 + \frac{1}{2}\bar{\bar{x}}^1\right) > u^2(\hat{x}^2) \tag{3.7}$$

Also,

$$\left[\frac{1}{2}\hat{x}^1 + \frac{1}{2}\bar{\bar{x}}^1\right] + \left[\frac{1}{2}\hat{x}^2 + \frac{1}{2}\bar{\bar{x}}^2\right] = \frac{1}{2}\left[\hat{x}^1 + \hat{x}^2\right] + \frac{1}{2}\left[\bar{\bar{x}}^1 + \bar{\bar{x}}^2\right]$$
$$= \frac{1}{2}e + \frac{1}{2}e = e.$$

This means $\hat{x}$ is not Pareto-optimal, and hence not an equilibrium, a contradiction.

There are then two possibilities to consider: (i) $u^1(\hat{x}^1) < u^1(\bar{\bar{x}}^1)$ and $u^2(\hat{x}^2) > u^2(\bar{\bar{x}}^2)$; (ii) $u^1(\hat{x}^1) > u^1(\bar{\bar{x}}^1)$ and $u^2(\hat{x}^2) < u^2(\bar{\bar{x}}^2)$. We will consider each case in turn.

Case (i) :
Note that $\hat{p} \neq \bar{\bar{p}}$. For if $\hat{p} = \bar{\bar{p}}$, then

$$\hat{p}\bar{\bar{x}}^1 = \bar{\bar{p}}\,\bar{\bar{x}}^1 = \bar{\bar{p}}\tilde{e}^1 = \hat{p}\tilde{e}^1 = \hat{p}\hat{x}^1 \tag{3.8}$$

But $u^1(\hat{x}^1) < u^1(\bar{\bar{x}}^1)$ implies $\hat{p}\bar{\bar{x}}^1 > \hat{p}\hat{x}^1$, a contradiction to (3.8). Thus $\hat{p} \neq \bar{\bar{p}}$. Since $\hat{p}e = \bar{\bar{p}}e = 1$ [see (2.2)], there exist i, j such that

$$\hat{p}_i > \bar{\bar{p}}_i \text{ and } \hat{p}_j < \bar{\bar{p}}_j \tag{3.9}$$

Choose $e_j - \tilde{e}_j^1 > \hat{\varepsilon}_j > 0$ such that $\hat{e}_i \equiv (\hat{p}_j/\hat{p}_i)\hat{\varepsilon}_j < \tilde{e}_i^1$; define $\bar{\bar{\varepsilon}}_i = (\bar{\bar{p}}_j/\bar{\bar{p}}_i)\hat{\varepsilon}_j$; then $\hat{\varepsilon}_i < \bar{\bar{\varepsilon}}_i < \tilde{e}_i^1$. Now, define e^1 as follows: $e_i^1 = \tilde{e}_i^1 - \hat{\varepsilon}_i$, $e_j^1 = \tilde{e}_j^1 + \hat{\varepsilon}_j$, $e_k^1 = \tilde{e}_k^1$ for $k \neq i, j$; define $e^2 = e - e^1$. Then, clearly $0 << e^1 << e$ and $0 << e^2 << e$. Also, $\hat{p}e^1 = \hat{p}\tilde{e}^1 - \hat{p}_i\hat{\varepsilon}_i + \hat{p}_j\hat{\varepsilon}_i + \hat{p}_j\hat{\varepsilon}_j = \hat{p}\tilde{e}^1$; so $\hat{p}e^2 = \hat{p}e - \hat{p}e^1 = \hat{p}e - \hat{p}\tilde{e}^2$. Thus $(\hat{X}, \hat{p})$ is an equilibrium, given $E \equiv (e^1, e^2)$.

Define $\bar{e}^1$ as follows: $\bar{e}_i^1 = \tilde{e}_i^1 - \bar{\bar{\varepsilon}}_i$, $\bar{e}_j^1 = \tilde{e}_j^1 + \hat{\varepsilon}_j$, $\bar{e}_k^1 = \tilde{e}_k^1$ for $k \neq i, j$; define $\bar{e}^2 = e - \bar{e}^1$. Then $0 << \bar{e}^1 << e$ and $0 << \bar{e}^2 << e$. Also $\bar{\bar{p}}\,\bar{e}^1 = \bar{\bar{p}}\tilde{e}^1 - \bar{\bar{p}}_i\bar{\bar{\varepsilon}}_2 + \bar{\bar{p}}_j\hat{\varepsilon}_j = \bar{\bar{p}}\bar{e}^2 = \bar{\bar{p}}e - \bar{\bar{p}}\,\bar{e}^1 = \bar{\bar{p}} - \bar{\bar{p}}\tilde{e}^2$. Thus $(\bar{\bar{x}}, \bar{\bar{p}})$ is an equilibrium given $\bar{E} = (\bar{e}^1, \bar{e}^2)$.

Define $(X, p) = (\hat{X}, \hat{p})$ and $(X, \bar{p}) = (\bar{\bar{X}}, \bar{\bar{p}})$. Then (X, p) is an eqilibrium given E, $(\bar{X}, \bar{p})$ is an equilibrium given $\bar{E}$. Also $\hat{\varepsilon}_i < \bar{\bar{\varepsilon}}_i$ implies that $\bar{e}^1 < e^1$ and so $\bar{e}^2 > e^2$. Hence, $(E, \bar{E}) \in T$. Since $u^1(x^1) = u^1(\hat{x}^1) < u^1(\bar{\bar{x}}^1) = u^1(\bar{x}^1)$ and $u^2(x^2) = u^2(\hat{x}^2) > u^2(\bar{\bar{x}}^2) = u(\bar{x}^2)$, the proposition is established.

Case (ii):
Note, again, that $\hat{p} \neq \bar{\bar{p}}$. For if $\hat{p} \neq \bar{\bar{p}}$,

$$\bar{\bar{p}}\hat{x}^1 = \hat{p}\hat{x}^1 = \hat{p}\tilde{e}^1 = \bar{\bar{p}}\tilde{e}^1 = \bar{\bar{p}}\bar{\bar{x}}^1 \tag{3.10}$$

But $u^1(\hat{x}^1) > u^1(\bar{\bar{x}}^1)$ implies $\bar{\bar{p}}\hat{x}^1 > \bar{\bar{p}}\bar{\bar{x}}^1$, a contradiction to (3.10). Thus, $\hat{p} \neq \bar{p}$. Since $\hat{p}e = \bar{p}e$, there exist i, j such that

$$\bar{\bar{p}}_i > \hat{p}_i \text{ and } \bar{\bar{p}}_j < \hat{p}_j \tag{3.11}$$

Choose $e_j - \tilde{e}_j^1 > \bar{\bar{\varepsilon}}_j > 0$ such that $\hat{\varepsilon}_j \equiv (\hat{p}_j/\hat{p}_j)\bar{\bar{\varepsilon}}_j < \tilde{e}_i^1$; define $\bar{\bar{\varepsilon}}_i = (\bar{\bar{p}}_j/\bar{\bar{p}}_i)\bar{\bar{\varepsilon}}_j$; then $\bar{\bar{\varepsilon}}_i < \hat{\varepsilon}_i < \tilde{e}_i^1$. Now, define e^1 as follows: $e_i^1 = \tilde{e}_i^1 - \bar{\bar{\varepsilon}}_i$, $e_j^1 = \tilde{e}_j^1 + \bar{\bar{\varepsilon}}_j$, $e_k^1 = \tilde{e}_k^1$ for $k \neq i, j$; write $e^2 = e - e^1$. Then clearly $0 << e^1 << e$, and $0 << e^2 << e$. Also, $\bar{\bar{p}}e^1 = \bar{\bar{p}}\tilde{e}^1 - \bar{\bar{p}}_i\bar{\bar{\varepsilon}}_j = \bar{\bar{p}}\tilde{e}^1$; so $\bar{\bar{p}}e^2 = \bar{\bar{p}}e - \bar{\bar{p}}e^1 = \bar{\bar{p}} - \bar{\bar{p}}\tilde{e}^1 = \bar{\bar{p}}\tilde{e}^2$. Thus $(\bar{\bar{X}}, \bar{\bar{p}})$ is an equilibrium given $E \equiv (e^1, e^2)$.

Define $\bar{e}^1$ as follows: $\bar{e}_i^1 = \tilde{e}_i^1 - \hat{\varepsilon}_i$, $\bar{e}_j^1 = \tilde{e}_j^1 + \bar{\bar{\varepsilon}}_j$, $\bar{e}_k^1 = \tilde{e}_k^1$ for $k \neq i, j$; write $\bar{e}^2 = e - \bar{e}^1$. Then $0 << \bar{e}^1 << e$ and $0 << \bar{e}^2 << e$. Also, $\hat{p}\bar{e}^1 = \hat{p}\tilde{e}^1 - \hat{p}_i\hat{\varepsilon}_i + \hat{p}_j\bar{\bar{\varepsilon}}_j = \hat{p}\tilde{e}^1$; so $\hat{p}\bar{e}^2 = \hat{p}e - \hat{p}\tilde{e}^1 = \hat{p}\tilde{e}^2$. Thus, $(\hat{X}, \hat{p})$ is an equilibrium given $\bar{E} = (\bar{e}^1, \bar{e}^2)$.

Define $(X, p) \equiv (\bar{\bar{X}}, \bar{\bar{p}})$ and $(\bar{X}, \bar{p}) \equiv (\hat{X}, \hat{p})$. Then (X, p) is an equilibrium given E and $(\bar{X}, \bar{p})$ is an equilibrium given $\bar{E}$. Also, $\hat{\varepsilon}_i > \bar{\bar{\varepsilon}}_i$ implies that $e^1 > \bar{e}^1$ and so $e^2 < \bar{e}^2$. Hence, $(E, \bar{E}) \in T$. Since $u^1(x^1) = u^1(\bar{\bar{x}}^1) < u^1(\hat{x}^1) = u^1(\bar{x}^1)$ and $u^2(x^2) = u^2(\bar{\bar{x}}^2) < u^2(\hat{x}^2) = u^2(\bar{x}^2)$, the proposition is established.

Even though there are tedious calculations involved in the proof of Proposition 3.2, the idea is essentially simple, and 'case (i)' in the proof can be illustrated as in Figure 2.

4. THE THREE-AGENT CASE: SOME EXAMPLES

The recent revival of interest in the transfer problem of international trade has its focus on the three-country, two-good case. The point that is being made in this literature is that in the three-country setting, the

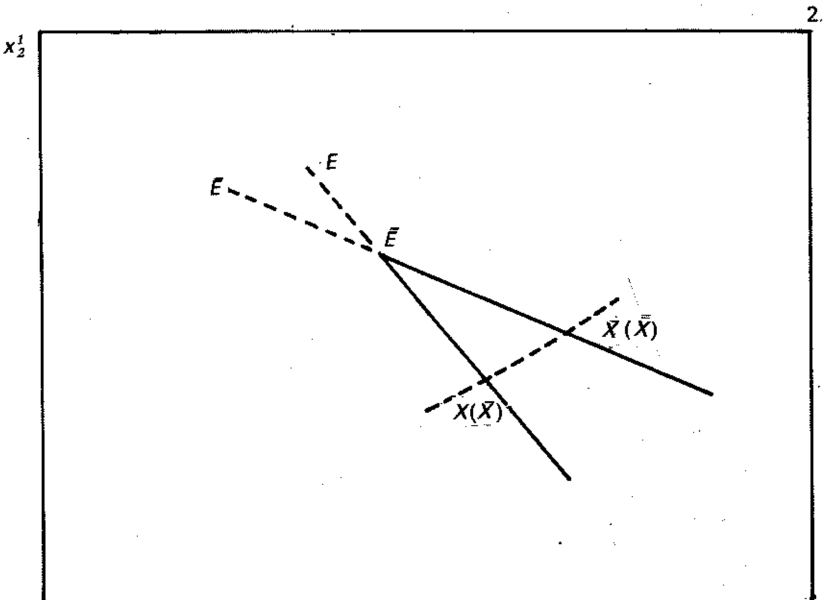

Figure 2

'donor' can improve its welfare after a 'small' transfer, even though the equilibrium is locally stable [see, e.g. Yano (1981) for details].

From the point of view of Section 3, the relevant question to be asked is somewhat different, since we do not restrict transfers to be 'small'. Specifically, the question to be asked is the following: 'Is it possible to construct an example of a three-agent, two-good economy, in which the donor can improve its welfare after a transfer, even though there is at most one equilibrium for every distribution of endowments?'

In Example 4.1, we follow essentially the Aumann-Peleg example (1974), modified suitably to make it easy to answer the above question in the affirmative. The lesson of the example is clear: the results of Section 3 will *not* generalize to the many-agent case. Additional assumptions are clearly needed to prove any definitive result in the many-agent setting.

Example 4.1

Let $e = (3, 1)$; $e^1 = (2, 0)$, $e^2 = (0, 1)$, $e^3 = (1, 0)$; $\bar{e}^1 = (1, 0)$, $\bar{e}^2 = (0, 1)$, $\bar{e}^3 = (2, 0)$. Denote (e^1, e^2, e^3) by E; $(\bar{e}^1, \bar{e}^2, \bar{e}^3)$ by $\bar{E}$. Observe that $\sum_i e^i = \sum_i \bar{e}^i = e$. The assumption $e^i \in R^n_{++}$ made earlier in Section 2 to sidestep the awkward problem of non-existence of a Walrasian equilibrium is *not* satisfied here. We shall show by direct computation that a Walrasian equilibrium exists.

Let $u^1(x_{\frac{1}{2}}, x_2) = [x_{\bar{2}}^{-2} + x_2^{-2}]^{-1/2}$ for $(x_2, x_2) >> 0$; $u^1(x_2, x_2) = 0$ for $x_2 = 0$ or $x_2 = 0$. Let $u^2(x_2, x_2) = u^1(x_2, x_2)$; $u^3(x_2, x_2) = x_2 + \ln(1 + x_2)$.

Define $x^1 = (2/5, 1/5)$, $x^2 = (8/5, 4/5)$, $x^3 = (1, 0)$; $p_2 = 1/11$, $p_2 = 8/11$; $X = (x^1, x^2, x^3)$; $p = (p_2, p_2)$. Then it can be checked that (X, p) is an equilibrium given E.

First, note that $pe = 1$. Denote p_2/p_2 by r. Then $r = 1/8$. Note that X is clearly an allocation. Also, at X,

$$\frac{\partial u^1(x^1)}{\partial x_1} \bigg/ \frac{\partial u^1(x^1)}{\partial x_2} = \left(\frac{x_2^1}{x_1^1}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8},$$

$$\frac{\partial u^2(x^2)}{\partial x_1} \bigg/ \frac{\partial u^2(x^2)}{\partial x_2} = \left(\frac{x_2^2}{x_1^2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Also, $pe^1 = 2/11$; $px^1 = \frac{1}{11}\left(\frac{2}{5}\right) + \frac{8}{11}\left(\frac{1}{5}\right) = \frac{10}{55} = \frac{2}{11}$ so x^1 is in B^1.

Similarly, $pe^2 = 8/11$; $px^2 = \frac{1}{11}\left(\frac{8}{5}\right) + \frac{8}{11}\left(\frac{4}{5}\right) = \frac{40}{55} = \frac{8}{11}$ so x^2 is in B^2;

$pe^3 = 1/11; px^3 = 1/11$ so x^3 is in B^3.

Since u^1 is quasi-concave, so $u^1(x^1) \geq u^1(x)$ for all x in B^1. Similarly, u^2 is quasi-concave, so $u^2(x^2) \geq u^2(x)$ for all x in B^2. Finally, for any x in B^3, $(1/11)x_2 + (8/11)x_2 \leq 1/11$. So $u^3(x_2, x_2) - u^3(x_2^3, x_2^3) = [x_2 + \ln(1 + x_2)] - 1 = (x_2 - 1) + \ln(1 + x_2) \leq [1 - 8x_2 - 1] + \ln(1 + x_2) = \ln(1 + x_2) - 8x_2 \leq 0$. Thus (X, p) is an equilibrium, given E.

Next, define $\bar{x}^1 = (1/2, 1/2)$, $\bar{x}^2 = (1/2, 1/2)$, $\bar{x}^3 = (2, 0)$; $\bar{p}_1 = 1/4$, $\bar{p}_2 = 1/4$; $\bar{X} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$; $\bar{p} = (\bar{p}_1, \bar{p}_2)$. Then it can be checked that $(\bar{X}, \bar{p})$ is an equilibrium given $\bar{E}$.

First, note that $\overline{pe} = 1$. Denote $(\bar{p}_1/\bar{p}_2)$ by $\bar{r}$; then $\bar{r} = 1$. Note that $\bar{X}$ is clearly an allocation.

Also, $\overline{px}^1 = 1/8 + 1/8 = 1/4 = \overline{pe}^1$; $\overline{px}^2 = 1/8 + 18 = 1/4 = \overline{pe}^2$; $\overline{px}^3 = 2/4 = \overline{pe}^3$.

So $\bar{x}^i \in B^i$ for $i = 1,2,3$. Also at $\bar{X}$,

$$\frac{\partial u^1(\bar{x}^1)}{\partial x_1} \Big/ \frac{\partial u^1(\bar{x}^1)}{\partial x_2} = \left(\frac{\bar{x}_2^1}{\bar{x}_1^2}\right)^3 = 1$$

$$\frac{\partial u^2(\bar{x}^1)}{\partial x_1} \Big/ \frac{\partial u^2(\bar{x}^1)}{\partial x_2} = \left(\frac{\bar{x}_2^2}{\bar{x}_1^2}\right)^3 = 1$$

Since u^1 is quasi-concave, $u^1(\bar{x}^1) \geq u^1(x)$ for all x in B^1. Similarly, since u^2 is quasi-concave, $u^2(\bar{x}^2) \geq u^2(x)$ for all x in B^2. Finally, for any x in B^3, $(1/4)\, x_2 \leq 1/2$. So $u^3(x_2, x_2) - u^3(\bar{x}_1^3, \bar{x}_2^3) = [x_2 + \ln(1 + x_2)] - 2 = [x_1 - 2] + \ln(1 + x_2) \leq [2 - x_2 - 2] + \ln(1 + x_2) = \ln(1 + x_2) - x_2 \leq 0$. Thus, $(\bar{X}, \bar{p})$ is an equilibrium, given $\bar{E}$.

Finally, note that

$$u^1(\bar{x}^1) > u^1(x^1)$$

$$u^2(\bar{x}^2) < u^2(x^2)$$

$$u^3(\bar{x}^3) > u^3(x^3)$$

Thus, agent 1, the donor, increases its welfare by transferring (1, 0) to agent 3. Agent 3, the recipient, also increases its welfare. Agent 2, which is neither the donor nor a recipient finds its welfare reduced after the transfer takes place!

The next step is to show that given the total endowment of the economy, $e = (3, 1)$, and the utility functions of the three agents, there is at most one equilibrium for every distribution of endowments. (By this, we will mean $e^1 > 0$, $e^2 > 0$, $e^3 > 0$, such that $e^1 + e^2 + e^3 = e$, since if $e^i = 0$ for some i in $\{1, 2, 3\}$, we really have a two-country model). We know that if (X, p) is an equilibrium given some E, $p >> 0$, $x^1 >> 0$ and $x^2 >> 0$.

For any $p >> 0$, defining $r = (p_1/p_2)$, we know that $(x_2^1/x_1^1)^3 = r$, and $re_1^1 + e_2^1 = rx_1^1 + rx_2^1$. Also, $(x_2^2/x_1^2)^3 = r$, and $re_2^2 = rx_1^2 + x_2^2$. So $re_1^1 + e_2^1 = rx_1^1 + r^{1/3}x_1^1 = (r + r^{1/3})x_1^1$. Hence $x_1^1 = [re_1^1 + e_2^1]/[r + r^{1/3}]$ and $x_2^1 = r^{1/3}[re_1^1 + e_2^1]/[r + r^{1/3}]$. Similarly, $x_1^2 = [re_1^2 + e_2^2]/[r + r^{1/3}]$; $x_2^2 = r^{1/3}[re_1^2 + e_2^2]/[r + r^{1/3}]$.

The derivation of the demand function for agent 3 requires somewhat more detailed analysis, depending on the pattern of its endowment. We distinguish two cases for ease of exposition: (i) $e_2^3 < 1$; (ii) $e_1^3 \geq 1$.

Case (i) $[e_1^3 < 1]$:

Note that if $x_1^3 = 0$ at a price ratio r, then by the budget constraint, $x_2^3 = re_1^3 + e_2^3$. For this point to be a demand point, we must have

$$r \geq \frac{u_1^3}{u_2^3} = 1 + x_2^3 = 1 + re_1^3 + e_2^3$$

so

$$r \geq (1 + e_2^3)/(1 - e_1^3) > 1.$$

From this it certainly follows that for the range $r \leq 1$, $x_1^3 > 0$. If x_1^3 is also positive, then $r = u_1^3/u_2^3 = 1 + x_2^3 > 1$, a contradiction. So for $r \leq 1$, $x_2^3 = 0$.

Next, for $1 < r < (1 + e_2^3)/(1 - e_1^3)$, we can also conclude that $x_1^3 > 0$. If x_2^3 is zero, then for this to be a demand point

$$r \leq \frac{u_1^3}{u_2^3} = 1 + x_2^3 = 1, \text{ a contradiction.}$$

So x_2^3 is also positive, and $r = u_1^3/u_2^3 = (1 + x_2^3)$.

So, for $1 < r < (1 + e_2^3)/(1 - e_2^3)$, $x_2^3 = (r - 1)$.

Finally, for $r \geq (1 + e_2^3)/(1 - e_1^3)$, note that x_2^3 is positive by the immediately preceding argument. If x_1^3 is also positive, then $r = u_1^3/u_2^3 = 1 + x_2^3$, so $x_2^3 = (r - 1)$. But $rx_1^3 + x_2^3 = re_1^3 + e_2^3$ implies that $x_2^3 < re_1^3 + e_2^3$. Hence $(r - 1) < re_1^3 + e_2^3$, and so $r < (1 + e_2^3)/(1 - e_1^3)$ a contradiction. Hence, for $r \geq (1 + e_2^3)/(1 - e_1^3)$, $x_1^3 = 0$, and by the budget constraint, $x_2^3 = re_1^3 + e_2^3$.

We can, therefore, summarize the demand information as follows:

$$\begin{aligned} x_2^3 &= 0 && \text{for } r \leq 1 \\ &= (r - 1) && \text{for } 1 \leq r \leq (1 + e_2^3)/(1 - e_1^3) \end{aligned}$$

$$= re_1^3 + e_2^3 \qquad \text{for } r \geq (1 + e_2^3)/(1 - e_1^3)$$

Case (ii) ($e_1^3 \geq 1$):
Note that if $x_1^3 = 0$ at a price-ratio, r, then by the budget constraint, $x_2^3 = re_1^3 + e_2^3$. For this point to be a demand point, we must have

$$r \geq \frac{u_1^3}{u_2^3} = 1 + x_2^3 = 1 + re_1^3 + e_2^3 \geq 1 + r$$

since $e_1^3 \geq 1$. This contradiction means that x_1^3 is always positive.

Turning to x_2^3, we note that if at a price ratio, r, $x_2^3 = 0$, then

$$r \leq \frac{u_1^3}{u_2^3} = 1 + x_2^3 = 1$$

so $r \leq 1$. On the other hand, if $x_2^3 > 0$, then

$$r = \frac{u_1^3}{u_2^3} = 1 + x_2^3 > 1.$$

Thus, we can summarize the demand information as follows:

$$x_2^3 = 0 \qquad \text{for } r < 1$$

$$= (r - 1) \qquad \text{for } r \geq 1$$

This completes our analysis of the third country's demand.

We now turn to the aggregate demand for commodity 2. Denote by $Z(r)$ the total demand for good 2 of the three agents at the price ratio, r. If r is an equilibrium price ratio, then $Z(r) = 1$, and $r > 0$.

For this analysis, too, we find it convenient to distinguish between the aforementioned two cases of the third agent's endowment holdings.

Case (i) ($e_1^3 < 1$):
First note that if r is an equilibrium price-ratio, then $r < 1$. For, if $r \geq 1$, then $Z(r) = 1$ yields

$$(1 + r^{2/3}) = r[e_1^1 + e_1^2] + [e_2^1 + e_2^2] + [1 + r^{2/3}]x_2^3$$

$$\therefore \quad (1 + r^{2/3})[1 - x_2^3] = r[e_1^1 + e_1^2] + [e_2^1 + e_2^2].$$

Now, $x_2^3 \geq 0$, so the left-hand side is less than or equal to $(1 + r^{2/3})$. Also $e_1^3 < 1$, so $e_1^1 + e_1^2 > 2$ and the right-hand side is greater than $2r$. That is, $(1 + r^{2/3}) > 2r$. Since $r \geq 1$, so $r^{2/3} \leq r$, and we finally have $1 + r > 2r$, which yields $r < 1$, a contradiction.

This simplifies matters greatly, for we now know that at an equilibrium price ratio, r, we must have $r < 1$, and furthermore

$$(1 + r^{2/3}) = r[e_1^1 + e_1^2] + [e_2^1 + e_2^2]$$

Denote $[e_1^1 + e_1^2]$ by d_1, and $[e_2^1 + e_2^2]$ by d_2. For $0 \leq r \leq 1$, let $f(r) = 1 + r^{2/3}$, and $g(r) = rd_1 + d_2$. Then, there is at most one solution to $f(r) = g(r)$, satisfying $0 < r < 1$. Suppose, on the contrary, there were two, call them r^* and r^{**}. Without loss of generality, suppose $0 < r^* < r^{**} < 1$. So there is $0 < \theta < 1$, such that $\theta r^{**} = r^*$. Note that f is strictly concave on [0,1].

So, $g[r^*] = f[\theta r^{**}] = f[\theta r^{**} + (1-\theta)\cdot 0] > \theta f(r^{**}) + (1-\theta)f(0)$

and $f[r^*] = g[\theta r^{**}] = d_1(\theta r^{**}) + d_2 = \theta d_1 r^{**} + \theta d_2 + (1-\theta)d_2$

$$= \theta g(r^{**}) + (1-\theta)d_2.$$

Since $f(r^*) = g(r^*)$, so we have

$$\theta f(r^{**}) + (1-\theta) = \theta f(r^{**}) + (1-\theta)f(0) < \theta g(r^{**}) + (1-\theta)d_2$$

and $$f(r^{**}) < g(r^{**}) + [(1-\theta)/\theta][d_2 - 1]$$

$$\leq g(r^{**}) = f(r^{**}), \text{ a contradiction.}$$

Case (ii) $(e_1^3 \geq 1)$:
If r is an equilibrium price ratio, then

$$r[e_1^1 + e_1^2] + [e_2^1 + e_2^2] = [1 + r^{2/3}] \text{ if } r \leq 1$$

and $r[e_1^1 + e_1^2] + [e_2^1 + e_2^2] = [1 + r^{2/3}][1 - (r-1)]$ if $r \geq 1$. Clearly, if r is an equilibrium price ratio, $0 < r < 2$. Denote $(e_1^1 + e_1^2)$ by d_1 and $(e_2^1 + e_2^2)$ by d_2. Also, let $g(r) = rd_1 + d_2$; let $h(r) = (1 + r^{2/3})$ for $0 \leq r \leq 1$ and $h(r) = (1 + r^{2/3})(2 - r)$ for $1 \leq r \leq 2$.

It can be checked that $h(0) = 1$, $h(1) = 2$, $h(2) = 0$; $h'^-(1) = 2/3$, $h'^+(1) = (-4/3)$; for $0 < r < 1$, $h''(r) < 0$, and for $1 < r < 2$, $h''(r) < 0$. Also, $g(0) = d_2 \leq 1$; $g'(r) = d_1 \leq 2$; $g''(r) = 0$.

We can now show that h is *strictly concave* on [0, 2]. To see this, pick r, r' in [0, 2], $r < r'$. If r, r' are both in [0, 1], then for every $0 < \lambda < 1$, we must have $h(\lambda r + (1-\lambda)r') > \lambda h(r) + (1-\lambda)h(r')$. Similarly, if r, r' are both in [1, 2], then for every $0 < \lambda < 1$, we must have $h(\lambda r + (1-\lambda)r') > \lambda h(r) + (1-\lambda)h(r')$. These follow easily from the observation that $h''(r) < 0$ for $0 < r < 1$, and $h''(r) < 0$ for $1 < r < 2$.

We have, therefore, to check only the case in which $r < 1 < r'$. Pick any λ in (0, 1), and let $r'' = \lambda r + (1-\lambda)r'$. There are then three cases to consider: (1) $r'' > 1$; (2) $r'' < 1$; (3) $r'' = 1$.

Case (1) $(r'' > 1)$: In this case, there is $0 < \theta < 1$, such that

$r'' = \theta + (1-\theta)r'$. So, $\lambda + (1-\lambda)r' > \lambda r + (1-\lambda)r' = r'' = \theta + (1-\theta)r'$. So $(\lambda - \theta) > (\lambda - \theta)r'$, and $(\lambda - \theta) < 0$ (since $r' > 1$). So $\theta > \lambda$. Now, we simply note the following sequence of inequalities.

$$
\begin{aligned}
h(r'') &= h[\theta + (1-\theta)r'] \\
&> \theta h(1) + (1-\theta)h(r') \text{ [since } h'' < 0 \text{ on } (1,2)] \\
&= (\theta - \lambda)h(1) + \lambda h(1) + (1-\theta)h(r') \\
&= (\theta - \lambda)h(1) + \lambda h(1) + (1-\lambda)h(r') + (\lambda - \theta)h(r') \\
&= (\theta - \lambda)[h(1) - h(r')] + \lambda h(1) + (1-\lambda)h(r') \\
&> \lambda h(1) + (1-\lambda)h(r') \text{ [since } h(1) > h(r') \text{ and } \theta > \lambda] \\
&> \lambda h(r) + (1-\lambda)h(r') \text{ [since } h(1) > h(r)].
\end{aligned}
$$

Case (2) $[r'' < 1]$: In this case, there is $0 < \theta < 1$, such that $r'' = \theta r + (1-\theta)$. So, $\lambda r + (1-\lambda) < \lambda r + (1-\lambda)r' = r'' = \theta r + (1-\theta)$. So $(\lambda - \theta)r < (\lambda - \theta)$, and $(\lambda - \theta) > 0$ (since $r < 1$). So $\lambda > \theta$. Now, note the following sequence of inequalities.

$$
\begin{aligned}
h(r'') &= h[\theta r + (1-\theta)] \\
&> \theta h(r) + (1-\theta)h(1) \text{ [since } h'' < 0 \text{ on } (0,1)] \\
&= (\theta - \lambda)h(r) + \lambda h(r) + (1-\theta)h(1) \\
&= (\theta - \lambda)h(r) + \lambda h(r) + (1-\lambda)h(1) + (\lambda - \theta)h(1) \\
&= (\lambda - \theta)[h(1) - h(r)] + \lambda h(r) + (1-\lambda)h(1) \\
&> \lambda h(r) + (1-\lambda)h(1) \text{ [since } h(1) > h(r) \text{ and } \lambda > \theta] \\
&> \lambda h(r) + (1-\lambda)h(r') \text{ [since } h(1) > h(r')]
\end{aligned}
$$

Case 3 $(r'' = 1)$: Here

$$
\begin{aligned}
h(r'') &= h(1) = \lambda h(1) + (1-\lambda)h(1) \\
&> \lambda h(r) + (1-\lambda)h(r')
\end{aligned}
$$

[Since $h(1) > h(r)$, and $h(1) > h(r')$].

Note that if r is an equilibrium price-ratio, then $0 < r < 2$, and

$$g(r) = h(r)$$

We claim now that there is at most one solution to $g(r) = h(r)$, satisfying $0 < r < 2$. Suppose, on the contrary, there were two, call them r^* and r^{**}. Without loss of generality, $0 < r^* < r^{**} < 2$. So there is $0 < \theta < 1$, such that $r^* = \theta r^{**}$. Now, following the method used in Case (i), we get a contradiction. [Replace f by h throughout].

We have thus established that, in both Case (i) and Case (ii), there is at most one positive value of r, satisfying $Z(r) = 1$.

Now, suppose (X^*, p^*) and (X^{**}, p^{**}) are equilibria, given a distribution of endowments, E. Then $Z(r^*) = 1 = Z(r^{**})$, and $r^* > 0$ and $r^{**} > 0$. Hence, $r^* = r^{**}$, and so $p^* = p^{**}$. But, then $x^{1*} = x^{1**}$, $x^{2*} = x^{2**}$, by using the demand functions of agents *1* and *2*. So $x^{3*} = e - x^{1*} - x^{2*} = e - x^{1**} - x^{2**} = x^{3**}$. Thus $X^* = X^{**}$, which shows that there is at most one equilibrium, given an arbitrary distribution of endowments, E.

Example 4.2

Instead of starting out with a complete numerical specification of the characteristics of the agents, we want to share the insights of Leonard and Manning (1983) on how to generate 'paradoxical' or 'unorthodox' examples. Suppose that agent 1 is characterized by

$$u^1(x_1, x_2) = x_1^{1/2} x_2^{1/2}$$
$$e^1 = (a, 0)$$

whereas agent 2 is described by

$$u^2(x_1, x_2) = x_1^{7/8} x_2^{1/8}$$
$$e^2 = (b, 0)$$

Let commodity 2 be the numeraire. Then the demands of the two agents are $(a/2, pa/2)$ and $(7b/8, pb/8)$ respectively. If agent 3 has the endowment $e^3 = (0, c)$, then his consumption must be $(a/2 + b/8, c - pa/2 - pb/8)$ *in equilibrium* (since he consumes what the others do not). Suppose $a = 10$, $b = 4$, $p = 1$; then, equilibrium consumption bundles must be (5, 5), (7/2, 1/2) and $(11/2, c - 11/2)$ respectively. Now consider a transfer of four units of the first commodity from agent 1 to agent 2. At the equilibrium price p, consumption bundles become $(3, 3p)$, $(7, p)$ and $(4, c - 4p)$. If agent 1 is to be *better-off* as a result of the transfer, then it must be that $p > 25/9$. Certainly, $p = 3$ permits the transfer to be advantageous to both agents 1 and 2; whether this can be an *equilibrium* depends on the utility function u^3 of the third agent: u^3 must be such that at $p = 1$, he consumes $(11/2, c - 11/2)$ whereas at $p = 3$ his consumption is $(4, c - 12)$. There are then only two restrictions on u^3. Attention may, therefore, be concentrated on a two-parameter family of utility functions. For instance, one can take

$$u^3(x_1, x_2) = \alpha x_1 - \frac{1}{3}\beta x_1^3 + x_2;\ \alpha, \beta > 0 \qquad (4.1)$$

For an appropriate range of p, compute the demand function of agent 3 for the first good:

$$x_1^3 = [(\alpha - p)/\beta]^{1/2} \tag{4.2}$$

The restrictions on u^3 explained above imply

$$5.5 = [(\alpha - 1)/\beta]^{1/2};\ 4 = [(\alpha - 3)/\beta]^{1/2} \tag{4.3}$$

One can solve (4.3) for α, β and substituting in (4.1) get a numerical specification of u^3 which will allow the particular advantageous reallocation of their initial endowment. Note that c can be chosen arbitrarily as long as $c > 12$. The excess demand function for good 1 is

$$a/2 + 7b/8 + (\alpha - p/\beta)^{1/2} - a - b$$

This has a negative first derivative, thereby ensuring uniqueness and Walrasian stability. Clearly, by reversing the transfer from the final to the initial situation (so that agent 2 becomes the 'donor' and 1 becomes the recipient), one constructs an example of a transfer that makes *both* the donor and the recipient worse-off (to the satisfaction of agent 3). For useful comments on linking this example to the necessary conditions of Bhagwati-Brecher-Hatta, the reader can turn to the paper of Leonard and Manning (1983, footnote 3).

5. The Many-Agent Case

This section investigates the transfer problem in a many-agent, many-commodity setting. It is well-known that definitive results are hard to come by at this level of generality. Thus, we will need some additional assumptions on our model in order to proceed.

Let us start by recalling a few facts from the theory of demand. We take these facts (and others to be noted later) to be sufficiently well-known that one need not go into the details of their derivations. [The reader can consult a standard text like Varian (1978).] For each $i = 1, \ldots, m$ given p in R^n_{++} and d in R^n_{++}, there is a unique solution $g^i(p, d)$ to the following problem:

$$v^i(p, d) \equiv \max u^i(x)$$
$$\text{subject to } px \leq pd,\ x \text{ in } R^n_+$$

That is, for each $i = 1, \ldots, m$, there is an *ordinary (Marshallian) demand function*, $g^i(p, d)$, and an *indirect utility function*, $v^i(p, d)$ for p in R^n_{++}, d in R^n_{++}.

For our next fact, it is convenient to assume:

(A.3) Given any w in R, there is x in R^n_+, such that $u^i(x) > w$ ($i = 1, \ldots, m$).

Then, for each $i = 1, \ldots, m$, given p in R^n_{++}, w in R, $w > u^i(0)$, there is a unique solution $h^i(p, u)$ to the following problem:

$$M^i(p, w) \equiv \min px$$
$$\text{subject to } u^i(x) \geq w, \ x \text{ in } R^n_+$$

That is, for each $i = 1, \ldots, m$, there is a *compensated (Hicksian) demand function*, $h^i(p, w)$ and an *expenditure function* $M^i(p, w)$ for p in R^n_{++}, w in R, $w > u^i(0)$.

In order to prove the main result of this section, we need the following additional assumptions. These assumptions are stated without assuming differentiability of the demand functions, following the style of Nikaido (1968, p. 305).

(A.4) For each i in $[2, \ldots, m]$, and for p, $\bar{p}$ in R^n_{++}, d in R^n_{++}, $\bar{p} \geq p$ implies

$$g^i_k(\bar{p}, d) \geq g^i_k(p, d) \ \text{ if } \ k \notin \{j \mid \bar{p}_j > p_j\}$$

(A.5) For $p, \bar{p}$ in R^n_{++}, w in R, $w > u^1(0)$, $\bar{p} \geq p$ implies

$$h^1_k(\bar{p}, w) \geq h^1_k(p, w) \ \text{ if } \ k \notin \{j \mid \bar{p}_j > p_j\}$$

(A.6) For each i in $[1, \ldots, m]$, for p in R^n_{++}, d, $\bar{d}$ in R^n_{++}, $\bar{d} > d$ implies

$$g^i_j(p, \bar{d}) > g^i_j(p, d) \ \text{ for } \ j = i, \ldots, n$$

Note that (A.4) says that all goods are 'gross substitutes' for agents $[2, \ldots, m]$, while (A.5) says that all goods are 'net substitutes' for agent 1 (which, it will be recalled, is the 'donor'). Finally, (A.6) says that all goods are 'normal' for all agents. [Assumptions (A.1)–(A.6) are consistent with arbitrary Cobb-Douglas utility functions for all agents.]

For p in R^n_{++}, and d in R^n_{++}, it is known that the indirect utility function, $v^1(p, d)$ is continuous in (p, d), by the Maximum Theorem [for a statement of Maximum Theorem, see Berge (1963), p. 116] and increasing in each component of d [by (A.2)]. Thus given any p in R^n_{++}, w in R, $w > u^1(0)$, and using (A.3), there is d in R^n_{++}, such that $v^1(p, d) = w$. Similarly, for $\bar{w}$ in R, $\bar{w} > w$, there is $\bar{d} > d$ such that $v^1(p, \bar{d}) = w$. Now, $h^1[p, v^1(p, d)] = g^1[p, d]$ and $h^1[p, v^1(p, \bar{d})] = g^1[p, \bar{d}]$. Using (A.6), we therefore have $\bar{w} > w$ implying

$$h_j^1[p, \overline{w}] > h_j^1[p, w] \text{ for } j = i, \ldots, n \tag{5.1}$$

Proposition 5.1: Let (X, p) be an equilibrium given E, and $(\overline{X}, \overline{p})$ be an equilibrium given $\overline{E}$. *If* $(E, \overline{E})$ *is in* T, *then*

$$u^1(\overline{x}^1) < u^1(x^1) \tag{5.2}$$

Proof: Since (X, p) is an equilibrium given E, so p is in R^n_{++}, x^i is in R^n_{++}, e^i is in R^n_{++}, and $u^i(x^i)$ $u^i(0)$ for i in $[1, \ldots, m]$. Similarly, $(\overline{X}, \overline{p})$ is an equilibrium given $\overline{E}$, so $\overline{p}$ is in R^n_{++}, $\overline{x}^i$ is in R^n_{++} e^i is in R^n_{++}, and $u^i(\overline{x}^i) > u^i(0)$. Using the fact that (X, p) and $(\overline{X}, \overline{p})$ are equilibria given E and $\overline{E}$ respectively,

$$\sum_{i=1}^{m} g^i(p, e^i) = e = \sum_{i=1}^{m} g^i(\overline{p}, \overline{e}^i) \tag{5.3}$$

Let $w^1 = u^1(g^1(p, e^1))$, $\overline{w}^1 = u^1(g^1(\overline{p}, \overline{e}^1))$. Then, since $g^1(p, e^1) = h^1(p, w^1)$ and $g^1(\overline{p}, \overline{e}^1) = h^1(\overline{p}, \overline{w}^1)$, so (5.3) becomes

$$h^1(p, w^1) + \sum_{i=2}^{m} g^i(p, e^i) = e = h^1(\overline{p}, \overline{w}^1) + \sum_{i=2}^{m} g^i(\overline{p}, \overline{e}^1) \tag{5.4}$$

Thus, we have

$$[h^1(p, w^1) - h^1(\overline{p}, \overline{w}^1)] + \sum_{i=2}^{m} [g^i(p, e^i) - g^i(\overline{p}, \overline{e}^i)] = 0 \tag{5.5}$$

Let $\min_j (\overline{p}_j/p_j) = (\overline{p}_k/p_k)$. Then defining $q = (p/p_k)$, $\overline{q} = (\overline{p}/\overline{p}_k)$, we have

$$[h^1(q, w^1) - h^1(\overline{q}, \overline{w}^1)] + \sum_{i=2}^{m} [g^i(q, e^i) - g^i(\overline{q}, \overline{e}^i)] = 0 \tag{5.6}$$

and

$$\overline{q}_j = \overline{p}_j / \overline{p}_k \geq p_j / p_k = q_j; j = 1, \ldots, n \tag{5.7}$$

Suppose now, contrary to (5.2), that $\overline{w}^1 \geq w^1$. Then note that $h_k^1(q, w^1) - h_k^1(\overline{q}, \overline{w}^1) = h_k^1(q, w^1) - h_k^1(\overline{q}, w^1) + h_k^1(\overline{q}, w^1) - h_k^1(\overline{q}, \overline{w}^1)$. Now using (5.7) and (A.5), $h_k^1(\overline{q}, w^1) \geq h_k^1(q, w^1)$. And, using (5.1) and $\overline{w}^1 \geq w_1$, $h_k^1(\overline{q}, \overline{w}^1) \geq h_k^1(\overline{q}, w^1)$. Hence

$$h_k^1(q, w^1) - h_k^1(\overline{q}, \overline{w}^1) \leq 0 \tag{5.8}$$

Next, for $i = 2, \ldots, m$, we have $g_k^i(q, e^i) - g_k^i(\overline{q}, \overline{e}^i) = g_k^i(q, e^i) - g_k^i(\overline{q}, e^i) + g_k^i(\overline{q}, e^i) - g_k^i(\overline{q}, \overline{e}^i)$. Now, using (5.7) and (A.4), $g_k^i(\overline{q}, e^i) \geq g_k^i(q, e^i)$. And, using (A.6), and $\overline{e}^i \geq e^i$ $[i = 2, \ldots, m]$, we have $g_k^i(\overline{q}, \overline{e}^i) \geq g_k^i(\overline{q}, e^i)$. Also for some i in $[2, \ldots, m]$, $\overline{e}^i > e^i$, so using (A.6), $g_k^i(\overline{q}, \overline{e}^i) > g_k^i(\overline{q}, e^i)$ for this i.

Thus,

$$g_k^i(q, e^i) - g_k^i(\overline{q}, \overline{e}^i) \leq 0 \text{ for all } i \text{ in } [2, \ldots, m] \tag{5.9}$$

and
$$g_k^i(q, e^i) - g_k^i(\overline{q}, \overline{e}^i) < 0 \text{ for some } i \text{ in } [2, \ldots, m] \tag{5.10}$$

Using (5.9) and (5.10),

$$\sum_{i=2}^{m} [g_k^i(q, e^i) - g_k^i(\overline{g}, \overline{e}^i)] < 0 \tag{5.11}$$

But (5.8) together with (5.11) contradict (5.6). Hence $\overline{w}_1 < w_1$, which establishes (5.2).

Remark

Clearly Proposition 5.1 shows that the donor is worse-off after a transfer, under a set of *sufficient conditions*. Alternative sufficient conditions can also be derived. For example, we have verified that if all countries have the same utility function, and furthermore this is a differentiable homothetic function, then the result of Proposition 5.1 will also hold. The reason is that in this case relative prices do not change from one equilibrium to another, given the total endowment of the world economy. In other words, the terms of trade effect is totally absent. We think that, for this reason alone, such sufficient conditions are somewhat less interesting in this context than the ones we have used to prove Proposition 5.1. Consequently, we have not gone into the detailed derivation of this additional result.

6. FURTHER REMARKS

We shall sketch the Samuelsonian comparative static approach to the transfer problem in a heuristic manner. Consider the two-agent, two-commodity model. As usual, agent i is characterized by (u^i, e^i). Furthermore, let $e^1 = (e^1_{x_1}, e^1_{x_2}) \in R^2_{++}$ and choosing the appropriate units of measurement, $e^2 = (1 - e^1_{x_1}, 1 - e^1_{x_2})$. Assume the relevant interiority and differentiability assumptions so that by invoking the implicit function theorem one can justify taking the derivatives in a neighbourhood of e^1. Let (X, p) be an equilibrium corresponding to the endowment pattern $e = (e^1, e^2)$. Choosing the first commodity as a numeraire, write $p = (1, r)$ where $r > 0$. The equilibrium condition in the second market is:

$$\overline{x}_2^1 + \overline{x}_2^2 = 1 \tag{6.1}$$

Let $\bar{u} = u(\bar{x}^1)$ and $\bar{u}^2 = u(\bar{x}^2)$. Write the expenditure functions M^1 and M^2 as:

$$M^1(p, w, \bar{u}_2^1) \equiv M^1(r, e^1_{x_1} + re^1_{x_2}, \bar{u}^1) = \bar{x}^1_1 + r\bar{x}^1_2 \tag{6.2}$$
$$M^2(p, w, \bar{u}^2) \equiv M^2(r, 1 - e^1_{x_1} + r(1 - e^1_{x_2}), \bar{u}^2] = \bar{x}^2_1 + r\bar{x}^2_2 \tag{6.3}$$

Using a standard result [Varian (1978, p. 123)], one has

$$\frac{\partial M^1}{\partial r} = \bar{x}^1_2 \quad \frac{\partial M^2}{\partial r} = \bar{x}^2_2$$

Hence, the equilibrium condition (6.1) is rewritten as:

$$\frac{\partial M^1}{\partial r}(r, \bar{u}^1) + \frac{\partial M^2}{\partial r}(r, \bar{u}^2) = 1 \tag{6.4}$$

From (6.2)–(6.4), after differentiation with respect to $e^1_{x_1}$ (denoting the derivative of a variable x with respect to $e^1_{x_1}$ by $\dot{x}$), one has:

$$M^1_r\,\dot{r} + \quad + M^1_{u^1}\,\dot{u}^1 = 1 + e^1_{x_2}\,\dot{r}$$
$$M^2_r\,\dot{r} + M^2_{u^2}\,\dot{u}^2 \quad = -1 + (1 - e^1_{x_2})\dot{r}$$
$$M^1_{rr}\,\dot{r} + M^1_{ru^1}\,\dot{u}^1 + M^2_{rr}\,\dot{r} + M^2_{ru^2}\,u^2 = 0$$

Let D be the determinant of the matrix

$$\begin{bmatrix} M^1_r - e^1_{x_2} & M^2_{u^1} & 0 \\ M^2_r - (1 - e^1_{x_2}) & 0 & M^2_{u^2} \\ M^1_{rr} + M^2_{rr} & M^1_{ru^1} & M^2_{ru^2} \end{bmatrix}$$

Solving for $\dot{r}$ and $\dot{u}^1$ we get

$$\dot{r} = [-M^1_{ru^1}\,M^2_{u^2} + M^1_{u^1}\,M^2_{ru^2}]/D$$
$$\dot{u}^1 = M^2_{u^2}(M^1_{rr} + M^2_{rr})/D$$

The signs of $\dot{r}$ and $\dot{u}$ depend, in particular, on the sign of D. This is interpreted as a stability condition (an application of the correspondence principle). For further analysis of necessary conditions for determining the signs of $\dot{u}^1$ (and $\dot{u}^2$) (as in other applications), the Samuelsonian approach leads to voluminous algebra, and often a taxonomic presentation of alternative conditions under which the signs of the derivatives can be unambiguously determined. While one can provide an interpretation of the sign of D as a 'local' stability condition in the 2×2 case, any interpretation for more than two goods must be in terms of uniqueness. Some clarification remarks on the issue of 'uniqueness' and 'stability' conditions are perhaps in order. It is recognized in general equilibrium theory that a comparative static analysis may not even get off the ground without postulating at least a local uniqueness condition. This is because without uniqueness the question that naturally arises is which equilibrium after an (infinitesinal) parameter change are we comparing to which equilibrium before change [see Arrow-Hahn (1971, pp. 207, 242, 245)]. Secondly, Samuelson was, of course, right in noticing the link between comparative statics and dynamics. However, there may well be alternative adjustment mechanisms (i.e., alternative models of disequilibrium behaviour) associated with a particular notion of equilibrium: and 'stability' conditions might be quite different. Finally, although, in general, local and global uniqueness and stability are conceptually different, in the particular case of a two-good model (with an arbitrary, finite number of agents), the various concepts, in effect, collapse into one (provided excess demand functions are continuous). That is, local stability of every equilibrium (under the Walrasian tatonnement adjustment process) implies there is only one equilibrium. This, in turn, implies that the equilibrium is globally (hence, locally!) stable. This explains why trade theorists using primarily the two-good model have focused on the local stability condition.

References

(The list of references is not intended to be complete. However, several items on the list have extended discussions of earlier (and often ignored) efforts.)

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