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INEQUALITY AND WELFARE IN MARKET ECONOMIES

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Atkinson (1970) and Kolm (1969) have shown how Lorenz rankings of distributions of a fixed amount of income (or a single commodity) may correspond to social welfare rankings: lower inequality indicates higher social welfare. Atkinson and Bourguignon (1982) and Kolm (1977) offer two ways of extending this result to multi-commodity environments. We investigate an alternative approach based on the existence of markets and market prices at which agents maximize utility. Our main result offers a welfare-based method of making real national income comparisons which takes into account the distribution of individual welfare.

1. Introduction

At the foundation of most normative analyses in public economics is the notion of a social welfare function, weighing the relative positions of the various participants in the economy. There is usually some general agreement on the characteristics of the social welfare function to be used in a particular context. However, the information and objectives of the hypothetical social planner are typically not so precise as to lead to a specific functional representation. As a result, economists often turn to a 'unanimity' partial ordering, comprising all rankings over which every admissible social welfare function would agree.[1] The Pareto ranking, for example, naturally arises when welfare is increasing in individual utilities but otherwise unspecified.

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[1]See, for example, Atkinson (1983, p. 3), Blackorby and Donaldson (1977), Sen (1970, 1973a) or Willig (1981).

A key result concerning unanimity partial orderings of social welfare was established by Atkinson (1970) in the context of the pure redistribution of a given total of a single commodity, say income. There, the social welfare functions are assumed to be expressible as the sum of identical individual utility (or evaluation) functions, where all that is known about these functions is that they are increasing and strictly concave in income. The surprising result he shows is that Lorenz rankings of income distributions correspond precisely to the unanimity rankings of the assumed class of social welfare functions. This conclusion is made all the more interesting by the fact that the Lorenz curve (which depicts the cumulative share of the total income held by each poorest k percent of the population) had been perceived primarily as a descriptive indicator of inequality, not as the basis of a normative evaluation system. Moreover, the result can be extended in several directions.[2] Dasgupta, Sen and Starrett (1973), for example, relax the assumption of an additive form of welfare function. Shorrocks (1983) describes how the Lorenz criterion may be generalized to accommodate changes in the aggregate level, as well as the distribution, of income.

An important direction that has received somewhat less attention is the extension to environments where there are *many* commodities influencing individual welfare. The principal articles in this regard, Kolm (1977) and Atkinson and Bourguignon (1982), adopt a 'direct' approach to the problem, deriving multidimensional analogues of the Lorenz criterion. While the main results of the two papers differ in important ways, one conclusion common to both is that it is not enough for each good separately to be more equally distributed according to the Lorenz criterion. Interaction between the jointly distributed variables proves to be crucial. For example, maintaining the assumption of strictly concave utility, Kolm finds that an unambiguous improvement in welfare can only be assured when each good's distribution is 'smoothed' in precisely the same way.[3] Noting that this criterion disallows certain comparisons, Atkinson and Bourguignon replace concavity with various sets of sign restrictions on second and higher order partial derivatives. Each set of restrictions leads to a different multivariate version of the Lorenz criterion, signalling unanimous welfare improvements relative to the given class of welfare functions.

Atkinson and Bourguignon also mention an alternative 'indirect' approach, applicable where market prices exist for each of the goods. This offers the possibility of reducing commodity bundles to a single indicator 'expenditure'

[2]Several generalizations can be found in the independent paper of Kolm (1969). Rothschild and Stiglitz (1973) also have correct extensions, but several of the proofs are flawed. See also Foster (1985).

[3]More precisely, each must be acted upon by the same bistochastic matrix. See section 2 and fig. 4.

or 'income' over which the usual Lorenz criterion could conceivably be applied. Indeed, under the assumption of utility maximization, prices can also offer useful information on 'locally relevant' weights [e.g. see Sen (1982, p. 34)] which may be an advantage over the direct approach. However, as noted by Atkinson and Bourguignon, if incomes and prices change simultaneously, we may be caught up in yet another multidimensional dilemma.

The goal of this paper is to investigate the extent to which the indirect approach to multi-commodity comparisons is useful in making welfare comparisons. The motivating question is: When do Lorenz-type comparisons of expenditure distributions have anything to say about the associated levels of social welfare? We consider this question in two distinct settings. The first is a pure-distribution model where agents may trade after the goods are allocated. In the resulting exchange economy, prices are endogenously set according to relative supply and demand as the initial allocation varies. The question is whether the social welfare levels of after-trade, equilibrium allocations might be reflected in Lorenz comparisons of equilibrium income distributions. We find that apart from certain notable special cases (two-agent economies or quasi-homothetic preferences), the equilibrium version of the Atkinson theorem does not necessarily hold. Our conclusions are emphasized by means of an example with well-behaved, identical, strictly concave utility in which greater income inequality leads to unambiguously higher welfare due to changes in equilibrium prices.

The second context examined is the general market economy common to analyses of real national income [e.g. see Sen (1976)]. The objects of comparison are now final allocations constrained only by the requirement that each agent's bundle can be supported as a utility-maximizing point given market prices. The mechanism for price determination is left unspecified, but there is every possibility that the price vectors associated with different allocations are different. We then look for a Lorenz-type criterion involving expenditure distributions that will indicate welfare improvements. The main result we obtain (Theorem 3) is perhaps best described as 'revealed preference meets the Atkinson theorem': If the expenditure distribution obtained from a given allocation at its own prices dominates a second allocation's expenditure distribution at the same prices, according to the generalized Lorenz criterion of Shorrocks (1983), then the first has a higher level of social welfare. The end-product is a welfare-based method of making real national income comparisons which takes into account the distribution of individual welfare. Interestingly, our critierion implies Hicks' (1940, 1958) real national income criterion and has an analogous index number interpretation.

We begin in section 2 with the basic definitions and notation used in the paper. Section 3 presents the results for the equilibrium, pure-distribution environment. The revealed preference approach to welfare comparisons is developed in section 4, while a final section discusses the potential for generalization.

2. Model and notation

We consider a market economy with m agents, each having the same consumption set R^n_+ and continuous utility function U. The utility function is assumed to be *monotonic* (more of all commodities implies higher utility while more of any commodity implies no lower) and *strictly concave* [for any distinct commodity bundles a and b, if $U(a) > U(b) \neq U(0)$, then $U(\lambda a + (1-\lambda)b) > \lambda U(a) + (1-\lambda)U(b)$ for every positive λ below 1]. These assumptions will be maintained throughout without explicit reference.

For any allocation $x = (x_1, \ldots, x_m) \in R^{mn}_+$ and price vector $p \geqq 0$, we define the association expenditure distribution $z = (z_1, \ldots, z_m) \in R^m_+$ by $z_i := px_i$ for each i. If the entries of z are reordered from poorest to richest, we obtain the ordered version $\hat{z}$. One distribution z is said to *Lorenz dominate* another z', written zLz', if

$$\sum_{i=1}^{k} \hat{z}_i \Big/ \sum_{i=1}^{m} z_i \geqq \sum_{i=1}^{k} \hat{z}'_i \Big/ \sum_{i=1}^{m} z'_i, \quad \text{for all } k = 1, \ldots, m, \tag{1}$$

with strict inequality for some k. In other words, the share of total expenditure held by the poorest k agents is no smaller in z than in z', and for some k it is larger. Alternatively, z *generalized Lorenz dominates* z', written $zGLz'$, if

$$\sum_{i=1}^{k} \hat{z}_i \geqq \sum_{i=1}^{k} \hat{z}'_i, \quad \text{for all } k = 1, \ldots, m, \tag{2}$$

with strict inequality for some k. The generalized Lorenz criterion is based on absolute expenditure levels held by the agents, rather than expenditure shares.

An $m \times m$ matrix B is a *bistochastic matrix* if its entries are non-negative and each column and each row sums to 1. Multiplication of a vector $v \in R^m$ by a bistochastic matrix has the effect of permuting and/or smoothing the entries of v. In fact, it can be shown that for any z and z' with the same total expenditure, zLz' holds if and only if $\hat{z}$ is different from $\hat{z}'$ and there exists a bistochastic matrix B such that $z = Bz'$. Similarly, where z and z' are arbitrary distributions, $zGLz'$ iff $\hat{z} \neq \hat{z}'$ and $z \geqq Bz'$ for some bistochastic matrix B.[4]

A *social welfare function* $W: R^m \to R$ aggregates utility levels into an overall

[4]For purposes of matrix multiplication, we shall take distributions like z to be column vectors. These results follow directly from Hardy, Littlewood and Polya (1952). See also Kolm (1969) and Shorrocks (1983).

level of social welfare. We shall focus on the class $\mathscr{W}$ of welfare functions that are strictly increasing in each utility level and S-concave, where the latter term requires $W(Bu) \geqq W(u)$ for any bistochastic matrix B. In other words, $\mathscr{W}$ is the class of Pareto consistent, symmetric welfare functions that (weakly) prefer more equal utility levels. In particular, it contains all strictly increasing, symmetric and concave welfare functions, including the utilitarian 'sum-of-utilities' function. Note that since the agent's own utility value is used in social evaluations, we are being explicitly 'welfarist' [in the terminology of Hicks (1959) and Sen (1979)] with all its associated attractions and difficulties.

The next section explores the pure-distribution case in which there is a fixed aggregate endowment vector $E \gg 0$ for the economy, and prices are set according to competitive forces. We note here a few definitions applicable to the resulting exchange economy which, without loss, is indexed by E. A *feasible allocation* is a vector of consumption bundles $x=(x_1,\ldots,x_m)$ for which $\sum_{i=1}^{m} x_i = E$. The symbol e will be used to denote the *initial allocation*, or the feasible allocation from which income is reckoned. A *competitive equilibrium* for e is a price vector p and a feasible allocation x at which each agent maximizes $U(\cdot)$ on the budget set $\{a \in R^n_+ : pa \leqq pe_i\}$. Under the above assumptions, we know that for every initial allocation there is at least one competitive equilibrium, and that equilibrium prices are always positive. It should be noted that each agent i will achieve a utility level greater than the minimum level so long as $e_i \neq 0$. Also, the first welfare theorem is valid in this environment, which implies that an equilibrium allocation is Pareto efficient in the usual strong sense.

3. Equilibrium welfare comparisons

Any equilibrium (p, x) from a given initial allocation e leads to an equilibrium or 'own-price' income distribution y, on one hand, and equilibrium utility allocation u and social welfare level $W(u)$ on the other. In this section we explore the possibility that comparisons of equilibrium income distributions may say something about the accompanying equilibrium levels of social welfare.

We can immediately identify at least one case in which such conclusions may be made. Let the equilibrium income distribution y associated with e be completely equal, and suppose that y' is an equilibrium income distribution of some other initial allocation e'. Then, where u and u' are the respective equilibrium utility allocations, it is easy to see that $W(u) \geqq W(u')$ for all welfare functions W in $\mathscr{W}$. For if y' is also completely equal, both x and x' must be the equal division allocation $x^e := (E/m,\ldots,E/m)$, and so $W(u) = W(u')$. Alternatively, if y' is not completely equal, then x' cannot be x^e, so that by the strict concavity of U:

$$u_i = U\left(\frac{E}{m}\right) = U\left(\frac{1}{m}x_1' + \cdots + \frac{1}{m}x_m'\right) > \frac{1}{m}U(x_1') + \cdots + \frac{1}{m}U(x_m')$$

$$= \frac{1}{m}u_1' + \cdots + \frac{1}{m}u_m'.$$

Therefore, $W(u) > W(u')$, since u vector-dominates a bistochastic transformation of u'. This conclusion is an equilibrium version of the classical utilitarian result [see Sen (1973a, p. 16)], which asserts that social welfare is maximized at equality.

Now suppose that e and e' are arbitrary initial allocations, each giving rise to an income distribution and a welfare level at equilibrium. Can we conclude from yLy' that $W(u) > W(u')$? The rest of this section is devoted to determining if and when the analogue of the Atkinson theorem can be obtained in this environment. Of course, if many different equilibrium prices are associated with a given initial allocation, there can be several distinct income share distributions and welfare levels, and it becomes necessary to specify a mechanism for deciding which of these is to be chosen. We can avoid this problem by assuming that for every initial allocation e the equilibrium price vector p is unique up to a positive multiple. It then follows that the equilibrium income distribution is unique up to a positive multiple, while the equilibrium allocation, utility allocation, and social welfare level are uniquely defined for every e. This *uniqueness assumption* will be invoked several times in the present section.

3.1. Two-agent economies

To see the basic elements of our approach, it is useful to begin with economies having $m=2$ agents. The nature of the relationship between an agent's equilibrium income share $\alpha_i := y_i/\Sigma y_k$ and his or her equilibrium utility level u_i will, in general, prove to be crucial. In the two-agent case, this relationship is straightforward.

Lemma 1. Let E be an exchange economy satisfying the uniqueness assumption, and suppose that e and e′ are two initial allocations for E. Then for $i=1,2$, *we have* $u_i > u_i'$ *if and only if* $\alpha_i > \alpha_i'$.

Proof. Let e and e' be two initial allocations with equilibrium prices p and p' and utility distributions u and u'. Suppose that $u_i > u_i'$ for some $i=1,2$. By definition of equilibrium we must have $p'x_i > y_i'$. If it were true that $\alpha_i' \geqq \alpha_i$ we would have $y_i' = p'(\alpha_i' E) \geqq p'(\alpha_i E)$. By the intermediate value theorem, there must exist some bundle b on the segment $(x_i, \alpha_i E]$ such that $p'b = y_i'$. But since $pb = y_i$, it follows that there would have to be two distinct equilibrium

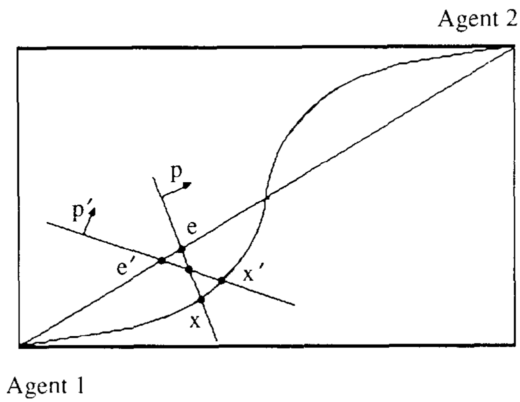

Fig. 1

prices associated with the initial allocation $(b, E-b)$, violating uniqueness. Hence, it must be true that $\alpha_i > \alpha_i'$.

Now suppose that $u_i' \geqq u_i$. If this holds with equality, then by strict concavity of U we have $x_i = x_i'$. Applying the uniqueness assumption at the initial allocation x yields $\alpha_i = \alpha_i'$. If the strict inequality $u_i' > u_i$ holds, then $\alpha_i < \alpha_i'$ by the first part of the proof. Hence, in either case, $\alpha_i \leqq \alpha_i'$.

The lemma ensures a positive monotonic relationship between an agent's income share and utility level at equilibrium. Although this might appear to be a natural enough conclusion, it is not necessarily true for economies admitting multiple equilibria or having more than two agents. The latter possibility is discussed in the next section. The former is suggested by fig. 1. In this diagram, both initial allocations are located along the diagonal of the Edgeworth box, implying that $e_i = \alpha_i E$ and $e_i' = \alpha_i' E$ for $i = 1, 2$. Clearly $\alpha_1 > \alpha_1'$ and yet $u_1 < u_1'$. The uniqueness assumption is violated where the two price lines cross. When this possibility is ruled out, we can establish the desired relationship between the income distribution and social welfare at equilibrium.

Theorem 1. Let E be an exchange economy satisfying the uniqueness assumption, and suppose that e and e′ are two initial allocations for E. If yLy′, then $W(u) > W(u')$ *for all* $W \in \mathscr{W}$.

Proof. Without loss of generality we may assume that $y_1 \leqq y_2$ and $y_1' \leqq y_2'$, so that agent 1 is poorer than agent 2 in each distribution. Let $e'' := x^e$ denote the equal-division initial allocation, and note that $\alpha_1' < \alpha_1 \leqq \alpha_1''$ by the

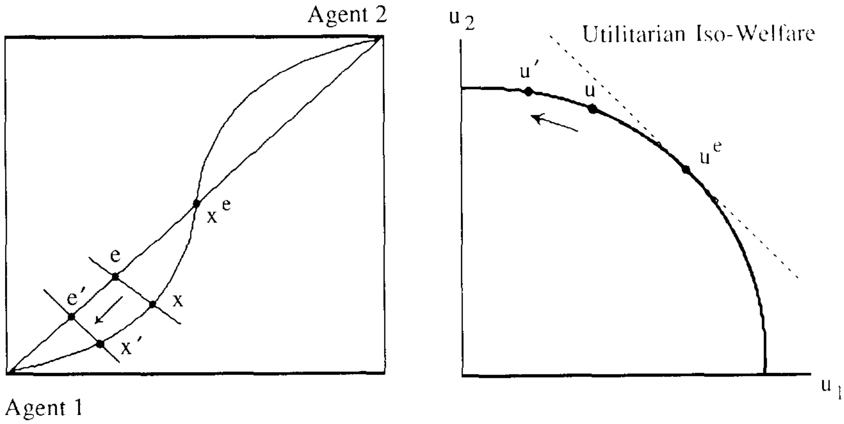

Fig. 2

Lorenz criterion. Lemma 1 implies that $u'_1 < u_1 \leqq u''_1$, so that by the continuity of the utility function, there is a bundle a_1 in the segment $(x'_1, x''_1]$ satisfying $u_1 = U(a_1)$. By the Pareto efficiency of the equilibrium allocation x, we must have $u_2 \geqq U(a_2)$, where $a_2 := E - a_1$.

Now let $\beta \in [0,1)$ be such that $a_1 = \beta x'_1 + (1-\beta)x''_1$. Substituting $(x'_1 + x'_2)/2$ for x''_1 and defining $\theta := (1-\beta)/2$, yields:

$$a_1 = (1-\theta)x'_1 + \theta x'_2; \qquad a_2 = \theta x'_1 + (1-\theta)x'_2.$$

By strict concavity of U, then

$$u_1 = U(a_1) > (1-\theta)u'_1 + \theta u'_2,$$

$$u_2 \geqq U(a_2) > \theta u'_1 + (1-\theta)u'_2.$$

Hence u is strictly larger than a bistochastic transformation of u', and so $W(u) > W(u')$ for all $W \in \mathcal{W}$, as desired.

An equilibrium version of the Atkinson result can therefore be obtained in the two-agent case. To see this graphically, examine fig. 2. As the initial allocation moves from x^e in the direction of e and e', the income share of agent 1 falls, and there is more inequality by the Lorenz criterion. By Lemma 1, for each share vector there corresponds a unique equilibrium allocation along the Pareto efficient set, so that a decrease in α_1 leads to a decrease in u_1. This corresponds to movements along the utility possibility frontier, as depicted in fig. 2. Since this frontier is concave and symmetric, it is clear that u^e has the highest utilitarian social welfare (depicted as linear iso-welfare sets) and welfare decreases in the direction indicated as the income share of the poorer person falls. This is even more pronounced when iso-welfare sets are curved, indicating an aversion to inequality in utility levels. Consequently, Lorenz-dominated equilibrium income distributions have lower welfare.

3.2. A three-agent example

When additional agents are present, however, this is no longer true. The conclusions of Lemma 1 may fail, so that the link between income share and utility is severed.[5] In fact, even in well-behaved economies it is possible for a policy of transfers from richest to poorest to leave the latter with more income, but less utility. And if the utility function is sufficiently steep at the lower utility levels, and flat at the upper levels, the utility loss may overpower the intended direct welfare gain from the redistribution.

This may be seen more easily in a concrete example. Suppose that agents share the 'quasi-homothetic' utility function $U(a,b):=g[\hat{U}(a,b)]$ where $\hat{U}(a,b):=[\frac{3}{4}a(b+24)]^{1/3}$ and

$$g(s)=\begin{cases}100s, & \text{for } s\leqq 6,\\ 594+s, & \text{for } s>6.\end{cases}$$

Clearly U is a continuous, montonic, and strictly concave utility function having the demand function:

$$d(p,Y)=\begin{cases}\left(\dfrac{Y+24p_2}{2p_1},\dfrac{Y-24p_2}{2p_2}\right), & \text{for } Y-24p_2\geqq 0,\\ \left(\dfrac{Y}{p_1},0\right), & \text{otherwise,}\end{cases}$$

where Y is the income of the agent. The income expansion path for $p_1=p_2$ is given in fig. 3.

Suppose that the initial allocation is e, where $e_1=e_2=(8,4)$ and $e_3=$

[5]Conditions under which this may occur are well documented in the trade literature [see Majumdar and Mitra (1985) and references therein] and the mathematical economics literature [see Gale (1974) or Aumann and Peleg (1974), for example]. In particular, Polterovich and Spivak (1980) call attention to a so-called budgetary paradox in which the income share of one agent rises and the shares of the rest fall in a specific way, and yet the utility of the first falls. Their example rests on the existence of an agent for whom one of the goods is Giffen.

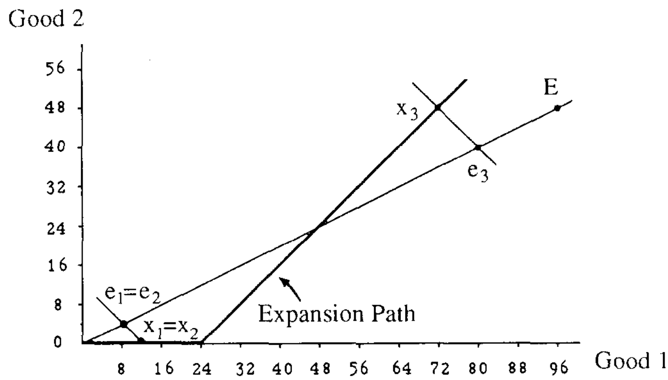

Fig. 3

(80, 40), so that irrespective of prices the income shares are $\alpha_1 = \alpha_2 = 1/12$ and $\alpha_3 = 10/12$. It is easily verified that $p_1/p_2 = 1$ is the unique equilibrium price ratio and the final demands are $x_1 = x_2 = (12, 0)$ and $x_3 = (72, 48)$, as depicted in fig. 3. Allowing for rounding error, the corresponding equilibrium utility vector is $u = (600, 600, 610)$, so that (for example) utilitarian welfare is 1810.

Now let agent 3 give a grant of (1/10, 1/20) to agent 1 and (8 − 1/10, 4 − 1/20) to agent 2. Clearly the income share of both poorer agents will rise, resulting in a Lorenz-superior distribution irrespective of prices. The new equilibrium price ratio is $p_1/p_2 = 7/6$, and the accompanying quantities are $x'_1 = (11\frac{4}{7}, 0)$, $x'_2 = (22\frac{5}{7}, 0)$, and $x'_3 = (61\frac{5}{7}, 48)$. Note that the position of agent 1 has worsened; the new equilibrium utility vector is about $u' = (593, 601, 608)$ implying a *lower* level of utilitarian welfare. Indeed, u is strictly larger than a bistochastic transformation of u', so that no matter which welfare function W is chosen from $\mathscr{W}$, the welfare level $W(u)$ is strictly higher than the new level $W(u')$. In other words, this is a pure-distribution example where *more* inequality is better.

3.3. A positive result

The example in the previous section hinges upon the differential consumption patterns of the various agents at the margin. Since the marginal propensity to consume good 1 is unity for the recipients and only 1/2 for the giver, the redistribution raises the aggregate demand for good 1 and leads to a higher equilibrium ratio p_1/p_2. The story would be different if all equilibrium bundles were located on the same portion of the linear expansion path. Then the marginal propensities would be the same for all agents and the redistribution would affect only the income distribution, not prices. The conclusion of Lemma 1 would once again hold.

Formally, suppose that $U(\cdot)$ is quasi-homothetic and smooth,[6] so that for any given price vector $q \gg 0$, demand functions have the form,

$$x(q, Y) = a(q) + b(q)Y,$$

on the interior of the comsumption set for some $a(q)$, $b(q) \in R^n$. Suppose that at the initial allocations e and e' the equilibrium allocations x and x' are *interior* in that $x_i, x'_i \gg 0$ for all i. Then p is a scalar multiple of p'. For if $e'' = x^e$ is the equal-division allocation, $E = \Sigma_i x_i = ma(p) + b(p)pE = m[a(p) + b(p)px_i^e]$ so that (p, x^e) is an equilibrium for e''. A similar argument implies that (p', x^e) is an equilibrium for e'', so that $p = \lambda p'$ for some $\lambda > 0$ by the smoothness of U. Without loss of generality we may assume that $p = p'$.

Now pick any $W \in \mathscr{W}$, and suppose that y Lorenz dominates y'. Let $[\underline{y}, \bar{y}]$ be the interval whose endpoints are respectively the minimum and maximum income from among all incomes in y and y'. Define the indirect utility function $v(\cdot)$ on $[\underline{y}, \bar{y}]$ by $v(Y) = U[a(p) + b(p)Y]$, and note that $v(\cdot)$ is strictly increasing and strictly concave by the properties of U. It is easy to show that the indirect welfare function $V(s_1, \ldots, s_m) := W(v(s_1), \ldots, v(s_m))$ is not only S-concave in $s = (s_1, \ldots, S_m)$, but also strictly S-concave [i.e. $V(Bs) > V(s)$ for any bistochastic B and distribution s for which Bs is not a permutation of s]. Therefore the Lorenz-superior distribution y has higher indirect welfare than y'. But since the indirect welfare level is the same as the level of welfare actually achieved at equilibrium, we have $W(u) = V(y) > V(y') = W(u')$, which establishes the following result.

Theorem 2. Suppose that $U(\cdot)$ is quasi-homothetic and smooth. Let e and e' be initial allocations leading to interior equilibrium allocations. If yLy', then $W(u) > W(u')$ for all $W \in \mathscr{W}$.

The condition on equilibrium allocations can be dropped in the special case where utility is homothetic. For then the equilibrium price is unchanged under any redistribution of income (even zero incomes), equilibrium demands are found along the segment from the origin to the aggregate endowment vector E, and the indirect utility function has the above properties over the entire interval $[0, pE]$. The rest of the proof proceeds as above.

Corollary. Suppose that $U(\cdot)$ is homothetic and smooth. Let e and e' be two initial allocations. If yLy', then $W(u) > W(u')$ for all $W \in \mathscr{W}$.

Each of the results in this section has been stated in terms of comparisons

[6]Quasi-homothetic preferences have linear expansion paths. By smooth we mean that first partial derivatives exist and are continuous. This assumption ensures that equilibrium prices are unique up to a positive multiple.

of equilibrium income distributions. It should be noted that each may also be interpreted in terms of (pre-trade) transfers of physical commodities. We say that allocation e' is obtained from e by a *regressive commodity transfer* if for some i and j we have $y_i \leqq y_j$ and $e_i - e'_i = e'_j - e_j > 0$, with $e'_k = e_k$ for all other $k \neq i, j$. In other words, a positive amount of some commodity (or set of commodities) is taken from agent i and given to a richer agent j. It is easy to show that in each of the above results, a regressive commodity transfer before trade will lead to a Lorenz-dominated income distribution after trade. Hence, under the hypothesis of Theorem 1, Theorem 2 or its corollary, a regressive commodity transfer leads to an unambiguously lower level of social welfare. Analogously, the example shows that it is possible for regressive commodity transfers before trade to alter prices sufficiently to obtain a more favorable social outcome.

4. Welfare comparisons and revealed preference

The previous section has shown that, apart from certain special cases, Lorenz comparisons of equilibrium income distributions may not indicate improvements in social welfare. This is perhaps not too unexpected, given that the link between individual utility and income depends crucially on market prices. If prices change too dramatically in the two equilibria under consideration, the monotonic relationship between income and utility may be broken, dashing all hopes for a welfare ranking of the equilibrium allocations.

Is there any alternative way of making distribution-based welfare comparisons? Some guidance may be found in the literature on comparisons of real national income [see Hicks (1940, 1958), Little (1950), Graaff (1957), or Sen (1976), for example]. There, use is made of common-base comparisons in which a single base price vector is used to evaluate the social product in two social states. A higher real national income (or alternatively, higher potential welfare) is indicated when state's aggregate value *at its own price* is higher than the value in the other state using the same price. Of course, higher real national income is not necessarily associated with higher actual welfare, since the comparison ignores, inter alia, how goods are distributed among the agents. In what follows, we adapt the common-base approach to obtain a criterion that explicitly accounts for the distribution.

We return now to the general environment described in section 2, where the aggregate output is unrestricted and prices are not necessarily set according to equilibrium conditions. Let x and x' be two allocations, and suppose that for each i, the bundle x_i is utility-maximizing given price $p \gg 0$ and income equal to the expenditure level $z_i := px_i$. Let $z = (z_1, \dots, z_m)$ denote the expenditure distribution at x and let $z' := (px'_1, \dots, px'_m)$ be the expenditure distribution of x' given p. We will show that if z dominates z' by the generalized Lorenz criterion, then x has a higher level of welfare than x' according to any welfare function in the class $\mathscr{W}$.

Let $zGLz'$. Then there exists a bistochastic matrix B such that $z \geqq Bz'$. By definition, there exist non-negative coefficients b_{ij} satisfying $\Sigma_k b_{kj} = \Sigma_k b_{ik} = 1$ for $i, j = 1, \ldots, n$, such that

$$\begin{aligned} z_1 &\geqq z_1'' := b_{11}z_1' + b_{12}z_2' + \cdots + b_{1m}z_m', \\ &\vdots \qquad\qquad \vdots \\ z_m &\geqq z_m'' := b_{m1}z_1' + b_{m2}z_2' + \cdots + b_{mm}z_m'. \end{aligned} \tag{5}$$

Moreover, z is not a permutation of z', which implies either (α) $z \neq z''$ or (β) $\hat{z}'' \neq \hat{z}'$. In other words, (α) $z_i > z_i''$ for some i, or (β) some row i of (5) contains at least two positive terms $b_{ij}z_j'$ and $b_{ik}z_k'$ for which z_j' is distinct from z_k'.

Now, restating system (5) in terms of the price vector p, we obtain:

$$px_i \geqq b_{i1}px_1' + b_{i2}px_1' + \cdots + b_{im}px_m', \quad \text{for all } i.$$

Thus, the consumption bundle $b_{i1}x_1' + \cdots + b_{im}x_m'$ lies in the budget set of agent i, so that by utility maximization:

$$U(x_i) \geqq U(b_{i1}x_1' + \cdots + b_{im}x_m'), \quad \text{for all } i, \tag{6}$$

and by strict concavity of $U(\cdot)$:

$$U(b_{i1}x_1' + \cdots + b_{im}x_m') \geqq b_{i1}U(x_1') + \cdots + b_{im}U(x_m'), \quad \text{for all } i. \tag{7}$$

Monotonicity of utility implies that (6) holds strictly for some i in case (α); while (7) is strict for some i in case (β) by strict concavity. Therefore, $u > Bu'$, and hence $W(u) > W(u')$ for all welfare functions W in $\mathscr{W}$. This establishes the following result.

Theorem 3. Let x and x' be allocations and let z and z' be their respective expenditure distributions given $p \gg 0$. If x_i is utility-maximizing at p for each i, and $zGLz'$, then $W(u) > W(u')$ for all W in $\mathscr{W}$.

The link between the generalized Lorenz criterion and social welfare has been known for some time [e.g. Shorrocks (1983)] for the special case where utility is a function of a single commodity (income) alone. Theorem 3 establishes that this connection still holds when utility is a function of many commodities, so long as individual expenditure levels are calculated at appropriate prices. The proof is based on a revealed preference argument in conjunction with the assumed concavity properties for individual utility and social welfare.

It is useful to contrast this theorem with the results we obtained in the

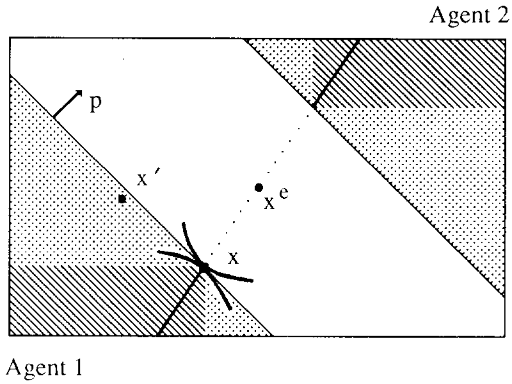

Fig. 4

previous section in the context of an exchange economy. In judging between two exchange equilibria, each allocation can be associated with two expenditure distributions, one for each price vector. Thus, there are four possible comparisons that could be made: one cross-price, one own-price and two common-base. The cross-price comparison may be discarded out of hand. The own-price comparison was the topic of the previous section, and was shown to be useful only in certain circumstances. A common-base comparison is what is used in Theorem 3. In the exchange economy the generalized Lorenz criterion reduces to the Lorenz criterion since aggregate expenditure is the same for any two feasible allocations evaluated at a common price. Thus, the issue depends on whether a pair of common-base expenditure distributions can be ranked by the Lorenz criterion.[7]

The potential comparability of this criterion is illustrated in fig. 4 using an Edgeworth box. Note that the expenditure distribution at equilibrium (p, x) is Lorenz-superior to that of x', or any other allocation in the shaded region, given p. This is because the price line through x is closer to the equal division point than the line through any given point in this region. Therefore, by Theorem 3, x must have higher welfare than any allocation in the shaded region. In contrast, the Kolm criterion (1977, Theorem 3), for example, has somewhat less cutting power. In order for one allocation to be ranked above another in that world, the distribution of each commodity must be 'smoothed' according to the same bistochastic matrix. In graphical terms this means that x is only ranked above those allocations in the shaded regions along the line through the equal-division point. The added power of our criterion over Kolm's derives from our use of the information that each

[7]It can be noted that Theorem 2 holds precisely because prices do not change; and in the case of unchanged prices, common-base comparison is just the same as own-price comparison.

agent is maximizing utility at the price in question. This constrains the possible forms that U may take and, in particular, the marginal rates of substitution at x. With a smaller class of possible utility functions the prospects for comparisons expand, and we obtain a larger set of dominated allocations.

The pure-distribution version of the criterion also admits an interpretation in terms of regressive commodity transfers, but this time the transfers take place after trade. For if x' is obtained from x by a regressive commodity transfer, the expenditure distribution at x must surely Lorenz dominate the expenditure distribution at x', given p. Hence, any post-trade regressive commodity transfer leads to a second allocation with a lower level of social welfare than the first one. This is depicted in the darker rectangular regions in fig. 4.

Of course, Theorem 3 goes beyond the simple pure-distribution exchange economy. The allocations x and x' could be equilibrium allocations from a classical production economy or, indeed, arbitrary allocations satisfying the conditions of the theorem. In any case, the approach might be usefully regarded as a natural extension of Hicks' common-base criterion to account for the distribution of the social product. Recall that the generalized Lorenz criterion requires the sum of the expenditures of the poorest k consumers to be larger (or no smaller) in x than in x'. So for $k=n$ the inequality is precisely Hicks' requirement that the aggregate value of the social product be no greater in x' than in x. Clearly, then, the additional $n-1$ inequalities of the generalized Lorenz criterion allow us to go from statements concerning potential welfare to the rather strong conclusions we obtain. Moreover, our criterion has a nice index number interpretation analogous to Hicks. The Paasche quantity index for the economy as a whole is the ratio of the total expenditure of the first allocation to that of the second allocation, where both are evaluated at the first price vector. Hicks' criterion says that if the Paasche quantity index is no less than 1, then the first allocation has higher real income. The generalized Lorenz criterion defines a Paasche index for *each* of n groups of consumers: the poorest consumer, the poorest two consumers, and so on, until we reach Hicks' aggregate Paasche index when we consider all n consumers. Theorem 3 then says that if each of these index numbers is no less than 1, and some index exceeds 1, then the welfare of the first allocation is higher.

5. Concluding remarks

In this paper we have presented several results linking expenditure (or income) distributions and social welfare via Lorenz criteria. The context has been the traditional market economy under the specialized assumptions of the original Atkinson theorem, appropriately generalized to the multi-commodity framework. We now explore the possibilities for relaxing some of the key assumptions on the welfare criteria and utility.

In each of our results, social welfare has been represented by a symmetric function of individual utilities. It should be noted, though, that similar conclusions apply to an apparently broader class of welfare criteria. In particular, symmetry may be relaxed if we are willing to use an expected social welfare criterion in conjunction with the Lerner equi-probability assumption.[8] Thus, using a weighted utilitarian criterion, for example, welfare is on average higher for the allocation having a Lorenz-superior expenditure distribution. Alternatively, the results also hold for certain other welfare criteria having no functional representation. The lexicographic generalization of a Rawls maximin criterion, known as the leximin criterion, is a case in point. Under leximin, states are compared on the basis of the utilities of the worst-off agents or, if equal, on the utilities of the next-worse-off agents, and so forth. From the definition of Lorenz dominance employed in our results, it is easy to see that the Lorenz criterion also signals higher leximin social welfare.

In addition, our results depend upon the assumption of identical utilities which, although not particularly restrictive at the aggregate level,[9] is certainly a constraint at the micro level. Perhaps the only justification for taking this approach is that it is an easy way to simplify the analysis to a manageable level.[10] It should be noted, though, that an assumption of a mass of representative consumers is certainly a step up from the single representative consumer not uncommon in public economics. One can extend the single-dimensional Atkinson theorem to non-identical utilities by applying once again the expected social welfare criterion in an equi-probable environment. But because of the special role that prices play within a market economy, our multi-commodity results would not benefit from this approach. On the other hand, even with non-identical utility functions, it may be possible to obtain a result along the lines of Sen (1973b) using the leximin criterion.

[8]See Sen (1973b). It should be noted that expected social welfare is a symmetric function of utilities under the equi-probability assumption.

[9]See Kirman and Koch (1986) for example.

[10]On this see Hicks (1958). For an insightful discussion on the perils of the identical utility assumption, see Sen (1989).

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