Advances in Mathematical Economics 23

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# Advances in Mathematical Economics

Volume 23

Springer

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ISSN 1866-2226 ISSN 1866-2234 (electronic)
Advances in Mathematical Economics
ISBN 978-981-15-0712-0 ISBN 978-981-15-0713-7 (eBook)
https://doi.org/10.1007/978-981-15-0713-7



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The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

# Contents

# Complicated Dynamics and Parametric Restrictions in the Robinson-Solow-Srinivasan (RSS) Model

**M. Ali Khan and Tapan Mitra**

**Abstract** The delineation of the optimal policy function (OPF) in the discounted setting has remained an open question since the 2005 demonstration of optimal topological chaos (OTC) in a particular instance of the 2-sector RSS model. This paper provides an explicit solution of the OPF when the discount factor is less than the labor/capital-output ratio $a$. With OTC conceived both as period-three cycles and turbulence, it establishes the existence of OTC for non-negligible parametric ranges of the model, shows the identified ranges also to be necessary, presents exact restrictions on $a$, and extends the 1996 Mitra-Nishumura-Yano theorems on discount-factor restrictions.

**Keywords** Two-sector model · Optimal policy function · Period-three cycles · Turbulence · Optimal topological chaos · Parametric restrictions

**Article type:** Research Article
**Received:** August 10, 2019
**Revised:** September 1, 2019

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A preliminary version of this paper was first circulated on October 15, 2009. It was presented at the *Mini Conference on Economic Theory* held at the University of Illinois at Urbana-Champaign, November 5–7, 2010, to honor the retirement of Professor Nicholas Yannelis. It was subsequently presented at Professor Maruyama's seminar at Keio University on January 28, 2011. Tapan Mitra, the Goldwin Smith Professor of Economics at Cornell University, passed away February 3, 2019 after a seven-year illness. During the last two years of his life, he worked on chaotic dynamics in active collaboration with Liuchun Deng of NUS-Yale and the author. The surviving author is submitting this manuscript for publication in its original form (with references updated) not

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T. Maruyama (ed.), *Advances in Mathematical Economics*, Advances in Mathematical Economics 23, https://doi.org/10.1007/978-981-15-0713-7_4

The Golden Ratio's attractiveness seems first and foremost from the fact that it has an almost uncanny way of popping up where it is least expected.[1]

Livio (1970, p. 7)

## 1 Introduction

It has been well-understood, at least since the early nineties, that solutions to single-agent intertemporal optimization problems can exhibit complicated dynamics in the form of *topological* and *ergodic* chaos precisely defined; see [2] and [23] for anthologies of the pioneering papers. A recent survey [31] of this work delineates how endogenous sources of chaos revolve around a variety of considerations: "upward inertia" as a consequence of zero consumption levels, "downward inertia" as a consequence of depreciating capital, supermodularity of the felicity functions, factor-intensity reversals in a two-sector technology, and high levels of impatience have all been given salience.[2] In [15], optimal topological chaos has been shown in a particularly parsimonious instance of a stripped-down version of the two-sector model, the so-called *borderline* case of the two-sector RSS model, one that involves a specific relationship between only two parameters: $\xi$, the marginal rate of transformation of capital today into that of tomorrow, given full employment of both factors; and $d$, the rate of depreciation.[3] The result is executed with linear felicities, a polar form of the factor intensity assumption, and a positive rate of

only because of its continuing analytical interest, but also in the hope that it'll facilitate the understanding and reception of the Deng-Khan-Mitra results as and when they are written up. In this connection, he thanks Professors Mukul Majumdar, Toru Maruyama, Debraj Ray and Santanu Roy for their kind encouragement. He also thanks Mordecai Kurz, Chris Metcalf and Paulo Sousa for correspondence and conversation on the original draft presented at Urbana-Champaign.

**JEL Classification**: D90, C62, O21

**Mathematics Subject Classification (2010)**: 91B62, 49J45, 49O2

[1]The reader uninitiated into the mysteries of the "golden-ratio" may want to check out [13, pp. 25–27] or [22, pp. 78–86]. A case could be made for singling out [28, 36] as the pioneering applications of the number in economic theory.

[2]A narrative is laid out in [31, Section 6] and it revolves around (1) capital depreciation and linear felicities but with factor intensity reversals in a two-sector model with Cobb-Douglas and Leontief technologies, as in the numerical results in [6], (2) fully circulating capital but with zero consumption levels on the optimal path, as in [33], (3) the inclusion of depreciation with Leontief technologies in both sectors, the so-called Leontief-Shinkai model, and supermodular felicities, as in [34], and with linear felicities, as in [35, 37], (4) non-zero optimal consumption levels in the extension in [40] of [35], and finally, (5) the establishment of ergodic chaos and geometric sensitivity in [39]. Also see the early attempt in [32].

[3]See [3] where the principal result involves eight parameters, and [40], where the result is whittled down to a simpler setting, but still with four parameters. This footnote is an obvious subscription to the simplicity imperative in [24, 45] and others. With Saari [45], it is also a resigned acceptance.

capital depreciation. It is striking on two counts. First, it goes against the continuous-time intuition, rather well-established in the early seventies, that optimal programs, even for a considerably generalized setting of such a model, exhibit saddle-point stability.[4] Second, relative to instances of the more recent literature, it overcomes the disadvantage of zero-consumption levels[5] (in two of the three periods in the period-three cycle established in [33]) without the tagging of somewhat ad-hoc felicity functions, as in [35] and [40].[6]

All this being said, it is important to be clear that the methodological import of the result in [15] lies not so much in the existence of optimal topological chaos (work done more than two decades earlier had already established this), but rather in the fact that complicated dynamics could be shown without any knowledge of the shape of the $OPF$ other than its continuity,[7] and to bring out the power of some sufficient tests for topological chaos, specifically those guaranteeing turbulence in the resulting system. However, as interest deepens in the two-sector RSS model, it is natural to ask whether the $OPF$ can be pinned down for a non-negligible range of parametric values; and if so, what additional light can be thrown on the question of the robustness of optimal topological chaos in the model, and in particular, its demonstration as a consequence of conditions that are easy to check. This question was explicitly left open in [19], where the authors asked for the optimal policy correspondence when the discount factor $\rho$ was less than the inverse of the marginal rate of transformation of capital $(1/\xi)$, and conjectured that "it is possible that ... the graph of the optimal policy correspondence is the lower boundary of the graph of $G$, which, following the terminology in [8], can be referred to as the "check-map" policy function since its graph resembles the standard check mark.[8] The first, and perhaps primary, contribution of this paper is to answer this question.

[4]See [50, 51] and [14] for further genealogical details; also [31, 161–162].

[5]A point of view insisted on in Joan Robinson's response to Stiglitz; see [41] and [51]. In [15], despite a linear utility function, consumption is *never* zero along any optimal path, except one starting at zero capital stock, and then only in the initial period. Put differently, Joan Robinson's criticism does not apply to the $OPF$ reported in this paper, and seems to be purely an artifact of the continuous-time formulation.

[6]In [35], the authors work with the Weitzman-Samuelson reduced-form utility function, and in [40], with a constant elasticity of substitution felicity function. The substantive motivation for either specification is not fully apparent.

[7]For complicated dynamics, see the textbook [7] and the pioneering papers [46, 47]. For the economic literature, see [29, 38]. Also note that Khan and Mitra [15] shares a similarity with [40] as regards this feature of working without a specific analytical form of the $OPF$.

[8]The concluding remarks in [19] conjecture the shape of the optimal policy correspondence at the threshold discount factor $\rho = (1/\xi)$, to be $G$, described in their Eq. (23), and seen to be a composite of the pan- and check-maps, and everything in between; also see Figs. 1 and 2 in this paper.

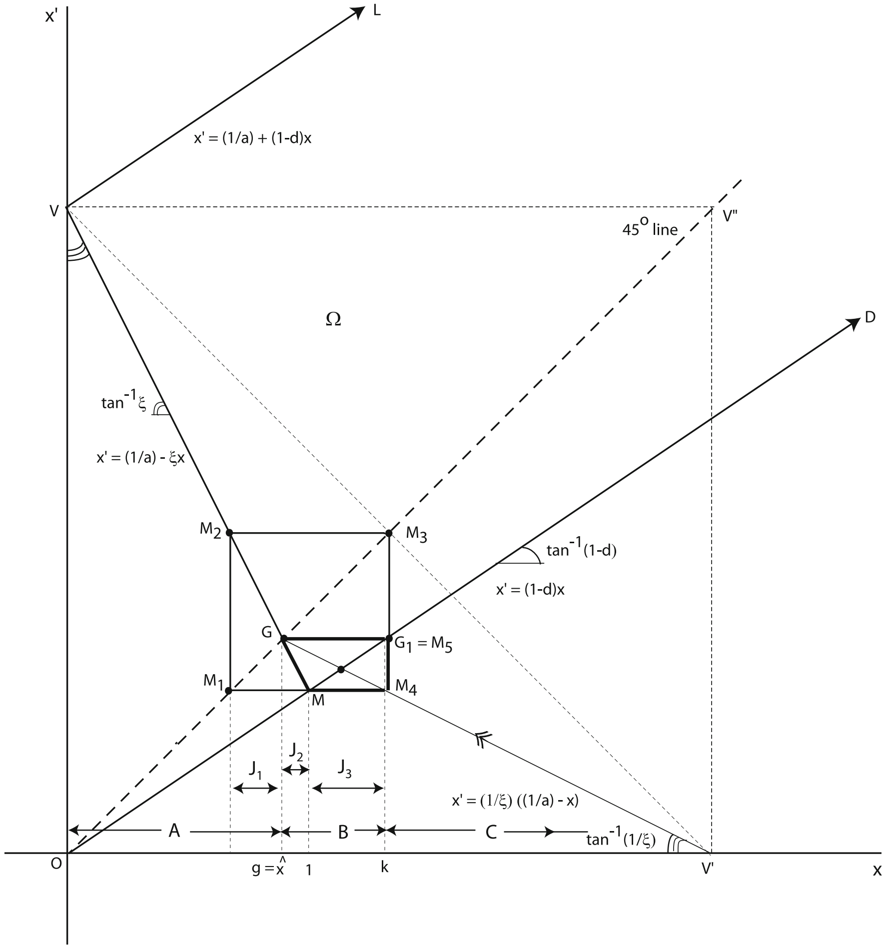


**Fig. 1** The 2-sector RSS model in the borderline case $(\xi-(1/\xi))(1-\mathrm{d})=1$ or $\mathrm{a}\xi^3 = (\xi^2-1)$ or $(1-\mathrm{d})((\mathrm{d}/\mathrm{a})+(1-\mathrm{d})^2)=(1/(1+\mathrm{ad}))$

In the first substantive section of the paper, Sect. 3, we show that for all values of the discount factor less than the labor-output ratio[9] $a$ in the investment goods sector,

$$\rho < a = (1/(\xi + 1 - d)) < (1/\xi), \quad \xi > 1, \quad 0 < d < 1, \tag{1}$$

[9]Note that the use of the abbreviation "labor-output ratio" is ambiguous since there are two outputs in the model; in this paper, we shall use it to refer to the labor required to produce a machine, the labor/capital-output ratio, so to speak.

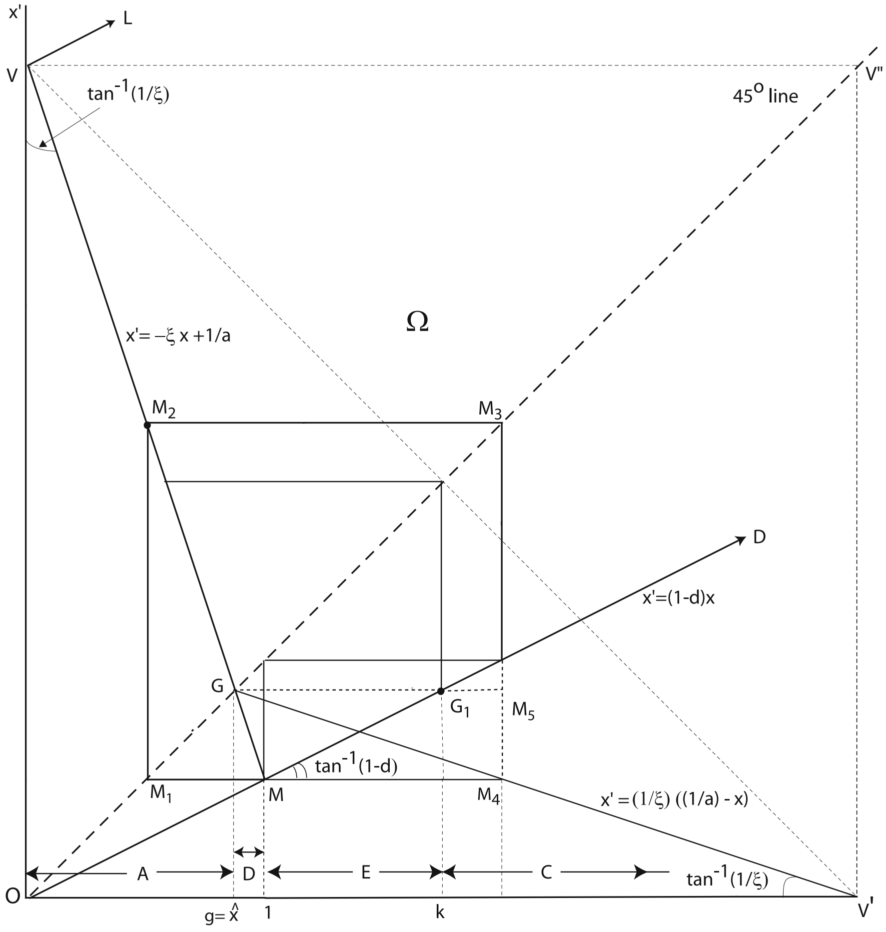


**Fig. 2** The 2-sector RSS model in the outside case $(\xi - 1)(1-\mathrm{d})=1$ or $\mathrm{a}(\xi^2 - \xi + 1) = (\xi - 1)$ or $(1/\mathrm{a})-1 = (1-\mathrm{d})+(1-\mathrm{d})^{-1}$

the $OPF$ is indeed given by the check-map.[10] And once this information is factored into the equation, we move, in the second substantive section of the paper, beyond the specific instance of the two-sector RSS model analyzed in [15] for turbulence, and give a rather complete treatment of both turbulence and period-three

[10] See [19] and earlier references for the straightforward details of the case when $-1 < \xi \leq 1$. The terminology *check-map* appears in [8, p. 46], but a detailed analysis goes back to [9], and subsequently, in an optimal intertemporal context, to [11]. For a more recent numerical attempt rooted in the RSS setting, but again without giving it an optimality underpinning, see [27]. It is of interest that of the five figures in [31], none concern the check-map, though Figure 6.5 comes closest to it.

cycles. In particular, we can go beyond the parameter equality restriction in [15] to show the existence of optimal topological chaos in an entire range of parametric combinations. This is to say, to move from

$$\left[\frac{1}{1+ad} - \frac{d(1-d)}{a} = (1-d)^3\right] \quad \text{to} \quad \left[\frac{1}{1+ad} - \frac{d(1-d)}{a} \leq (1-d)^3\right].$$

On rewriting the above inequality somewhat more explicitly, we can now work with the range given by[11]

$$(1-d)\left(\frac{1}{a} - \xi(1-d)\right) \geq \frac{1}{1+ad} \Longleftrightarrow \left(\xi - \frac{1}{\xi}\right)(1-d) \geq 1 \Longleftrightarrow a \leq \frac{\xi^2 - 1}{\xi^3} \tag{T}$$

Technically conceived, using the same argument that the second iterate of the $OPF$ is turbulent, this simply generalizes the result in [15]. But knowing the $OPF$, we can do more. We can show that the existence of a period-three cycle is guaranteed by the following restrictions.

$$(1-d)\left(\frac{1}{a} - \xi(1-d)\right) \geq 1 \Longleftrightarrow (\xi - 1)(1-d) \geq 1 \Longleftrightarrow a \leq \frac{\xi - 1}{\xi^2 - \xi + 1} = \frac{\xi^2 - 1}{\xi^3 + 1} \tag{PT}$$

Since we are working under the restriction that $\xi$ is greater than unity, it is clear that any of the inequalities in $(PT)$ imply the corresponding inequality in $(T)$. And so in hindsight, it is clear that the particular restriction used in [15] is *weaker* than the one ensuring period-three cycles; weaker in that turbulence of the second iterate of the $OPF$ is lower down in the Sharkovsky order than its period-three property.[12] Put differently, given the continuity of the $OPF$, $(PT)$, being a sufficient condition for the period-three cycle, ensures turbulence as well. The bottom line is that both cases (turbulence and the period-three property) are very easy to describe once one has the explicit form of the $OPF$. These results are presented in Sect. 4.

While these results provide robust parameter configurations for which the second iterate of the $OPF$ is turbulent, or the $OPF$ satisfies the Li-Yorke condition, they

[11] The derivation of these formulae is relegated to the Appendix 8.1. The numbering $(T)$ and $(PT)$ is dictated by the words *turbulence* and *period-three:* as we shall see in the sequel, the specification $(T)$, and guaranteeing turbulence actually places a weaker restriction on the parameters than the specification $(PT)$, guaranteeing and optimal period-three cycle. It may also be worth pointing out that already in 2005, the authors had shown the existence of an optimal program with period-three cycles in another instance of the two-sector RSS model, in which the inequality is replaced by an equality in $(PT)$.

[12] It comes after all the odd-period cycles greater than a single period, but before those of period-six cycles; see [30] for extended discussion. It is also worth emphasizing here that $(T)$ is sufficient and not a necessary condition for turbulence.

also show that concrete restrictions on the parameters of the RSS model appear to be involved in exhibiting such phenomena. The third substantive section of this paper takes this as a point of departure, and turns to what, following the *anything-goes* theorems of Sonnenschein-Mantel-Debreu, has come to be referred to as the *rationalizability* theory.[13] The basic rationale of this theory in terms of the problematic at hand is a simple one: unlike the situation for the Arrow-Debreu-McKenzie (ADM) model that involves the construction of an economy with a *given* excess demand function, here one constructs here a single-agent, Gale-McKenzie (GM) intertemporal optimization model with a policy function identical to a *given* one; see [12, 25]. However, in the context of the GM model, one can ask sharper questions, and furnish sharper answers. In particular, given that the discount factor stands on its own in the GM model, one can focus on it, and ask for necessary and sufficient conditions on its magnitude under which a given policy function (the tent- or logistic map, for example) exhibiting period-three cycles, turbulence of the second iterate of the policy function, or more generally positive topological entropy, can be rationalized. In the literature, *discount-factor restrictions* associated with such phenomena have been obtained in a variety of intertemporal allocation models: these are the so-called "exact MNY discount-factor restrictions" of [28, 36] which involve, in the context of the GM model, the golden number.[14] The three results reported in [48, Theorems 4.4–4.6] summarize the state of the art, though substantial ongoing work continues; see [49] and his references. A key element of the class of intertemporal allocation models studied in this context is that the reduced form utility function exhibits (beyond the standard concavity assumption) some form of strict concavity on its domain. The function might be required to be strictly concave in both arguments, or at least in one of them (that is, either in the initial stock or the terminal stock). The strict concavity requirement plays two roles. It ensures the existence of an optimal policy function. But, in addition, it is seen to be indispensable to the methods used to derive the discount-factor restrictions for complicated optimal behavior.

With the OPC determined for a non-negligible range of discount factors, and with topological chaos in the form of turbulence and period-three cycles robustly identified, this work then prompts two sets of questions in the context of the two-sector RSS model. First, do the existing theorems apply to the restricted setting of the two-sector RSS model? and if not, are there reformulated counterparts of these theorems that can be proved? The first question is easily answered. The point is that even strict concavity of the felicity function does imply strict concavity of the reduced-form utility function of the RSS model, and therefore discount-factor restrictions for complicated optimal behavior in the RSS model, if any, will have to be established by methods different from those employed in the literature. As

[13] For references to the Sonnenschein-Mantel-Debreu theorem, and for a recent survey of the available theory that gives references to the pioneering papers, see [48].

[14] See for example [13, 22] for the fascinating career of the golden number; also the footnoted epigraph.

regards the second, one has to take into account the fact that the two-sector RSS model cannot generate the maps (for example, the usual tent maps)that have been used in the derivation of the (MNY) bound, and consider the bounds on the discount factor that arise from the check-map.[15] In particular, we show that $\rho < (1/3)$ implies that there exist $(a, d)$ such that the RSS model $(a, d, \rho)$ has an optimal policy function which generates a period three cycle. Conversely, if the RSS model $(a, d, \rho)$ has an optimal policy function which generates a period three cycle, then $\rho < (1/2)$. Although not *exact* restrictions, it is quite remarkable how closely the restriction on $\rho$ on the sufficiency side compares with that on the necessity side. Further, the restriction of $\rho < (1/2)$ on the necessity side is really a very strong discount-factor restriction for period-three optimal cycles to occur, since it involves a discount rate of a 100%. Thus, "period three implies heavy discounting" turns out to be a robust conclusion, valid for a broad class of intertemporal allocation models, *including* the RSS model. In addition, we offer discount-factor restrictions arising from optimal turbulence. We show that $\rho$ being less than $(\sqrt{\mu})^3$, where $\mu$ is the golden-number $(\sqrt{5} - 1)/2$, implies that there exist $(a, d)$ such that the RSS model $(a, d, \rho)$ has an optimal policy function whose second iterate exhibits turbulence. Conversely, if the RSS model $(a, d, \rho)$ has an optimal policy function whose second iterate exhibits turbulence, then $\rho < (1/2)$. These results are new, and even though irreversible investment has been shown to bring in the role of depreciation of capital, as in [35], we have been able to go further by exploiting the special structure of the two-sector RSS model. These results are an important third contribution of the paper, and constitute Sect. 6.

Indeed, given the rather specific technological structure—the toy-nature of the two-sector RSS setting, so to speak—there is the obvious suggestion that it might be possible to exploit this structure to make even further progress on this topic. The fact that the model can be completely summarized by the two parameters $(a, d)$, makes it possible to address the problem of identifying *technological restrictions* involved when the $OPF$ generates, for instance, optimal turbulence or period-three cycles. Because intertemporal allocation models are phrased in terms of a general convex technology set, a similar exercise with respect to technological parameters has not been attempted before, to the best of our knowledge. In any case, the results here are especially satisfying. We provide an exact labor-output ratio (in the investment good sector) restriction for period-three cycles in the two-sector RSS model of optimal growth in the following sense. We show that if the labor-output ratio $a < (1/3)$, then there exist $\rho \in (0, 1)$ and $d \in (0, 1)$, such that the RSS model defined by $(\rho, a, d)$ has an optimal policy function, $h$, which generates a period-three cycle. Conversely, we show that if there is an RSS model, defined by parameters $(\rho, a, d)$,which has an optimal policy function that generates a period-three cycle, then $a < (1/3)$. It is useful to note in this context that regardless of the value of the depreciation

[15] It is interesting that despite being sidelined by the tent-map and the logistic function in the earliest economic applications, the check-map has been resiliently present from the very inception of the work. Also see Footnote 8 and its references.

factor, $d$, (and the value of the discount factor $\rho$) our result indicates that it is *not* possible to have an optimal policy function generating period three cycles when $a \geq (1/3)$. However, more to the point, we also supplement this result by showing that condition $(T)$ above is necessary and sufficient for the optimal policy function of the RSS model to exhibit optimal turbulence. These results are the important fourth contribution of the paper, and constitute Sect. 5.

This extended introduction to the problem and the results obtained has already outlined the paper to a substantial extent. In terms of a summary, after a brief introduction to the two-sector RSS model in Sects. 2 and 3 establishes the $OPF$ under the restriction $\rho < a$, and with this pinned down, Sect. 4 turns to optimal topological chaos formalized in terms of both turbulence and period-three cycles. Subsequent parts of the paper turn to the parametric restrictions: Sect. 5 to *exact* restrictions on the labor-output coefficient $a$, for optimal period-three cycles and to (T) for optimal turbulence; and Sect. 6 to discount-factor restrictions, again in the context of both optimal turbulence and period-three cycles. Section 7 concludes the paper with open questions and complementary analyses that will be reported elsewhere. The substantial technicalities of the paper lie in the proofs of the necessity results, and their details are relegated to an Appendix so that they may not interfere with the reader primarily interested in the substantive contribution of this work.

## 2 The Two-Sector RSS Model

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in [25, 26], the collection of production plans $(x, x')$, the amount $x'$ of machines in the next period (tomorrow) from the amount $x$ available in the current period (today), is given by the *transition possibility set.* Here it takes the specific form

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\},$$

where $z \equiv (x' - (1-d)x)$ is the number of machines that are produced, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on the irreversibility of investment and the use of labor. Associated with $\Omega$ is the transition correspondence, $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\Gamma(x) = \{x' \in \mathbb{R}_+ : (x, x') \in \Omega\}$. For any $(x, x') \in \Omega$, one can also consider the amount $y$ of the machines available for the production of the consumption good, leading to a correspondence

$$\Lambda : \Omega \longrightarrow \mathbb{R}_+ \text{ with } \Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1-d)x)\}.$$

Welfare is derived only from the consumption good and is represented by a linear function, normalized so that $y$ units of the consumption good yields a welfare level $y$. A *reduced form utility function*

$$u : \Omega \to \mathbb{R}_+ \text{ with } u(x, x') = \max\{y \in \Lambda(x, x')\}$$

indicates the maximum welfare level that can be obtained today, if one starts with $x$ of machines today, and ends up with $x'$ of machines tomorrow, where $(x, x') \in \Omega$. Intertemporal preferences are represented by the present value of the stream of welfare levels, using a discount factor $\rho \in (0, 1)$.

A *2-sector RSS model* $\mathcal{G}$ consists of a triple $(a, d, \rho)$, and the following concepts apply to it. A *program from* $x_o$ is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max \Lambda((x(t), x(t+1))$. A *program* $\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t+1)\}$, defined by $z(t+1) = (x(t+1) - (1-d)x(t))$ for all $t \in \mathbb{N}$. It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $\max\{x(0), 1/ad\} \equiv M(x(0))$, and so $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \infty$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from $x_o$ is called *optimal* if

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

for every program $\{x(t), y(t)\}$ from $x_o$. A program $\{x(t), y(t)\}$ is called *stationary* if for all $t \in \mathbb{N}$, we have $(x(t), y(t)) = (x(t+1), y(t+1))$. A *stationary optimal program* is a program that is stationary and optimal.

The parameter $\xi = (1/a) - (1-d)$ plays an important role in the subsequent analysis. It represents the marginal rate of transformation of capital today into that of tomorrow, given full employment of both factors. In what follows, and without further mention, we always assume that the parameters $(a, d)$ of the RSS model are such that

$$\xi > 1 \Longrightarrow a \in (0, 1). \tag{2}$$

For more details, technical and bibliographic, the reader is referred to Khan-Mitra [14] and its further elaboration in [15, 19]. For the basic geometric representation of the model, see Figs. 1 and 2 also detailed in [16, 17] and their references.

### 2.1 Constructs from Dynamic Programming

Using standard methods of dynamic programming, one can establish that there exists an optimal program from every $x \in X \equiv [0, \infty)$, and then use it to define a *value function*, $V : X \to \mathbb{R}$ by:

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)), \tag{3}$$

where $\{\bar{x}(t), \bar{y}(t)\}$ is an optimal program from $x$. Then, it is straightforward to check that $V$ is concave, non-decreasing and continuous on $X$. Further, it can be verified that $V$ is, in fact, increasing on $X$; see [14] for the verification.

It can be shown that for each $x \in X$, the Bellman equation

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\} \tag{4}$$

holds. For each $x \in X$, we denote by $h(x)$ the set of $x' \in \Gamma(x)$ which maximize $\{u(x, x') + \delta V(x')\}$ among all $x' \in \Gamma(x)$. That is, for each $x \in X$,

$$h(x) = \arg \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}.$$

Then, a program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from $x$ if and only if it satisfies the equation

$$V(x(t)) = u(x(t), x(t+1)) + \delta V(x(t+1) \text{ for } t \geq 0;$$

that is, if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$. We call $h$ the *optimal policy correspondence (OPC)*. When this correspondence is a function, we refer to it as the *optimal policy function (OPF)*.

It is easy to verify, using $\rho \in (0, 1)$, that the function $V$, defined by (3), is the *unique* continuous function on $Z \equiv [0, (1/ad)]$ which satisfies the functional equation of dynamic programming, given by (4).

### 2.2 The Modified Golden Rule

A *modified golden rule* is a pair $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega. \tag{MGR}$$

The existence of a modified golden-rule has already been established in [16, 19]. We reproduce that result here (without proof) for ready reference. A distinctive feature of our model is that we can describe the modified golden-rule stock explicitly in terms of the parameters of the model, and that it is independent of the discount factor.

**Proposition 1** *Define* $(\hat{x}, \hat{p}) = (1/(1+ad),\ 1/(1+\rho\xi))$. *Then* $(\hat{x}, \hat{x}) \in \Omega$, *where* $\hat{x}$ *is independent of* $\rho$, *and satisfies (MGR).*

The connection between the value function and the modified golden-rule may be noted as follows. Given a modified golden-rule $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$, we know that $\hat{x}$ is a stationary optimal stock (see, for example, [25, p. 1305]. Consequently, it is easy to verify that $V(\hat{x}) = \hat{x}/(1-\rho)$ and that

$$V(x) - \hat{p}x \leq V(\hat{x}) - \hat{p}\hat{x} \ \text{ for all } x \geq 0. \tag{5}$$

On choosing $x = \hat{x}+\varepsilon$, with $\varepsilon > 0$, in (5), and letting $\varepsilon \to 0$, we obtain $V'_+(\hat{x}) \leq \hat{p}$, and hence from (MGR),

$$V'_+(\hat{x}) \leq \hat{p} = 1/(1+\rho\xi) < (a/\rho). \tag{6}$$

## 2.3 A Failure of Strict Concavity

As emphasized in the introduction, a key element of the class of intertemporal allocation models studied in the literature in the substantive context of this work is that the reduced form utility function $u$ exhibits some form of strict concavity on its domain. As has been well-understood, this assumption fails in the RSS model that we study here. We provide a formal argument for the reader new to the model. Consider $x, \bar{x}$ with $1 < x < \bar{x} < k$, and $(x', \bar{x}') = (1-d)(x, \bar{x})$. Then, $(x, x') \in \Omega$, and $(\bar{x}, \bar{x}') \in \Omega$, and $u(x, x') = 1 = u(\bar{x}, \bar{x}')$. One can now choose $\tilde{x} = \lambda x + (1-\lambda)\bar{x}$ and $\tilde{x}' = \lambda x' + (1-\lambda)\bar{x}'$ with any $\lambda \in (0, 1)$. Then, it is easy to check that $\tilde{x}' = (1-d)\tilde{x}$, and $(\tilde{x}, \tilde{x}') \in \Omega$, and $u(\tilde{x}, \tilde{x}') = 1$. Thus, while $u$ is concave on $\Omega$, as noted above, it is not strictly concave in either the first or the second argument.

## 2.4 Basic Properties of the OPC

The basic properties of the OPC, with no additional restrictions on the parameters of our model, have already been described in [19]. We summarize these properties below. This helps us to present an explicit solution of the optimal policy correspondence in the next section.

To this end, we describe three regions of the state space; see Fig. 1.

$$A = [0, \hat{x}],\ \ B = (\hat{x}, k),\ \ C = [k, \infty)$$

where $k = \hat{x}/(1-d)$. In addition, we define a function, $g : X \to X$, by:

$$g(x) = \begin{cases} (1-d)x & \text{for } x \in C \\ \hat{x} & \text{for } x \in B \\ (1/a) - \xi x & \text{for } x \in A \end{cases}$$

We refer to $g$ as the pan-map, in view of the fact that its graph resembles a pan. In Figs. 1 and 2, it is given by $VGG_1D$.

We further subdivide the region $B$ into two regions as follows:

$$D = (\hat{x}, 1), \;\; E = [1, k)$$

and define a correspondence, $G : X \to X$, by:

$$G(x) = \begin{cases} \{(1-d)x\} & \text{for } x \in C \\ [(1-d)x, \hat{x}] & \text{for } x \in E \\ [(1/a) - \xi x, \hat{x}] & \text{for } x \in D \\ \{(1/a) - \xi x\} & \text{for } x \in A \end{cases} \tag{7}$$

**Proposition 2** *The optimal policy correspondence, $h$, satisfies:*

$$h(x) \subset \begin{cases} \{g(x)\} \; \textit{for all } x \in A \cup C \\ G(x) \quad \textit{for all } x \in B \end{cases} \tag{8}$$

It should be clear from this result that the only part of the optimal policy correspondence for which we do *not* have an explicit solution is for the middle region of stocks, given by $B = (\hat{x}, k) = D \cup E$; see Fig. 2.

Two useful implications of Proposition 2 are that (i) one must have positive optimal consumption levels in all programs that start from positive capital stocks, and (ii) the slope of the value function cannot exceed unity.

**Corollary 1**

*(i) If $\{x(t), y(t)\}$ is an optimal program from $x_o > 0$, then $y(0) > 0$.*
*(ii) If $0 < z' < z < \infty$, then*

$$\frac{V(z) - V(z')}{z - z'} \leq 1 \tag{9}$$

***Proof*** To see (i), note that for $x_o \in (0, \hat{x}]$, (8) implies that $y(0) = x_o > 0$, while for $x_o \in [k, \infty)$, (8) implies that $y(0) = 1$. For $x_o \in (\hat{x}, k)$, (8) implies that if $x' \in h(x_o)$, then $x' \leq \hat{x}$, and this means that $\hat{x} \in \Lambda(x_o, x')$. Thus $y(0) \geq \hat{x} > 0$.

To see (ii), pick any $x > 0$, and let $\{x(t), y(t)\}$ be an optimal program from $x > 0$. Since $y(0) > 0$ by (i), we can choose $0 < x' < x$, so that $\varepsilon \equiv [x - x'] < y(0)$, and define $\tilde{y}(0) = y(0) - \varepsilon > 0$.

Note that $x(1) \geq (1-d)x(0) \geq (1-d)x'$, and:

$$\begin{aligned}\tilde{y}(0) + a[x(1) - (1-d)x'] &= \tilde{y}(0) + a[x(1) - (1-d)x(0)] + a(1-d)\varepsilon \\ &= \tilde{y}(0) + 1 - y(0) + a(1-d)\varepsilon \\ &= 1 - \varepsilon[1 - a(1-d)] < 1,\end{aligned}$$

so that $(x', x(1)) \in \Omega$ and $\tilde{y}(0) \in \Lambda(x', x(1))$. Since $V(x')$ is at least as large as the sum of discounted utilities generated by the program $(x', x(1), x(2), \ldots)$, we have:

$$V(x) - V(x') \leq [y(0) - \tilde{y}(0)] = \varepsilon = (x - x').$$

This yields the desired bound on the slope of the value function, namely $(V(x) - V(x')/(x - x') \leq 1$. Since $V$ is concave on $\mathbb{R}_+$, (9) follows. ■

## 3 An Explicit Solution of the OPF

In this section we present an explicit solution of the optimal policy function when the discount factor is smaller than the labor-output ratio in the investment good sector. Specifically, we show that in this case, the map

$$H(x) = \begin{cases} (1/a) - \xi x & \text{for } x \in [0, 1] \\ (1-d)x & \text{for } x \in (1, \infty) \end{cases} \tag{10}$$

is the $OPF$. We refer to the map $H$ as a check-map.[16]

**Proposition 3** *Suppose the RSS model* $(a, d, \rho)$ *satisfies* $\rho < a$. *Then, its optimal policy correspondence,* $h$, *is the function given by* $H$ *in (10).*

***Proof*** Using Proposition 2, it is clear that we only need to show that $H$, given by (10), is the $OPF$ for $x \in (\hat{x}, k)$. To this end, let us define $c : X \to X$ by:

$$c(x) = \begin{cases} x & \text{for } x \in (\hat{x}, 1) \\ 1 & \text{for } x \in [1, k) \end{cases}$$

Note that $H(x) \geq (1-d)x > 0$ and $c(x) > 0$ for all $x \in (\hat{x}, k)$. Also, for $x \in (\hat{x}, 1)$, we have $c(x) = x$, and so

$$c(x) + a[H(x) - (1-d)x] = x + a[(1/a) - \xi x - (1-d)x] = x + 1 - a(1/a)x = 1.$$

[16] See Footnote 8, and the text it footnotes.

Thus, $(x, H(x)) \in \Omega$ and $c(x) \in \Lambda(x, H(x))$. For $x \in [1, k)$, we have $c(x) = 1$, and so

$$c(x) + a[H(x) - (1-d)x] = 1 + a[(1-d)x - (1-d)x] = 1.$$

Thus, we have again $(x, H(x)) \in \Omega$ and $c(x) \in \Lambda(x, H(x))$.

Let $\{x(t), y(t)\}$ be an optimal program from $x > 0$. We establish that $x(1) = H(x)$ for $x \in (\hat{x}, k)$. We know from Proposition 2 that $x(1) \geq H(x)$. So, it remains to rule out $x(1) > H(x)$. To this end we break up the verification into two parts corresponding to the two ranges of $x$, namely, $(i)$ $x \in (\hat{x}, 1)$ and $(ii)$ $x \in [1, k)$.

We begin with case (i). Suppose $x(1) = H(x) + \varepsilon$, where $\varepsilon > 0$. Note that $y(0) + a[x(1) - (1-d)x] \leq 1 = x + a[g(x) - (1-d)x]$ so that $y(0) \leq x + a[g(x) - x(1)] = x - a\varepsilon$. Using the optimality principle, we obtain

$$\begin{aligned} V(x) = y(0) + \rho V(x(1)) &\leq x - a\varepsilon + \rho[V(x(1)) - V(H(x))] + \rho V(H(x)) \\ &\leq x - a\varepsilon + \rho\varepsilon + \rho V(H(x)) < x + \rho V(H(x)), \quad (11) \end{aligned}$$

the second inequality following from Corollary 1, and the last inequality following from the fact that $\rho < a$ and $\varepsilon > 0$. But, since $(x, H(x)) \in \Omega$ and $c(x) = x \in \Lambda(x, H(x))$, we must have $V(x) \geq x + \rho V(H(x))$, which contradicts (11).

Next we turn to case (ii). Suppose $x(1) = H(x) + \varepsilon$, where $\varepsilon > 0$. Note that

$$\begin{aligned} y(0) + a[x(1) - (1-d)x] &= [y(0) - 1] + 1 + a[H(x) + \varepsilon - (1-d)x] \\ &= [y(0) - 1 + a\varepsilon] + 1, \end{aligned}$$

so that $y(0) \leq 1 - a\varepsilon$. Using the optimality principle,

$$\begin{aligned} V(x) = y(0) + \rho V(x(1)) &\leq 1 - a\varepsilon + \rho[V(x(1)) - V(H(x))] + \rho V(H(x)) \\ &\leq 1 - a\varepsilon + \rho\varepsilon + \rho V(H(x)) < 1 + \rho V(H(x)), \quad (12) \end{aligned}$$

the second inequality following from an analogue of Corollary 1, and the last inequality following from the fact that $\rho < a$ and $\varepsilon > 0$. But, since $(x, H(x)) \in \Omega$ and $c(x) = 1 \in \Lambda(x, H(x))$, we must have $V(x) \geq 1 + \rho V(H(x))$, which contradicts (12). ■

*Remark* Our sufficient condition ($\rho < a$) for an explicit solution of the $OPF$ as the check map (given by (10)) does not directly involve the depreciation factor, $d$. In view of this, one should not expect this sufficient condition to be a sharp one, even for the instances delineated in $(T)$ and $(PT)$ above. In particular, it has already been established in [16, 17] that for the case $\xi \leq 1/(1-d)$, the optimal policy function is the check map whenever $\rho < (1/\xi)$. Since $(1/\xi) = (a/(1 - a(1-d)) > a$, this shows that when $\xi \leq 1/(1-d)$, the $OPF$ is the check-map even for $\rho \in (a, (1/\xi))$.

## 4 Optimal Topological Chaos

With the optimal policy function explicitly determined (albeit in the specific case of the discount factor $\rho$ being less than the labor-output coefficient $a$), we can provide robust sets of parameter configurations for which (1) the second iterate of the optimal policy function exhibits turbulence, and (2) the optimal policy function satisfies the Li-Yorke condition. The set of parameter configurations for which (1) holds (stated in $(T)$ above) generalizes the result obtained in [15]. The set of parameter configurations for which (2) holds (stated in $(PT)$ above is clearly stronger than $(T)$. Both sets of parameter configurations ensure that the optimal policy function exhibits topological chaos.

We recall a few definitions relating to the concepts appearing in the previous paragraph. Let $X$ be a compact interval of the reals $\mathbb{R}$, and $f$ a continuous function from $X$ to itself. The pair $(X, f)$ is said to be a *dynamical system* with *state space* $X$ and *law of motion* $f$. A dynamical system $(X, f)$ is said to be *turbulent* if there exist points $a, b, c$ in $X$ such that

$$f(b) = f(a) = a, \ \ f(c) = b, \ \text{and either } a < c < b \text{ or } a > c > b$$

(see Fig. 3). It satisfies the *Li-Yorke condition* if there exists $x^* \in X$ such that

$$f^3(x^*) \leq x^* < f(x^*) < f^2(x^*) \text{ or } f^3(x^*) \geq x^* > f(x^*) > f^2(x^*).$$

The *topological entropy* of a dynamical system $(X, f)$ is denoted by $\psi(X, f)$, and the dynamical system itself is said to exhibit *topological chaos* if its topological entropy is positive.

**Proposition 4** *Suppose the RSS model $(a, d, \rho)$ satisfies $\rho < a$, and $(T)$ above. Then, the optimal policy correspondence, $h$, is the function given by $H$ in (10), and $h^2$ is turbulent.*

***Proof*** The proof naturally splits up into three parts. The first part involves verifying that

$$H^2(1) \geq k \Longleftrightarrow \left[\xi - \frac{1}{\xi}\right](1 - d) \geq 1$$

where $H$ is the check map, given by (10), and the right hand side of the implication is $(T)$. The second part involves showing that, when $(T)$ is satisfied, $f$ is turbulent, where $f(x) = H^2(x)$ for all $x \in \mathbb{R}_+$. The third part is to observe that when we combine these two parts with Proposition 3 we can conclude that when $\rho < a$, and $(T)$ holds, then the optimal policy correspondence $h$ is a function, given by the check map $H$, and $h^2$ is turbulent.

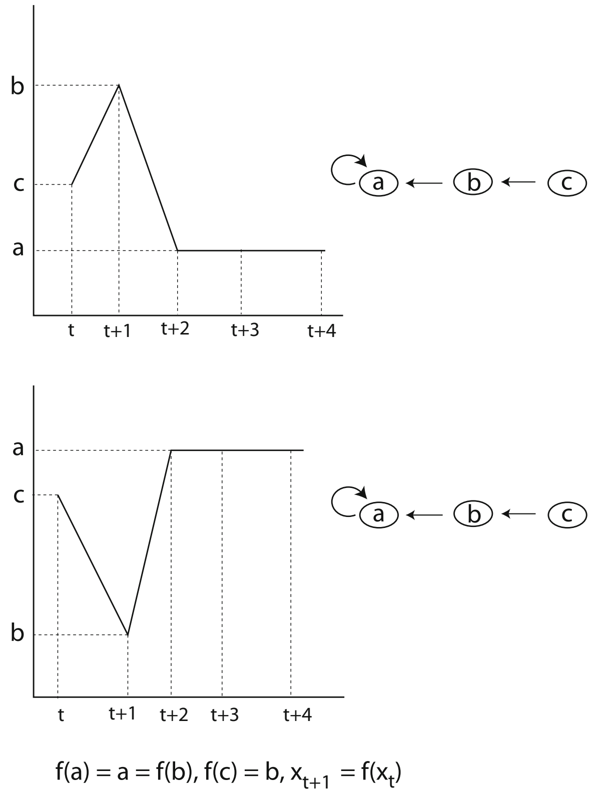


**Fig. 3** Turbulence of f(.)

For the first part, let us define the closed intervals (see Fig. 1),

$$J_1 = [1 - d, \hat{x}];\ J_2 = [\hat{x}, 1];\ J_3 = [1, k],$$

and denote $H^2(1)$ by $k'$. Denote the length of the interval $J_2$ by $\theta$. Notice that $H$ maps $J_2$ onto $J_1$, and the relevant slope for this domain is $(-\xi)$, so that the length of $J_1$ is $\xi\theta$. Further, $H$ maps $J_3$ onto $J_1$, and the relevant slope for this domain is $(1-d)$, so that the length of $J_3 = \xi\theta/(1-d)$. Thus, the length of $J_2 \cup J_3 = [\hat{x}, k]$ is $\{\theta + [\xi\theta/(1-d)]\}$. On the other hand, $H$ maps $J_1$ onto $[\hat{x}, k']$, and the relevant slope for this domain is $(-\xi)$, so that $k' > \hat{x}$, and $[k' - \hat{x}] = \xi^2\theta$. Thus, we obtain

$$k' \geq k \Longleftrightarrow \xi^2 \geq 1 + \frac{\xi}{(1-d)}. \tag{13}$$

One can rewrite the right-hand inequality in (13) as

$$1 \geq \frac{1}{\xi^2} + \frac{1}{\xi(1-d)} = \frac{1}{\xi}\left[\frac{1}{\xi} + \frac{1}{(1-d)}\right],$$

which is equivalent to

$$\xi \geq \left[\frac{1}{\xi} + \frac{1}{(1-d)}\right] \Longleftrightarrow \left(\xi - \frac{1}{\xi}\right)(1-d) \geq 1 \Longleftrightarrow (T),$$

thereby completing the first part of the demonstration.

For the second part, note that when $(T)$ holds, we have $H^2(1) \geq k$, while $H^2(\hat{x}) = \hat{x} < k$. Thus, by continuity of $H$, there is $z \in (\hat{x}, 1]$ such that $H^2(z) = k$. Defining $F(x) = H^2(x)$ for all $x \in \mathbb{R}_+$, we obtain $F(z) = k$, $F(k) = \hat{x} = F(\hat{x})$ and $\hat{x} < z < k$. This implies that the function $F$ is turbulent (see [4, p. 25]).

For the third part, assume that $\rho < a$. Then, by Proposition 3, the optimal policy correspondence $h$ is a function, given by the check map $H$. If in addition $(T)$ holds, then $h^2 = H^2$ is turbulent. ■

*Remark* When $h^2$ is turbulent $h^2$ has periodic points of all periods (see [4, Lemma 3, p. 26]). In particular, $h^2$ has a period three point, and so $h$ has a period six point. The fact that $h^2$ is turbulent implies that the topological entropy of $h^2$ , $\psi(h^2) \geq \ln 2$. This in turn implies that the topological entropy of $h$, $\psi(h) = (1/2)\psi(h^2) \geq \ln\sqrt{2}$ $> 0$, so that $h$ exhibits topological chaos.

**Proposition 5** *Suppose the RSS model $(a, d, \rho)$ satisfies $\rho < a$, and $(PT)$ above. Then, the optimal policy correspondence, $h$, is the function given by $H$ in (10), and $h$ satisfies the Li-Yorke condition.*

***Proof*** The proof again naturally splits up into three parts. The first part involves verifying that

$$H^2(1) \geq \frac{1}{1-d} \Longleftrightarrow [\xi - 1](1-d) \geq 1,$$

where $H$ is the check map, given by (10), and the right hand side of the implication is $(PT)$. The second part involves showing that, when (PT) is satisfied, $H$ satisfies the Li-Yorke condition. The third part is to observe that when we combine these two parts with Proposition 3 we can conclude that when $\rho < a$, and $(PT)$ holds, then the optimal policy correspondence $h$ is a function, given by the check map $H$, and $h$ satisfies the Li-Yorke condition.

For the first part, let us define the closed intervals (mark on Fig. 2),

$$I_1 = [1-d, \hat{x}];\ I_2 = [\hat{x}, 1];\ I_3 = [1, \tilde{k}],$$

where $\tilde{k} = [1/(1-d)] > k$,and denote $H^2(1)$ by $k''$. Denote the length of the interval $I_2$ by $\theta$. Notice that $H$ maps $I_2$ onto $I_1$, and the relevant slope for this domain is $(-\xi)$, so that the length of $I_1$ is $\xi\theta$. Further, $H$ maps $I_3$ onto $[1-d, 1] = I_1 \cup I_2$, and the relevant slope for this domain is $(1-d)$, so that the length of $I_3 = (\xi+1)\theta/(1-d)$. On the other hand, $H$ maps $I_1$ onto $[\hat{x}, k'']$, and the relevant slope for this domain is $(-\xi)$, so that $k'' > \hat{x}$, and $[k'' - \hat{x}] = \xi^2\theta$. Thus, we obtain

$$k'' \geq \tilde{k} \Longleftrightarrow \xi^2 \geq 1 + \frac{\xi + 1}{(1-d)} \tag{14}$$

One can rewrite the right-hand inequality in (14) as

$$\xi^2 - 1 \geq \frac{\xi + 1}{(1-d)} \Longleftrightarrow (\xi - 1)(1-d) \geq 1 \Longleftrightarrow (PT),$$

thereby completing the first part of our demonstration.

For the second part, note that we have $H(\tilde{k}) = (1-d)\tilde{k} = 1 < \tilde{k}$, and $H^2(\tilde{k}) = H(1) = (1-d) < H(\tilde{k})$. Thus, when (PT) holds, $H^3(\tilde{k}) = H^2(1) \geq [1/(1-d)] = \tilde{k}$, and Thus, we obtain

$$H^3(\tilde{k}) \geq \tilde{k} > H(\tilde{k}) > H^2(\tilde{k}),$$

which clearly means that the Li-Yorke condition is satisfied.

For the third part, assume that $\rho < a$.Then, by Proposition 3, the optimal policy correspondence $h$ is a function, given by the check map $H$. If in addition (PT) holds, then $h = H$ satisfies the Li-Yorke condition. ■

*Remark* When $h$ satisfies the Li-Yorke condition, $h$ has periodic points of all periods; see [21]. In particular, $h$ has a period three point. The fact that $h$ has a period three point implies by a result of [5] that the topological entropy of $h$ , $\psi(h) \geq \ln[(\sqrt{5}+1)/2] > 0$, so that $h$ exhibits topological chaos.

## 5 Technological Restrictions for Optimal Topological Chaos

We present "necessary and sufficient" conditions on technology, the so-called *technological restrictions,* for optimal topological chaos conceived as optimal period-three cycles and as optimal turbulence.

### *5.1 Technological Restrictions for Optimal Period-Three Cycles*

We begin with the sufficiency result.

**Proposition 6** *Let* $0 < a < (1/3)$. *Then, there exist* $\rho \in (0, 1)$ *and* $d \in (0, 1)$ *such that the two-sector RSS model with parameters* $(\rho, a, d)$ *has an optimal policy function,* $h$, *which generates a period-three cycle.*

***Proof*** Given $0 < a < (1/3)$, choose any $\rho \in (0, a)$. Then, since $\rho < a$, the analysis presented in Sect. 3 implies that for every $d \in (0, 1)$, the two-sector RSS model with parameters $(\rho, a, d)$ has an optimal policy function, $h$, given by the check map, $H$.

We now proceed to choose $d \in (0, 1)$ such that the optimal policy function $h$ exhibits a period-three cycle. Towards this end, define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(z) = (1 - z)^3 + [z(1 - z)/a],$$

and observe the following:

$$f(0) = 1, \; f(1) = 0 \text{ and } f'(z) = -3(1 - z)^2 + (1/a)(1 - 2z).$$

Since $a \in (0, (1/3))$ guarantees that $f'(0) = -3 + (1/a) > 0$, we have, for $z$ positive and close enough to 0, $f(z) > 1$. Since $f(1) < 1$, we can appeal to the intermediate value theorem, to assert the existence of $d \in (0, 1)$ for which $f(d) = 1$. This means that the RSS model with parameters $(\rho, a, d)$ satisfies:

$$(1 - d)^3 + [d(1 - d)/a] = 1. \tag{15}$$

Since the RSS model $(\rho, a, d)$ has the optimal policy function $h = H$, (15) implies

$$h^2(1) = H^2(1) \equiv (1 - d)^2 + (d/a) = [1/(1 - d)].$$

Since $H[1/(1 - d)] = 1$, we have $h^3(1) = h(h^2(1)) = 1$, and we obtain the period-three cycle

$$h(1) = 1 - d, \; h^2(1) = [1/(1 - d)], \; h^3(1) = 1. \qquad \blacksquare$$

Next, we turn to the necessity result

**Proposition 7** *Let $(\rho, a, d)$ be the parameters of a two-sector RSS model such that there is an optimal policy function h which generates a period-three cycle from some initial stock. Then $a < (1/3)$.*

***Proof*** Denote the optimal policy function by $h$, and the period-three cycle stocks by $\alpha, \beta, \gamma$. Without loss of generality we may suppose that $\alpha < \beta < \gamma$. There are then two possibilities to consider: (1) $\beta = h(\alpha)$, (2) $\gamma = h(\alpha)$.

In case (i), we must have $\alpha \in A$, and $\alpha \neq \hat{x}$ since $\beta > \alpha$. Consequently, $\beta = (1/a) - \xi\alpha$, and $\gamma \neq h(\alpha)$. Thus, we must have $\gamma = h(\beta)$, and since $\gamma > \beta$, we must have $\beta \in A$. But, since $\beta = (1/a) - \xi\alpha$ with $\alpha \in A$, $\alpha \neq \hat{x}$, we must have $\beta \in B \cup C$, a contradiction. Thus, case (i) cannot occur.

Thus case (ii) must occur. In this case, since $\gamma > \alpha$, we must have $\alpha \in A$, and $\alpha \neq \hat{x}$. Consequently, $\gamma = (1/a) - \xi\alpha$, and $\beta \neq h(\alpha)$. Thus, we must have $\beta = h(\gamma)$; it also follows that we must have $\alpha = h(\beta)$. Since $\beta < \gamma$, we must have

$\gamma \in B \cup C$; similarly, since $\alpha < \beta$, we must have $\beta \in B \cup C$. Also, note that $\beta$ cannot be $\hat{x}$ or $k$, and similarly $\gamma$ cannot be $\hat{x}$ or $k$.

We claim now that $\gamma > k$. For if $\gamma \leq k$, then, we must have $\hat{x} < \beta < \gamma < k$. But, then, $\gamma \in (\hat{x}, k)$, and so $\beta = h(\gamma)$ implies by (8) that $\beta \leq \hat{x}$. Since $\beta \neq \hat{x}$, we must have $\beta \in A$, a contradiction. Thus, the claim that $\gamma > k$ is established. But, then, by (8), we can infer that $\beta = (1-d)\gamma$.

We claim, next, that $\beta \in (\hat{x}, k)$. Since $\beta \in B \cup C$, and $\beta$ cannot be $\hat{x}$ or $k$, we must have $\beta > k$ if the claim is false. But, then, by (8), we can infer that $\alpha = (1-d)\beta > \hat{x}$, a contradiction, since $\alpha \in A$. Thus, our claim that $\beta \in (\hat{x}, k)$ is established.

Since $\beta \in (\hat{x}, k)$ and $\alpha = h(\beta)$, we can infer from (8) that $\alpha \geq (1-d)$. To summarize our findings so far, we have:

$$(i) \;\; \gamma > k > \beta > \hat{x} > \alpha \geq (1-d) \text{ and } (ii) \;\; \gamma = (1/a) - \xi\alpha, \beta = (1-d)\gamma, \tag{16}$$

from which we can infer that

$$\gamma = (1/a) - \xi\alpha \leq (1/a) - \xi(1-d) \text{ and } \beta = (1-d)\gamma \leq [(1-d)/a] - \xi(1-d)^2.$$

On simplifying the right-hand side of the inequality for $\beta$, we obtain the important inequality

$$\beta \leq [d(1-d)/a] + (1-d)^3. \tag{17}$$

Now suppose that contrary to the assertion of the Proposition, we have $a \geq (1/3)$. Then we can appeal to Lemma 1 in the Appendix to conclude that

$$\beta \leq [d(1-d)/a] + (1-d)^3 < 1. \tag{18}$$

Clearly, (18) implies that $\beta \in (\hat{x}, 1)$. Thus, by (8), $\alpha = h(\beta) \geq (1/a) - \xi\beta$, while $\hat{x} = (1/a) - \xi\hat{x}$, so that

$$(\hat{x} - \alpha) \leq \xi(\beta - \hat{x}). \tag{19}$$

Using (16)(ii), we have $(\beta - \hat{x}) = (1-d)(\gamma - k) \leq (1-d)(\gamma - \hat{x})$. Also, using (16)(i), we have $(\gamma - \hat{x}) = \xi(\hat{x} - \alpha)$, so that

$$(\beta - \hat{x}) \leq \xi(1-d)(\hat{x} - \alpha). \tag{20}$$

Combining (19) and (20) yields

$$\xi^2(1-d) \geq 1. \tag{21}$$

We now use (18) in conjunction with (21) to complete the argument. Since $\beta \in (\hat{x}, 1)$, (8) yields $\alpha = h(\beta) \geq [(1/a) - \xi\beta]$ and

$$\gamma = (1/a) - \xi\alpha \Longrightarrow \gamma \leq (1-\xi)(1/a) + \xi^2\beta.$$

Finally, since $\beta = (1-d)\gamma$, the above inequality for $\gamma$ implies

$$\beta \leq (1-d)(1-\xi)(1/a) + (1-d)\xi^2\beta \Longrightarrow (\xi^2(1-d)-1)\beta \geq (\xi-1)(1-d)(1/a).$$

By (1) and the specification of the two-sector RSS model, the left hand side cannot be zero. And so by (21), it is positive. On appealing to the identities presented as Lemmas 2 and 3, we conclude that $\beta \geq 1$, and contradict (18), establishing the Proposition. ■

### *5.2 Technological Restrictions for Optimal Turbulence*

We look at technological restrictions on the RSS model when complicated behavior takes the form of the second iterate of the optimal policy function exhibiting turbulence.

**Proposition 8** *(i) Let $(a, d)$ be such that $(T)$ holds. Then, there exist $\rho \in (0, 1)$ such that the two-sector RSS model with parameters $(\rho, a, d)$ has an optimal policy function, $h$, whose second iterate exhibits turbulence. (ii) Let $(\rho, a, d)$ be the parameters of a two-sector RSS model such that there is an optimal policy function $h$ whose second iterate exhibits turbulence. Then $(T)$ holds.*

***Proof*** We provide a proof of part (i) of the proposition and relegate the proof of part (ii) to the Appendix. Towards this end, given $(a, d)$ satisfying $(T)$, choose any $\rho \in (0, a)$. Then, since $\rho < a$, the analysis presented in Sect. 3 implies that the two-sector RSS model with parameters $(\rho, a, d)$ has an optimal policy function, $h$, given by the check map, $H$. Then, by Proposition 4, $h^2$ exhibits turbulence. ■

## 6 Discount-Factor Restrictions for Optimal Topological Chaos

We present "necessary and sufficient" conditions on the discount factor, the so-called *discount-factor restrictions*, for optimal topological chaos conceived as optimal period-three cycles and as optimal turbulence. These conditions are not exact in the sense that they are for the technological restrictions presented in Sect. 5.

## 6.1 *Discount-Factor Restrictions for Optimal Period-Three Cycles*

As in Sect. 5, we begin with period-three cycles and then turn to turbulence.

**Proposition 9** *(i) Let* $0 < \rho < (1/3)$. *Then, there exist* $a \in (0, 1)$ *and* $d \in (0, 1)$ *such that the two-sector RSS model with parameters* $(\rho, a, d)$ *has an optimal policy function,* $h$, *which generates a period-three cycle. (ii) Let* $(\rho, a, d)$ *be the parameters of a two-sector RSS model such that there is an optimal policy function* $h$ *which generates a period-three cycle from some initial stock. Then* $\rho < (1/2)$.

***Proof*** We begin with the proof of (i). Given $\rho \in (0, 1/3)$, pick $a \in (\rho, 1/3)$, and choose $d \in (0, 1)$ such that condition $(PT)$ is satisfied. Towards this end, define a function $f : [0, 1] \to \mathbb{R}$ by:

$$f(z) = (1 - z)^3 + [z(1 - z)/a],$$

and observe the following

$$f(0) = 1, \ f(1) = 0 \text{ and } f'(z) = -3(1 - z)^2 + (1/a)(1 - 2z).$$

Since $a \in (0, (1/3))$ guarantees that $f'(0) = -3 + (1/a) > 0$, we have, for $z$ positive and close enough to 0, $f(z) > 1$. Since $f(1) < 1$, we can appeal to the intermediate value theorem, to assert the existence of $d \in (0, 1)$ for which $f(d) = 1$. This means that the RSS model with parameters $(a, d)$ satisfies

$$(1 - d)^3 + [d(1 - d)/a] = 1 \tag{22}$$

Using (22), we obtain:

$$(d/a) + (1 - d)^2 = H^2(1) = [1/(1 - d)]$$

so that by the equivalence in the proof of Proposition 5, we obtain $(\xi - 1)(1 - d) = 1$, and ensure that $(PT)$ is satisfied. Since $\rho < a$, the analysis presented in Sect. 3 implies that the two-sector RSS model with parameters $(\rho, a, d)$ has an optimal policy function, $h$, given by the check map, $H$. Then, by Proposition 5, $h$ satisfies the Li-Yorke Condition, and therefore $h$ has a period-three cycle.

Next we turn to the proof of (ii). Towards this end, we claim that $\rho\xi \leq 1$. Suppose to the contrary, we have $\rho\xi > 1$. Then, the RSS model $(a, d, \rho)$ has an optimal policy function, $h$, given by the pan map. Since the $OPF$ generates a period-three cycle, let us denote the cycle by $\alpha, \beta, \gamma$, and without loss of generality suppose that $\alpha < \beta < \gamma$. Clearly, none of these values can be equal to $\hat{x}$.

We have either (a) $h(\alpha) = \beta$, or (b) $h(\alpha) = \gamma$. In case (a), noting that $\beta > \alpha$, we must have $\alpha \in A$. In this case, since $\alpha \neq \hat{x}$, $\beta \in B \cup C$. Since $h(\alpha) = \beta$, we must have $h(\beta) = \gamma$; and, since $\gamma > \beta$, we must have $\beta \in A$, a contradiction.

Thus, (a) cannot occur. In case (b), we must have $h(\gamma) = \beta$ and so $h(\beta) = \alpha$. Since $h(\alpha) = \gamma$ and $\gamma > \alpha$, we must have $\alpha \in A$, and since $\alpha \neq \hat{x}$, $\gamma \in B \cup C$. But since $h$ is a pan map, we must have $h(\gamma) \geq \hat{x}$; that is $\beta \geq \hat{x}$. Thus, since $\beta \neq \hat{x}$, $\beta \in B \cup C$ as well, and $h(\beta) \geq \hat{x}$; that is, $\alpha \geq \hat{x}$. Since $\alpha \in A$, we must then have $\alpha = \hat{x}$, a contradiction. Thus, case (b) cannot occur, and this establishes our claim.

Proposition 7 guarantees that $(PT)$ holds, thereby implying

$$(\xi - 1)(1 - d) \geq 1 \Longleftrightarrow \xi \geq 1 + \frac{1}{(1 - d)} > 2,$$

and furnishing the required conclusion that $\rho < (1/2)$. ■

## 6.2 Discount-Factor Restrictions for Optimal Turbulence

We show that if $\rho < \mu^{3/2}$, then there exist $(a, d)$, such that the RSS model $(a, d, \rho)$ has an optimal policy function whose second iterate exhibits turbulence. Conversely, if the RSS model $(a, d, \rho)$ has an optimal policy function whose second iterate exhibits turbulence, then $\rho < \mu$. Here $\mu$ is given by:

$$\mu = \frac{\sqrt{5} - 1}{2}$$

**Proposition 10** *(i) Let* $0 < \rho < \mu^{3/2}$*. Then, there exist* $a \in (0, 1)$ *and* $d \in (0, 1)$ *such that the two-sector RSS model with parameters* $(\rho, a, d)$ *has an optimal policy function,* $h$*, whose second iterate exhibits turbulence. (ii) Let* $(\rho, a, d)$ *be the parameters of a two-sector RSS model such that there is an optimal policy function* $h$ *whose second iterate exhibits turbulence. Then,* $\rho < \mu$*.*

***Proof*** We begin with the proof of (i). Towards this end, consider the quadratic equation

$$g(x) \equiv x^2 - x - 1 = 0,$$

with its two roots given by $x = (1 \pm \sqrt{5})/2$, and the positive root denoted by $R$. Since $g(0) = -1$, $g(R) = 0$, we obtain for all $x > R$, $x^2 - 1 > x$ which implies $(x - (1/x)) > 1$. Hence for any $\xi > R$, we can find $d(\xi) \in (0, 1)$, and subsequently $a(\xi) \in (0, 1)$ such that

$$(\xi - \frac{1}{\xi})(1 - d(\xi)) = 1 \text{ and } \frac{1}{a(\xi)} = \xi + (1 - d(\xi)). \tag{23}$$

We now simplify the notation by setting $(a(\xi), d(\xi)) \equiv (a, d)$, and obtain

$$(i) \ \ \xi = (1/a) - (1-d) \text{ and } (ii) \ \ (\xi - \frac{1}{\xi})(1-d) = 1.$$

and, furthermore, $\xi > R$.

Since $g(R) = 0$, we obtain $R(R^2 - R - 1) = 0$, and hence $R^3 = R^2 + R = R(R+1)$. Since $\mu R = [(\sqrt{5}-1)/2][(\sqrt{5}+1)/2] = 1$, we obtain

$$\mu^{3/2} = \sqrt{\mu^3} = \sqrt{\frac{1}{R^3}} = \sqrt{\frac{1}{R(R+1)}}.$$

Now for $\xi > R$, as $\xi \to R$, we have by (23) that $d(\xi) \to 0$, and $a(\xi) \to 1/(1+R)$. Thus, given $0 < \rho < \mu^{3/2}$, we can choose $\xi > R$ with $\xi$ sufficiently close to $R$, so that

$$\rho < \sqrt{\frac{a(\xi)}{\xi}}$$

Then, for the economy $(a, d, \rho) \equiv (a(\xi), d(\xi), \rho)$, we have by the above construction and Lemma 1 in the Appendix that the optimal policy function, $h$, coincides with the check map $H$. Further, by the Proposition in Sect. 5, $h^2$ exhibits turbulence.

Next, we turn to the proof of (ii). Towards this end, we claim that $\rho\xi \leq 1$. Suppose to the contrary that $\rho\xi > 1$. Then, the RSS model $(a, d, \rho)$ has an optimal policy function, $h$, given by the pan map. Since the second iterate of the $OPF$ exhibits turbulence, there exist $a, b, c \in X$ such that

$$h^2(b) = h^2(a) = a, \ \ h^2(c) = b, \text{ and either (I) } a < c < b \text{ or (II) } a > c > b.$$

Consider the possibility (I). Either we have (a) $a \leq \hat{x}$, or (b) $a > \hat{x}$. If (a) holds, then $h(a) \geq \hat{x}$, and since $h$ is the pan map, $h^2(a) \geq \hat{x}$. But this means $a = h^2(a) \geq \hat{x}$. Thus $a = \hat{x}$, and $h^2(b) = a = \hat{x}$. Since $b > a = \hat{x}$, we have $h(b) \geq \hat{x}$. Since $h(h(b)) = \hat{x}$, we must have $\hat{x} \leq h(b) \leq k$. Since $\hat{x} = a < c < b$, and $h$ is the pan map, $\hat{x} \leq h(c) \leq h(b)$. Thus $h^2(c) = h(h(c)) = \hat{x}$ also. But this contradicts the fact that $h^2(c) = b > a = \hat{x}$. In case (b), we have $b > c > a > \hat{x}$. Then, since $h$ is the pan map, and $b > c$, we have $a = h^2(b) \geq h^2(c) = b$, a contradiction.

A similar argument establishes a contradiction when possibility (II) occurs, and thereby establishes the claim.

Proposition 8(ii) ensures that $(T)$ is satisfied. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = x^2 - mx - 1$, where $m = [1/(1-d)]$. Clearly $F(0) = -1$, and $F(x) \to \infty$ as $x \to \pm\infty$. Thus, $F(x) = 0$ has a negative root and a positive root. The unique positive root of $F(x) = 0$ is given by $x' = (\sqrt{m^2+4}+m)/2$, and thus by continuity of $F$, we have $F(x) \geq 0 \Leftrightarrow x \geq x'$. But $(T)$ implies that $F(\xi) \geq 0$, and so, using the fact that $x \geq x'$, we obtain $\xi \geq x' > (\sqrt{5}+1)/2$. Given the claim $\rho\xi \leq 1$, this yields,

$$\rho \leq (1/\xi) < \frac{\sqrt{5}-1}{2} = \mu. \qquad \blacksquare$$

## 7 Concluding Remarks

We end the paper with two concluding remarks. First, in focusing on the magnitude of the labor-output ratio, $a$, in the investment good sector (a key technological parameter), our exercise might be seen as neglecting the role of other technological parameters of the two-sector RSS model: the marginal rate of transformation $\xi$, and the rate of depreciation $d$. This is certainly the case, and a similar exercise focusing on complementary restrictions in force for the other two parameters, and especially on the depreciation factor, would be extremely valuable. We hope to turn to it in future work. Second, our focus has been exclusively on topological chaos represented by period-three cycles and turbulence, and it would be interesting to consider other representations such as potential necessary and sufficient conditions for optimal period-six cycles, for example, and then to go beyond them to the consideration of ergodic chaos. The point is that these parametric restrictions are important in that they give precise quantitative magnitudes when turnpike theorems and those relating to asymptotic convergence do *not* hold; see [1, 26] for such theorems in both the deterministic and stochastic settings. More generally, as argued in [42, 43], and earlier in [8, 9], these questions have relevance for macroeconomic dynamics, and we hope to turn to them in future work.

## Appendix

This appendix collects a medley of results with the principal motivation that they do not interrupt and hinder a substantive reading of the results reported in the text above. The technical difficulty of the results resides principally in what could be referred to as the "necessity theory," which is to say, the proofs of Propositions 7 and 8(ii). The argument for the former can be furnished in a fairly straightforward way if some basic identities, routine but important, are taken out of the way. These identities are gathered here as Lemmas 1–3. The proof of Proposition 8(ii) is long and involved, with a determined verification of a variety of cases. This verification draws on results on the OPF that have not been reported before: (1) a monotone property, and (2) a straight-down-the-turnpike property. These are presented as Lemmas 4–8. The proof also draws on an unpublished result on the OPF that we reproduce for the reader's convenience: this reported as Lemma 9.

## *Lemmata for the Proof of Proposition 7*

We state with their proofs the three lemmata utilized in the proof of Proposition 7.

**Lemma 1** *If* $a \geq 1/3$, *then* $[d(1-d)/a] + (1-d)^3 < 1$.

***Proof*** Define $f : [0, 1] \to \mathbb{R}$, $f(z) = [z(1-z)/a] + (1-z)^3$, and note that Taylor's expansion for $f$ around $z \in (0, 1)$, for some $\bar{z} \in [0, z]$, is given by

$$f(z) = f(0) + f'(0)z + (1/2)f''(0)z^2 + (1/6)f'''(\bar{z})z^3.$$

Observe that $f(0) = 1$, $f(1) = 0$, and that $f'(z) = -3(1-z)^2 + (1/a)(1-2z)$, $f''(z) = 6(1-z) - (2/a)$, $f'''(z) = -6$. On factoring the information $f'(0) = -3 + (1/a)$ and $f''(0) = 6 - (2/a)$, into Taylor's expansion yields

$$f(z) = 1 + [-3 + (1/a)]z + (1/2)[6 - (2/a)]z^2 + (1/6)(-6)z^3, \tag{24}$$

and furnishes for all $z \in (0, 1)$,

$$[-3+(1/a)]z+(1/2)[6-(2/a)]z^2 = [-3+(1/a)](z-z^2) = [-3+(1/a)]z(1-z) \leq 0.$$

Using this information in (24), we obtain

$$f(z) \leq 1 - z^3 < 1 \text{ for all } z \in (0, 1),$$

which completes the proof. ∎

Next we turn to two useful identities.

**Lemma 2** $[d(1-d)/a] + (1-d)^3 = (1-d)[\xi - (\xi - 1)(1-d)]$.

***Proof*** The right hand side equals

$$\begin{aligned}(1-d)[\xi - (\xi-1) + d(\xi-1)] = (1-d)[1 + d(\xi-1)] &= (1-d)[1 + d(\frac{1}{a} - (1-d) - 1]\\ &= (1-d)[\frac{d}{a} + 1 - d(1-d) - d]\\ &= (1-d)[\frac{d}{a} + (1-d)^2] = [d(1-d)/a] + (1-d)^3\end{aligned}$$ ∎

**Lemma 3** $(\xi - 1)(1-d)/a = (\xi^2(1-d) - 1) + 1 - ([d(1-d)/a] + (1-d)^3$.

***Proof*** The left hand side equals

$$\begin{aligned}(\xi-1)(1-d)(\xi + (1-d)) &= (\xi-1)(1-d)^2 + \xi(\xi-1)(1-d)\\ &= (\xi-1)(1-d)^2 + \xi^2(1-d) - \xi(1-d)\end{aligned}$$

$$= [\xi^2(1-d)-1]+1-(1-d)[\xi-(\xi-1)(1-d)]$$
$$= [\xi^2(1-d)-1]+1-([d(1-d)/a]+(1-d)^3),$$

the last equality following from Lemma 2. ■

## *Lemmata for the Proof of Proposition 8(ii)*

### A Monotone Property

We present a monotone property of the OPF as a consequence of argumentation in [10], and first appealed to in the context of the two-sector RSS model in [18].

**Lemma 4** *For $x \in E$, the optimal policy function is monotone non-decreasing.*

***Proof*** Let $x(0)$ and $x'(0)$ belong to $E$, with $x'(0) > x(0)$. Let $\{x(t)\}$ be the optimal program from $x(0)$ and let $\{x'(t)\}$ be the optimal program from $x'(0)$. Denote $h(x(0))$ by $x(1)$ and $h(x'(0))$ by $x'(1)$. We want to show that $x'(1) \geq x(1)$. Suppose, on the contrary,

$$x'(1) < x(1) \tag{25}$$

Following [10], we now construct two alternative programs. The first goes from $x(0)$ to $x'(1)$ and then follows the optimal program from $x'(1)$; the second goes from $x'(0)$ to $x(1)$ and then follows the optimal program from $x(1)$. A crucial aspect of this technique in the current context (given the various production constraints) is that one be able to go from $x(0)$ to $x'(1)$, and from $x'(0)$ to $x(1)$. That is, one needs to show that $(x(0), x'(1)) \in \Omega$ and $(x'(0), x(1)) \in \Omega$.

We first check that $(x(0), x'(1)) \in \Omega$. Note that the irreversibility constraint is satisfied, since $x'(1) \geq (1-d)x'(0) > (1-d)x(0)$. Further, using (25), we have

$$a[x'(1)-(1-d)x(0)] < a[x(1)-(1-d)x(0)] \leq 1$$

so that the labor constraint is satisfied if

$$\bar{y} = 1-a[x'(1)-(1-d)x(0)] > 1-a[x(1)-(1-d)x(0)] = y(0) \geq 0$$

is the amount of labor devoted to the production of the consumption good. Finally, the capital constraint is satisfied, since $\bar{y} \leq 1 \leq x(0)$, since $x(0) \in E$, the set $E$ as in Fig. 2.

Next, we check that $(x'(0), x(1)) \in \Omega$. Note that the irreversibility constraint is satisfied, since (by using (25)), we have $x(1) > x'(1) \geq (1-d)x'(0)$. Further, since $x'(0) > x(0)$,

$$a[x(1)-(1-d)x'(0)] < a[x(1)-(1-d)x(0)] \leq 1$$

so that the labor constraint is satisfied if

$$\tilde{y} = 1 - a[x(1) - (1-d)x'(0)] > 1 - a[x(1) - (1-d)x(0)] = y'(0) \geq 0$$

is the amount of labor devoted to the production of the consumption good. Finally, the capital constraint is satisfied, since

$$\tilde{y} = 1 - ax(1) + a(1-d)x'(0) < 1 - ax'(1) + a(1-d)x'(0) = y'(0) \leq x'(0),$$

the strict inequality following from (25).

We can now present a self-contained exposition utilizing the techniques of [10]. First, from the definition of the OPF, we obtain

$$V(x(0)) = y(0) + \rho V(x(1)) \text{ and } V(x'(0)) = y'(0) + \rho V(x'(1)). \tag{26}$$

Second, by the principle of optimality, we have

$$V(x(0)) \geq \bar{y} + \rho V(x'(1)) \text{ and } V(x'(0)) \geq \tilde{y} + \rho V(x(1)).$$

In fact, if $V(x(0)) = \bar{y} + \rho V(x'(1))$, then $(x(0), x'(1), x'(2), \ldots)$ would be an optimal program from $x(0)$. Since (25) holds, and $(x(0), x(1), x(2), \ldots)$ is an optimal program from $x(0)$, this would contradict the fact that an optimal policy function exists. Thus, we must have $V(x(0)) > \bar{y} + \rho V(x'(1))$. For similar reasons, $V(x'(0)) > \tilde{y} + \rho V(x(1))$. This is to say

$$V(x(0)) > \bar{y} + \rho V(x'(1)) \text{ and } V(x'(0)) > \tilde{y} + \rho V(x(1)). \tag{27}$$

Clearly, (26) and (27) yield the inequality $y(0) + y'(0) > \bar{y} + \tilde{y}$. However, note that

$$\bar{y} + \tilde{y} = 1 - a[x'(1) - (1-d)x(0)] + 1 - a[x(1) - (1-d)x'(0)] > y(0) + y'(0),$$

which furnishes a contradiction, and establishes the claim. ■

It will be noted that the only place we make use of the fact that $x(0) \in E$ is in checking that $\bar{y} \leq x(0)$. Thus, if one can verify that this inequality holds, the optimal policy function can be shown to be monotone non-decreasing on an extended domain. The next result exploits this idea, and establishes a "local" monotonicity property of the $OPF$.

**Lemma 5** *Suppose $x^* \in B$, and $h(x^*) > (1/a) - \xi x^*$, then there is $\varepsilon > 0$, such that $N(\varepsilon) \equiv (x^* - \varepsilon, x^* + \varepsilon) \subset B$, and the optimal policy function is monotone non-decreasing on $N(\varepsilon)$.*

***Proof*** Denote $\{1 - a[h(x^*) - (1-d)x^*]\}$ by $y^*$. Then, we have

$$y^* = 1 - ah(x^*) + a(1-d)x^* < 1 - a[(1/a) - \xi x^*] + a(1-d)x^*$$
$$= a\xi x^* + a(1-d)x^* = a[(1/a) - (1-d)]x^* + a(1-d)x^* = x^*$$

Denote $(x^* - y^*)$ by $\mu$. Then, $\mu > 0$, and by continuity of $h$, we can find $\varepsilon > 0$, such that $N(\varepsilon) \equiv (x^* - \varepsilon, x^* + \varepsilon) \subset B$, and $[1 + a(1-d)]\varepsilon \leq (\mu/2)$, and $|h(x) - h(x^*)| < (\mu/2a)$ for all $x \in N(\varepsilon)$.

Now, let $x(0)$ and $x'(0)$ belong to $N(\varepsilon)$, with $x'(0) > x(0)$. We have to show that $x'(1) \equiv h(x'(0)) \geq h(x(0)) \equiv x(1)$. Suppose, on the contrary, that $x'(1) < x(1)$. Define $\bar{y}$ and $\tilde{y}$ as in the proof of Lemma 4. Then, one can arrive at a contradiction by following exactly the proof of Lemma 4 if one can show that $\bar{y} \leq x(0)$. Towards this end, note that

$$\bar{y} = 1 - ax'(1) + a(1-d)x(0) \leq 1 - ah(x^*) + (\mu/2) + a(1-d)x^* + a(1-d)\varepsilon$$
$$= y^* + (\mu/2) + a(1-d)\varepsilon = x^* - (\mu/2) + a(1-d)\varepsilon$$
$$= x^* - \varepsilon + [1 + a(1-d)]\varepsilon - (\mu/2) \leq x^* - \varepsilon < x(0).$$

This completes the proof of the Lemma. ■

As an application of the monotonicity property, we can say a bit more about the nature of the optimal policy function on the domain $(\hat{x}, 1]$.

**Lemma 6** *Suppose there is some* $\tilde{x} \in (\hat{x}, 1]$ *such that* $h(\tilde{x}) = H(\tilde{x})$. *Then,* $h(x) = H(x)$ *for all* $x \in [\hat{x}, \tilde{x}]$.

***Proof*** If not, there is some $x' \in [\hat{x}, \tilde{x}]$ such that $h(x') > H(x')$. Let $x'' = \inf\{x \in [x', \tilde{x}] : h(x) = H(x)\}$. Since $h(\tilde{x}) = H(\tilde{x})$, this is well defined, and by continuity of $h$ and $H$, we have $x'' > x'$, $h(x'') = H(x'')$ and $h(x) > H(x)$ for all $x \in (x', x'')$. Then by Proposition 2, we have $D_+h(x) \geq 0$ for all $x \in (x', x'')$. Thus, using the continuity of $h$, we have $h(x'') \geq h(x')$, see [44, Proposition 2, page 99]. But since $H(x'') = h(x'')$ and $h(x') > H(x')$, this implies that $H(x'') > H(x')$, which contradicts the fact that $H$ is decreasing on $[\hat{x}, 1]$. ■

We can collect together the above findings as the following result.

**Lemma 7** *Let* $\tilde{x} \in (\hat{x}, 1)$. *Then, exactly one of the following alternatives holds: (1)* $h(\tilde{x}) = H(\tilde{x})$, *(2)* $h(x) \geq h(\tilde{x})$ *for all* $x \in [\tilde{x}, k]$

***Proof*** If (i) does not hold, then $h(\tilde{x}) > H(\tilde{x})$. We claim first that $h(x) > H(x)$ for all $x \in [\tilde{x}, 1]$. If not, there is some $x' \in (\tilde{x}, 1]$ such that $h(x') = H(x')$. But, then by Lemma 6, we must have $h(\tilde{x}) = H(\tilde{x})$, since $\tilde{x} \in (\hat{x}, x')$, a contradiction.

Next, we turn to (ii). Using Lemma 5, we have $D_+h(x) \geq 0$ for all $x \in (\tilde{x}, 1)$. Using the continuity of $h$, we have $h(x) \geq h(\tilde{x})$ for all $x \in [\tilde{x}, 1]$. For $x \in E = [1, k)$, using Lemma 4, we have $h(x) \geq h(1)$, and since $h(1) \geq h(\tilde{x})$, we must have $h(x) \geq h(\tilde{x})$. This establishes (ii) by continuity of $h$. ■

**A Straight-Down-the-Turnpike Property**

**Lemma 8** *If* $\hat{x} \in h(x)$ *for some* $x \in (\hat{x}, k)$*, then* $\hat{x} \in h(x)$ *for all* $x \in (\hat{x}, k)$.

***Proof*** Let $x' \in (\hat{x}, k)$ be given such that $\hat{x} \in h(x')$. Then, there is $\varepsilon > 0$, such that for all $z \in I \equiv (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$, we have $(x', z) \in \Omega$ and $\{1 - a[z - (1 - d)x']\} < x'$, so that

$$u(x', z) = 1 - a[z - (1 - d)x'].$$

Define $F(x') = \{z : (x', z) \in \Omega\}$, and for $z \in F(x')$, define:

$$W(z) = u(x', z) + \rho V(z)$$

For $z \in I$, we have

$$W(z) = 1 - az + a(1 - d)x' + \rho V(z).$$

Since $\hat{x} \in I$, we obtain

$$W'_{-}(\hat{x}) = -a + \rho V'_{-}(\hat{x}) \tag{28}$$

For $z \in I$, with $z < \hat{x}$, we must have:

$$W(z) = u(x', z) + \rho V(z) \le V(x') = W(\hat{x})$$

the second equality following from the fact that $\hat{x} \in h(x')$. Thus, we have the first-order necessary condition $W'_{-}(\hat{x}) \ge 0$. Using this in (28), we obtain

$$V'_{-}(\hat{x}) \ge (a/\rho). \tag{29}$$

Next, let $x \in (\hat{x}, k)$ be given. Then we have $\hat{x} = (1-d)[\hat{x}/(1-d)] > (1-d)x$, and $a[\hat{x} - (1 - d)x] < a[\hat{x} - (1 - d)\hat{x}] = ad\hat{x} = ad/(1 + ad) < 1$. Further, we have

$$\begin{aligned} 1 - a[\hat{x} - (1 - d)x] &= 1 - a[\hat{x} - (1 - d)\hat{x}] + a(1 - d)(x - \hat{x}) \\ &= \hat{x} + a(1 - d)(x - \hat{x}) < \hat{x} + (x - \hat{x}) = x \end{aligned}$$

Thus, there is $\varepsilon > 0$, such that for all $z \in I \equiv (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$, we have $(x, z) \in \Omega$ and $\{1 - a[z - (1 - d)x]\} < x$, so that

$$u(x, z) = 1 - a[z - (1 - d)x]$$

Define $F(x) = \{z : (x, z) \in \Omega\}$, and for $z \in F(x)$, define

$$W(z) = u(x, z) + \rho V(z).$$

Clearly, $W$ is concave on its domain. For $z \in I$, we have

$$W(z) = 1 - az + a(1 - d)x + \rho V(z).$$

Since $\hat{x} \in I$, we obtain from (6) that

$$W'_+(\hat{x}) = -a + \rho V'_+(\hat{x}) \leq 0. \tag{30}$$

And, we can obtain from (29)

$$W'_-(\hat{x}) = -a + \rho V'_-(\hat{x}) \geq 0. \tag{31}$$

Now for all $z \in F(x)$ with $z > \hat{x}$, we obtain by (30) and the concavity of $W$,

$$W(z) - W(\hat{x}) \leq W'_+(\hat{x})(z - \hat{x}) \leq 0.$$

Similarly, for $z \in F(x)$ with $z < \hat{x}$, we obtain by (31) and the concavity of $W$,

$$W(z) - W(\hat{x}) \leq W'_-(\hat{x})(z - \hat{x}) \leq 0.$$

Thus, we have $W(z) \leq W(\hat{x})$ for all $z \in F(x)$. This means

$$\max_{(x,z)\in\Omega} [u(x, z) + \rho V(z)] = u(x, \hat{x}) + \rho V(\hat{x}).$$

Since the expression on the left hand side is $V(x)$, we obtain, by the optimality principle, $V(x) = u(x, \hat{x}) + \rho V(\hat{x})$, which means that $\hat{x} \in h(x)$, and completes the proof. ■

## *An OPF for a Special Case*

We now reproduce for the convenience of the reader a result from [20].

**Lemma 9** *If $(a, d)$ satisfies the restriction for the so-called borderline case, $(\xi - (1/\xi))(1 - d) = 1$, then for all values of $\rho < \sqrt{(a/\xi)}$, the OPF $h$ is given by the check-map, $H$.*

### *Proof of Proposition 8(ii)*

We now turn to a complete proof of Proposition 8(ii).

Since $h^2$ is turbulent, then there exist $a, b, c \in X$ such that

$$h^2(b) = h^2(a) = a,\ h^2(c) = b, \text{ and either (I) } a < c < b \text{ or (II) } a > c > b.$$

We consider specifically that (I) holds in (1), and analyze this case first. Suppose, contrary to the assertion of Proposition 8(ii), $(\xi - (1/\xi))(1-d) < 1$. Then, by (13) of Proposition 4, we have $H^2(1) < k$.

Let us denote $h(c)$ by $c'$, $h(b)$ by $b'$ and $h(a)$ by $a'$. Note that all the elements of the set $S = \{a, b, c, a', b', c'\}$ are in the range of $h$ on $X$. Thus the minimum element of the set must be greater than or equal to $(1-d)$. And the maximum element of the set must be less than or equal to $H(1-d) = H^2(1) < k$. Thus, the set $S \subset [1-d, k)$. We define $A' = [1-d, \hat{x})$, and first claim that

$$a \neq \hat{x} \tag{32}$$

For if $a = \hat{x}$, then since $h(b') = a = \hat{x}$, we cannot have $b'$ in $A'$. Thus, $b' \in [\hat{x}, k)$. Since $k > b > a = \hat{x}$, and $h(b) = b'$, we must have $b' \leq \hat{x}$. Thus, $b' = \hat{x}$, and so $h(b) = \hat{x}$, with $b \in (\hat{x}, k)$. Since $\hat{x} = a < c < b < k$, we must have $h(c) = \hat{x}$ by Lemma 8. Thus, $h^2(c) = h(\hat{x}) = \hat{x}$. But $h^2(c) = b > \hat{x}$, a contradiction. This establishes the claim (32). It also follows from (32) that $b \neq \hat{x}$, otherwise we get $a = h^2(b) = \hat{x}$, contradicting (4). Further, $c \neq \hat{x}$, otherwise we get $b = h^2(c) = \hat{x}$, a contradiction.

Since $h^2(a) = a$, and (32) holds, we must have $a' = h(a) \neq a$. Thus, we need to consider the two cases: (a) $a' > a$, and (b) $a > a'$.

Consider case (a) $[a' > a]$. Here $h(a) = a' > a$, and so $a \in A'$ and $a' \in B$. Then since $h(b') = h^2(b) = a$ and $a \in A'$, $b' \notin A$ and so $b' \in B$. Since $b' = h(b)$, $b \notin B$, and so $b \in A'$. Finally, since $h(c') = h^2(c) = b$, and $b \in A'$, $c' \notin A$ and so $c' \in B$. Since $c' = h(c)$, $c \notin B$, and so $c \in A'$. To summarize, we have

$$a, b, c \in A' \text{ and } a', b', c' \in B. \tag{33}$$

And since $b > c > a$, (33) implies that

$$b' = h(b) = H(b) < H(c) = h(c) = c' \text{ and } c' = h(c) = H(c) < H(a) = h(a) = a' \tag{34}$$

We show next that $c' < 1$. If not, then since $a' > c'$ by (34), we have $h(a') \geq h(c')$ by Lemma 4. But this yields:

$$a = h^2(a) = h(a') \geq h(c') = h^2(c) = b$$

which contradicts the fact that $b > a$. This establishes the claim. Also, given (34), we have $b' < c' < 1$. Thus $H(b') > H(c')$. Note now that $h(c') = h^2(c) = b > a = h^2(b) = h(b') \geq H(b') > H(c')$, so that $h(c') > H(c')$. Since $k > a' > c'$ by (34), we can use Lemma 7 to obtain $h(a') \geq h(c')$. But this implies:

$$a = h^2(a) = h(a') \geq h(c') = h^2(c) = b$$

which contradicts the fact that $b > a$. Thus, case (a) cannot arise.

Next, we turn to case (b) $[a > a']$. Here $h(a) = a' < a$, and so $a \in B$ and $a' \in A$. If $a' = \hat{x}$, then $a = h^2(a) = h(a') = h(\hat{x}) = \hat{x}$, contradicting (32). Thus $a' \in A'$.Then since $h(b') = h^2(b) = a$ and $a \in B$, $b' \notin B \cup \{\hat{x}\}$, and so $b' \in A'$. Since $b' = h(b)$, $b \notin A$, and so $b \in B$. Finally, since $h(c') = h^2(c) = b$, and $b \in B, c' \notin B \cup \{\hat{x}\}$ and so $c' \in A'$. Since $c' = h(c), c \notin A$, and so $c \in B$. To summarize, we have:

$$a, b, c \in B \ \text{ and } \ a', b', c' \in A'$$

And since $b > c > a$, (33) also implies that

$$H(c') = h(c') = h^2(c) = b > a = h^2(a) = h(a') = H(a').$$

Clearly, this implies that $c' < a'$ since $a', c' \in A'$. We now claim that $a < 1$. If not, then since $c > a$, Lemma 4 implies that $c' = h(c) \geq h(a) = a'$, which contradicts $c' < a'$, and establishes the claim.

If $h(a) = H(a)$, then we get $[\hat{x} - a'] = H(\hat{x}) - H(a) = (-\xi)(\hat{x} - a) = (-\xi)(h(\hat{x}) - h(a')) = (-\xi)(H(\hat{x}) - H(a')) = \xi^2(\hat{x} - a')$, so that $\xi = 1$, a contradiction. Thus, we must have $h(a) > H(a)$. Then, using the fact that $a < 1$, and Lemma 7, we obtain $c' = h(c) \geq h(a) = a'$ by virtue of the fact that $k > c > a > \hat{x}$. But this again contradicts $c' < a'$, and establishes that case (b) cannot arise.

Since cases (a) and (b) were the only possible cases, we can conclude that our initial hypothesis was false, and thereby establish Proposition 8(ii) under the possibility (I) in (1).

Next we turn to the consideration of possibility (II) holds in (1). Suppose, contrary to the assertion of part (ii) of the Proposition 8(ii), $(\xi - (1/\xi))(1 - d) < 1$. Then, by (13) of Proposition 4, we have $H^2(1) < k$.

Let us denote $h(c)$ by $c'$, $h(b)$ by $b'$ and $h(a)$ by $a'$. Note that all the elements of the set $S = \{a, b, c, a', b', c'\}$ are in the range of $h$ on $X$. Thus the minimum element of the set must be greater than or equal to $(1 - d)$. And the maximum element of the set must be less than or equal to $H(1 - d) = H^2(1) < k$. Thus, the set $S \subset [1 - d, k)$. We define $A' = [1 - d, \hat{x})$. Denote, as before, $h(b)$ by $b'$, $h(a)$ by $a'$ and $h(c)$ by $c'$. We first establish that

$$a \neq \hat{x} \tag{35}$$

Suppose, contrary to (35) that $a = \hat{x}$. Since $h(b') = h^2(b) = a = \hat{x}$, we can infer that $b' \notin A'$. Thus, $b' \in B \cup \{\hat{x}\}$. We consider the two cases: (a) $b' = \hat{x}$, (b) $b' \in B$.

Under case (a), $b' = \hat{x}$, then $h(b) = b' = \hat{x}$. But, $b < a = \hat{x}$, and so $h(b) > \hat{x}$, furnishing a contradiction.

Under case (b), $b' \in B$, then $h(b') = a = \hat{x}$. Thus, $h(x) = \hat{x}$ for all $x \in B$ by Lemma 8. Since $c < a = \hat{x}$, we have $c' = h(c) > \hat{x}$. Thus $c' \in B$, and so $h(c') = \hat{x}$. But $h(c') = h^2(c) = b < a = \hat{x}$, a contradiction. Thus (35) is established. It follows that $b \neq \hat{x}$, otherwise $a = h^2(b) = \hat{x}$, a contradiction. Further, $c \neq \hat{x}$, otherwise $b = h^2(c) = \hat{x}$, a contradiction. Furthermore, since (35) holds, and $h^2(a) = a$, we must have $a' \neq a$, and $h(a) = a'$ and $h(a') = a$. Thus, we need to consider the following two subcases: (A) $a' > a$, (B) $a > a'$.

We begin with the subcase (A) where $a' = h(a) > a$. Thus, $a \in A'$, and $a' \in B$. Since $h(b') = a$ and $a \in A'$, we must have $b' \in B$. Further, since $b' = h(b)$, we must have $b \in A'$. Finally, since $h(c') = b$ and $b \in A'$, we must have $c' \in B$; further since $c' = h(c)$, we must have $c \in A'$. To summarize, we have

$$a, b, c \in A' \text{ and } a', b', c' \in B.$$

Note that $h^2(b') = h(h^2(b)) = h(a) = a'$; $h^2(a') = h(h^2(a)) = h(a) = a'$, and $h^2(c') = h(h^2(c)) = h(b) = b'$. Further, since $a, b, c \in A'$, and $a > c > b$, we have $a' = h(a) < h(c) = c' < h(b) = b'$. Thus, the analysis of Case (I) can be applied to arrive at a contradiction. Consequently subcase (A) cannot occur.

Next, we turn to subcase (B) where $a > a' = h(a)$. Thus, $a \in B$, and $a' \in A$. Since $a' \neq a = h(a')$, we cannot have $a' = \hat{x}$. Thus, $a' \in A'$. Since $h(b') = a$ and $a \in B$, we must have $b' \in A'$. Further, since $b' = h(b)$, we must have $b \in B$. Finally, since $h(c') = b$ and $b \in B$, we must have $c' \in A'$; further since $c' = h(c)$, we must have $c \in B$. To summarize, we have

$$a, b, c \in B \text{ and } a', b', c' \in A'. \tag{36}$$

We now claim that $c < 1$. If not, we must have $a > c \geq 1$. Then by Lemma 4, $a' = h(a) \geq h(c) = c'$. But then by (36), $a = h(a') \leq h(c') = b$, a contradiction to the given condition in (II). This establishes the claim.

If $h(c) = H(c)$, then we get $(\hat{x} - b) = H(\hat{x}) - H(c') = (-\xi)(\hat{x} - c') = (-\xi)(h(\hat{x}) - h(c)) = (-\xi)(H(\hat{x}) - H(c)) = \xi^2(\hat{x} - c)$, so that, using $\xi > 1$, and $c \in (\hat{x}, k)$,

$$(b - \hat{x}) = \xi^2(c - \hat{x}) > (c - \hat{x}) \Longrightarrow b > c,$$

a contradiction. Thus $h(c) > H(c)$. Then, using $a > c$ and Lemma 7, we obtain

$$a' = h(a) \geq h(c) = c'.$$

On using this and (36), we get $a = h(a') \leq h(c') = b$. But this again contradicts the given condition (II). Thus, subcase (B) cannot arise.

Since subcases (A) and (B) were the only possible cases, we can conclude that our initial hypothesis must be false, and thereby establish Proposition 8(ii) under the possibility (II) in (1).

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