On Price Characterization of Optimal Plans in a Multi-Sector Economy

SWAPAN DASGUPTA and TAPAN MITRA

1. Introduction

In the literature on optimal intertemporal allocation theory, a major result is that (when future utilities are discounted) optimal programmes may be characterized by the existence of dual variables, interpreted as 'shadow prices', such that at these prices the given programme satisfies the so-called 'competitive conditions' and the 'transversality condition'. (See section 2b for appropriate definitions of terms used.)

The result of Weitzman (1973) (later modified by McKenzie (1986)) provides us with an extremely useful price characterization theory in the context of a general intertemporal model (referred to sometimes in the literature as the model in 'reduced form', and in this paper as the 'general model') where the intraperiod utility function is defined over technologically feasible pairs of initial and terminal states.

A model in which a (period) welfare function is defined on the (period) consumption vector alone (referred to henceforward as the 'consumption model') has received a considerable amount of independent attention. (Here, the consumption vector is the difference between an output vector and an input vector: see section 3a for details.) In particular, Peleg (1970) and Peleg and Ryder (1972) provide price characterization results for this model, using mathematical techniques quite different from those of Weitzman.

The difference between the two sets of results referred to above stems from the alternative definitions of a 'competitive' programme.

* This paper is dedicated, with respect, to Professor Dipak Banerjee on the occasion of his 60th birthday. Research on this paper was started while the first author was on sabbatic leave at Cornell University and Instituto Torcuato DiTella, Buenos Aires. Research of the second author was supported by a National Science Foundation Grant.

In the general model, utility along with terminal state variables can be regarded as the 'output' vector, and the initial state variables as the 'input' vector. The competitive conditions simply say that the present-value profits are maximized in each period along the programme over the set of all technically feasible triplets of initial state, terminal state and utility (see (2.1) for a formal definition).

It is, of course, clear that the consumption model can always be reduced to an equivalent model where the characterization result for the general model applies. However, in the consumption model the competitive conditions more naturally split up into two parts: (i) the present value of outputs (of goods) minus the present value of inputs (of goods) is maximized over technically feasible input-output pairs; (ii) the present value of welfare net of the present value of consumption is maximized over the consumption possibility set, which can be interpreted as maximizing welfare subject to an appropriate budget constraint. (See (3.2) and (3.3) for formal definitions.) This separation of the competitive conditions into two parts, one which can be thought of as being followed by 'producers', and the other as being followed by 'consumers', is particularly important if we would like to interpret the competitive programme as being attainable in a decentralized environment, where no agent has access to the information of *both* the technology set *and* the welfare function.

If we treat the consumption model as a special case of the general model, then the competitive conditions (3.2) and (3.3) for the consumption model readily yield the competitive condition (2.1) in the format of the general model. It has not been clear from the literature whether the converse is also true; that is, whether (3.2) and (3.3) can, in fact, be deduced from (2.1), by treating the consumption model as a special case of the general model. The main purpose of this paper is to demonstrate that this can be done. In so doing, we believe we provide a synthesis of price characterization results which have been developed independently for the two frameworks, in the existing literature.

Dual variables have been used very effectively in the literature on optimal intertemporal allocation in obtaining another major result, namely the existence of a 'stationary optimal state', supported by 'quasi-stationary' shadow prices. (For precise definitions, see sections 2a and 3a.)

For the general model, Flynn (1980) and McKenzie (1982) use duality theory to provide us with such an existence result on non-trivial stationary optimal states. For the consumption model, Peleg and Ryder (1974) have also used duality theory to develop a similar result, but their technique of proof is quite distinct from that of Flynn-McKenzie.

The difference between these two sets of results parallels the difference between the two sets of results on the 'price characterization of optimality' which we have discussed earlier. Specifically, the Flynn-McKenzie theory gives us the existence of what we call a 'discounted golden-rule equilibrium', where a stationary state is 'supported' by quasi-stationary shadow prices in the sense that among technically feasible triplets of initial state, terminal state and utility, the present-value profits are maximized at the triplet corresponding to the stationary state. In the Peleg-Ryder version, the existence of a 'modified golden-rule equilibrium' is proved. That is, in their framework, there is a stationary input-output pair which is 'supported' by 'quasi-stationary' shadow prices in the sense that (i) among technically feasible input-output pairs, the present value of output minus the present value of input is maximized; and (ii) the value of welfare net of the value of the stationary consumption (corresponding to the stationary input-output pair) is a maximum among all consumption vectors in the consumption possibility set.

We show that if the consumption model is viewed as a special case of the general model, the above result for the consumption model can be obtained from the corresponding result for the general model. One remark on this translation is worth making. The consumption model can be viewed as a special case of the general model if the utility function (used in the general model) is assumed to be upper semicontinuous, but it cannot be so viewed if the utility function is continuous (on this point see, especially, Peleg (1973), and Dutta and Mitra (1986)). Since the Flynn-McKenzie theory assumes the continuity of the utility function their result cannot be directly applied to the consumption model. One has to first prove the existence of a quasi-stationary price supported stationary optimal state in the general model under the assumption that the utility function is upper semicontinuous. This is done in Dasgupta and Mitra (1987), following the approach of Khan and Mitra (1986). (See result (R.6) in section 2c.)

The approach we have taken to obtain the results on price characterization of optimality and the existence of a modified golden-rule equilibrium in the framework of the consumption model is new. It would appear that the results themselves are not new, since they are independently obtained (by using a different approach) by Peleg and Ryder (1972, 1974). However, this impression is incorrect: it turns out that the results on the consumption model that we report here *cannot* be obtained from the Peleg-Ryder results. In both their papers, Peleg-Ryder assume that the welfare function is strictly increasing in each component of the consumption vector. And, it turns out that in both their papers, this assumption is quite critical to their proofs, so that it cannot be easily relaxed, while retaining the essence of their approach. We do not make this assumption (see assumption (B.6) in section 3a), because it is not even satisfied (at the boundary) by the commonly used Cobb-Douglas type of functions. We are nevertheless able to obtain the relevant results, because in our approach (or the approach followed in the literature on the general model) this assumption is quite unnecessary.

2. Duality Theory in a General Intertemporal Allocation Model

2a. *The Model*

The framework is described by a triplet (Ω, u, δ), where Ω, a subset of $\mathbf{R}^m_+ \times \mathbf{R}^m_+$, is a *transition possibility set*, $u: \Omega \to \mathbf{R}$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (a, b): this means that if the current state is a, then it is possible to be in the state b in one period.

We will need the following assumptions:

(A.1) (i) $(0,0) \in \Omega$; (ii) $(0,b) \in \Omega$ *implies* $b = 0$.
(A.2) Ω *is* (i) *closed, and* (ii) *convex.*
(A.3) *There is* ξ *such that* '$(a, b) \in \Omega$ *and* $\|a\| > \xi$' *implies* '$\|b\| < \|a\|$'.
(A.4) *If* $(a, b) \in \Omega$ *and* $a' \geqslant a, 0 \leqslant b' \leqslant b$, then (i) $(a', b') \in \Omega$ *and* (ii) $u(a', b') \geqslant u(a, b)$.
(A.5) u *is* (i) *upper semicontinuous and* (ii) *concave.*
(A.6) *There is* ζ *such that* $(a, b) \in \Omega$ *implies* $u(a, b) \geqslant \zeta$.

A *programme* from $b \in \mathbf{R}^m_+$ is a sequence $\{b(t)\}_0^\infty$ such that $b(0) = b$, and $(b(t), b(t+1)) \in \Omega$ for $t \geqslant 0$.

A programme $\{b(t)\}_0^\infty$ from $b \in \mathbf{R}_+^m$ is an *optimal programme* if

$$\sum_{t=0}^{\infty} \delta^t u(b'(t), b'(t+1)) \leq \sum_{t=0}^{\infty} \delta^t u(b(t), b(t+1))$$

for every programme $\{b'(t)\}_0^\infty$ from b.

An optimal programme $\{b(t)\}_0^\infty$ from $b \in \mathbf{R}_+^m$ is a *stationary optimal programme* if $b(t) = b(t+1)$ for $t \geq 0$. A *stationary optimal state* is an element $b \in \mathbf{R}_+^m$, such that $\{b\}_0^\infty$ is a stationary optimal programme. It is *non-trivial* if $u(b,b) > u(0,0)$.

A *discounted golden-rule equilibrium* is a pair $(\hat{b},\hat{p})$ with $(\hat{b},\hat{b}) \in \Omega$, $\hat{p} \in \mathbf{R}_+^m$, such that for all $(a,b) \in \Omega$,

$$u(\hat{b},\hat{b}) + \delta\hat{p}\hat{b} - \hat{p}\hat{b} \geq u(a,b) + \delta\hat{p}\hat{b} - \hat{p}a$$

The following 'boundedness properties' of our model are well-known.

(R.1) *Under assumptions* (A.3) *and* (A.4)(i),
(i) *If* $(a,b) \in \Omega$, *then* $\|b\| \leq \max[\xi, \|a\|]$
(ii) *If* $\{b(t)\}_0^\infty$ *is a programme from* $b \in \mathbf{R}_+^m$, *then* $\|b(t)\| \leq \max[\xi, \|b\|]$ *for* $t \geq 0$.

The existence of an optimal programme in this framework is also a standard result.

(R.2) *Under assumptions* (A.1), (A.2)(i), (A.3), (A.4)(i), (A.5)(i) *and* (A.6), *if* $b \in \mathbf{R}_+^m$, *there exists an optimal programme from* b.

Given (R.2), there is an optimal programme $\{b^*(t)\}_0^\infty$ from each $b \in \mathbf{R}_+^m$. We define

$$V(b) = \sum_{t=0}^{\infty} \delta^t u(b^*(t), b^*(t+1))$$

V is generally known as the *value function.*

2b. *Characterization of optimal programmes in terms of dual variables*

A sequence $\{b(t), p(t)\}_0^\infty$ is a *competitive programme* from $b \in \mathbf{R}_+^m$ if $\{b(t)\}_0^\infty$ is a programme from b, $p(t) \in \mathbf{R}_+^m$ for $t \geq 0$, and for all $t \geq 0$ we have

$$\begin{aligned}&\delta^t u(b(t), b(t+1) + p(t+1)b(t+1) - p(t)b(t)\\&\geq \delta^t u(a,b) + p(t+1)b - p(t)a \text{ for all } (a,b) \in \Omega\end{aligned} \tag{2.1}$$

A competitive programme $\{b(t), p(t)\}_0^\infty$ from $b \in \mathbf{R}_+^m$ is said to satisfy the *transversality condition* if

$$\lim_{t \to \infty} p(t)\, b(t) = 0$$

It is by now fairly well known that (roughly speaking) optimal programmes can be characterized as competitive programmes satisfying the transversality condition. We state this more precisely below in terms of three results.

(R.3) *Under assumptions* (A.3), (A.4)(i), (A.5)(i) *and* (A.6), *if* $\{b(t), p(t)\}_0^\infty$ *is a competitive programme from* $b \in \mathbf{R}_+^m$, *which satisfies the transversality condition, then* $\{b(t)\}_0^\infty$ *is an optimal programme from* b.

It is worth noting that the above result does not depend on the convexity of the transition possibility set or the concavity of the utility function. The converse of this result, stated below, relies heavily on the 'convex structure' of the model. The version we report here can be obtained by following the approach of Weitzman (1973): the interesting features of his technique of proof are (a) the use of an induction argument to obtain the 'dual variables', and (b) the combination of the dynamic programming approach exploiting the value function, with the duality approach exploiting the separation theorem. The same result can also be obtained by following the original Malinvaud procedure of selecting an infinite dual variable sequence as a limit point of finite dual variable sequences obtained for every finite horizon. This latter approach is followed in Dasgupta and Mitra (1987).

A vector a in $\mathbf{R}_+^m$ is *sufficient* if there is b in $\mathbf{R}_{++}^m$ such that (a,b) is in Ω. We shall need

(A.7) *There exists a sufficient vector in* $\mathbf{R}_{++}^m$.

(R.4) *Under assumptions* (A.1)–(A.7), *if* $\{b(t)\}_0^\infty$ *is an optimal programme from* b *in* $\mathbf{R}_{++}^m$, *then there exists a sequence* $\{p(t)\}_0^\infty$ *of dual variables, with* $p(t) \in \mathbf{R}_+^m$ *for* $t \geq 0$, *such that*

(i) $\{b(t), p(t)\}_0^\infty$ *is a competitive programme*

(ii) $\delta^t V(b(t)) - p(t)b(t) \geq \delta^t V(b) - p(t)b$ *for all* $b \in \mathbf{R}_+^m$, $t \geq 0$ (2.2)

and (iii) $\lim_{t \to \infty} p(t)b(t) = 0$ (2.3)

Conditions (2.2) and (2.3) in the above result are not 'independent'. For a competitive programme it can be shown that (2.2) is equivalent to (2.3).

(R.5) *Under assumptions* (A.1)–(A.7), *if* $\{b(t), p(t)\}_0^\infty$ *is a competitive programme from b in* $\mathbf{R}_+^m$, *then it satisfies* (2.2) *if and only if it satisfies* (2.3).

2c. *Existence of a stationary optimal state via duality theory*

The question of existence of a non-trivial stationary optimal state has been discussed extensively in the literature. Two treatments of the subject can be found in Sutherland (1970) and Khan and Mitra (1986), who use a purely primal approach, and Flynn (1980) and McKenzie (1982) who use the dual variable approach: an excellent survey of this question and related issues can be found in McKenzie (1986).

Let us call the transition possibility set Ω *δu-productive* if there exists $(\bar{a}, \bar{b})$ in Ω such that $\delta\bar{b} >> \bar{a}$, and $u(\delta\bar{b}, \bar{b}) > u(0, 0)$. The main existence result of the above literature using the duality approach can then be stated as follows. If Ω is δu-productive there is a discounted golden-rule equilibrium $(\hat{b}, \hat{p})$. Furthermore, it can be seen, using (R.3), that $\{\hat{b}\}_0^\infty$ is a stationary optimal programme from $\hat{b}$ and $\hat{b}$ is a *non-trivial* stationary optimal state. We state this formally below in result (R.6). This result can be obtained by following the approach of Khan-Mitra (1986), and is proved in Dasgupta and Mitra (1987). It is worth noting that this existence result cannot be directly obtained by following Flynn (1980) or McKenzie (1982) because they assume that the utility function is continuous, while we assume that it is only upper semicontinuous. The reason for using this weaker assumption in our theory is that it is only in this case that the consumption model can be viewed as a special case of the general model. (For elaboration on this point, see especially Peleg (1973) and Dutta and Mitra (1986).)

(R.6) *Under* (A.1)–(A.6), *if* Ω *is* δu*-productive, then there is a pair* $(\hat{b}, \hat{p})$ *such that* $(\hat{b}, \hat{p})$ *is a discounted golden-rule equilibrium. Furthermore,* $\hat{b}$ *is a non-trivial stationary optimal state.*

3. Duality Theory in the Consumption Model

3a. *The Model*

Consider a framework described by a triplet (Ω, w, δ), where Ω, a subset of $\mathbf{R}^m_+ \times \mathbf{R}^m_+$, is the *technology set*, w: $\mathbf{R}^m_+ \to \mathbf{R}$ is the period *welfare function*, and δ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of Ω is written as an ordered pair (x,y), where x represents the inputs of the m goods, and y the outputs producible with inputs x.

We will need the following assumptions:

(B.1) (i) $(0, 0) \in \Omega$; (ii) $(0, y) \in \Omega$ *implies* $y = 0$.

(B.2) Ω *is* (i) *closed, and* (ii) *convex.*

(B.3) *There is* ξ *such that* '$(x, y) \in \Omega$ *and* $\|x\| > \xi$' *implies* '$\|y\| < \|x\|$'.

(B.4) *If* $(x, y) \in \Omega$ *and* $x' \geqslant x$, $0 \leqslant y' \leqslant y$, *then* $(x', y') \in \Omega$.

(B.5) w *is* (i) *continuous, and* (ii) *concave.*

(B.6) *If* c, c' *are in* $\mathbf{R}^m_+$, *then* (i) $c' \geqslant c$ *implies* $w(c') \geqslant w(c)$, *and* (ii) $c' >> c$ *implies* $w(c') > w(c)$.

A *plan* from $y \in \mathbf{R}^m_+$ is a sequence $\{x(t), y(t)\}_0^\infty$ such that

$$y(0) = y;\ 0 \leqslant x(t) \leqslant y(t) \text{ and } (x(t), y(t+1)) \in \Omega \text{ for } t \geqslant 0$$

Associated with a plan $\{x(t), y(t)\}_0^\infty$ from y is a *consumption sequence* $\{c(t)\}_0^\infty$ defined by

$$c(t) = y(t) - x(t) \text{ for } t \geqslant 0$$

A plan $\{\bar{x}(t), \bar{y}(t)\}_0^\infty$ from y is an *optimal plan* if

$$\sum_0^\infty \delta^t w(\bar{c}(t)) \geqslant \sum_0^\infty \delta^t w(c(t)) \tag{3.1}$$

for every plan $\{x(t), y(t)\}_0^\infty$ from y.

An optimal plan $\{x(t),y(t)\}_0^\infty$ from y is a *stationary optimal plan* if $(x(t),y(t)) = (x(t+1), y(t+1))$ for $t \geqslant 0$. In this case we refer to a stationary optimal plan as $\{x,y\}_0^\infty$ with obvious interpretation, and to its associated stationary consumption sequence as $\{c\}_0^\infty$, where $c = y - x$. A *stationary optimal stock* is an element $y \in \mathbf{R}^m_+$ such that there is a stationary optimal plan from y. It is *non-trivial* if $w(c) > w(0)$.

A *modified golden-rule equilibrium* is a triple $(\hat{x}, \hat{y}, \hat{p})$ with $(\hat{x}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbf{R}^m_+$, such that denoting $(\hat{y}-\hat{x})$ by $\hat{c}$, we have

(i) $\hat{c} \geqslant 0$
(ii) $w(\hat{c}) - \hat{p}\hat{c} \geqslant w(c) - \hat{p}c$ for all c in $\mathbf{R}^m_+$
(iii) $\hat{p}(\delta\hat{y}-\hat{x}) \geqslant \hat{p}(\delta y - x)$ for all $(x,y) \in \Omega$

A sequence $\{x(t), y(t), p(t)\}_0^\infty$ is a *competitive plan* from y if $\{x(t), y(t)\}_0^\infty$ is a plan from y, $p(t) \in \mathbf{R}^m_+$ for $t \geqslant 0$, and for $t \geqslant 0$,

$$\delta^t w(c(t)) - p(t)\, c(t) \geqslant \delta^t w(c) - p(t)c \text{ for all } c \in \mathbf{R}^m_+ \tag{3.2}$$

and

$$p(t+1)\, y(t+1) - p(t)x(t) \geqslant p(t+1)y - p(t)x \quad \text{for all } (x,y) \in \Omega \tag{3.3}$$

3b. *Conversion to the format of the general model*

Our objective is to view the consumption model as a particular case of the general model of section 2 and to apply the results which have been developed for that model.

To this end, we define a *feasible input correspondence*, $g\colon \Omega \to \mathbf{R}^m_+$ by

$$g(a,b) = \{x\colon (x,b) \in \Omega \text{ and } x \leqslant a\}$$

Note that for each $(a, b) \in \Omega$, $a \in g(a, b)$ so g is non-empty valued. Also, for each $(a, b) \in \Omega$, $g(a, b)$ is, by definition, a bounded set. And, clearly $g(a, b)$ is a closed set for each $(a, b) \in \Omega$.

Next, we define a *utility function*, $u\colon\Omega \to \mathbf{R}$ by

$$u(a,b) \equiv \text{Max } w(a-x) \quad \text{subject to } x \in g(a, b)$$

Note that for each $(a, b) \in \Omega$, $g(a, b)$ is non-empty, compact, and w is continuous. Thus, defining $h(a,b) = \{\bar{x}\colon \bar{x} \in g(a, b)$, and $w(a-\bar{x}) \geqslant w(a-x)$ for all $x \in g(a, b)\}$, we note that h is a non-empty valued correspondence on Ω, and $u(a, b)$ $[\equiv w(a-\bar{x})$ for $\bar{x} \in h(a, b)]$ is well-defined on Ω.

We now show that, given (B.1)–(B.6), (Ω, u) satisfies (A.1)–(A.6) of section 2a. Clearly (B.1), (B.2), (B.3) and (B.4) imply respectively (A.1), (A.2), (A.3) and (A.4)(i). To establish (A.5)(i), let $(a^0,b^0) \in \Omega$. Suppose $(a^n,b^n) \in \Omega$ for $n = 1,2,3, \ldots$, and $(a^n,b^n) \to (a^0,b^0)$ as $n \to \infty$. Let $x^n \in g(a^n,b^n)$ for $n = 1,2,3,\ldots$. Since $a^n \to a^0$, there is $\beta^0 > 0$, such that $\|a^n\| \leqslant \|a^0\| + \beta^0$ for all $n \geqslant 1$. Thus, $\|x^n\| \leqslant \|a^n\| \leqslant \|a^0\| + \beta^0$ for all $n \geqslant 1$, and there is a convergent subsequence $(x^{n'})$ of (x^n) converging to $\bar{x}$, say. Then since $(x^{n'}, b^{n'}) \to (\bar{x}, b^0)$ as $n' \to \infty$, and Ω is closed, so $(\bar{x}, b^0) \in \Omega$. Also, $x^{n'} \leqslant a^{n'}$ for all n' implies $\bar{x} \leqslant a^0$. Thus $\bar{x} \in$

$g(a^0, b^0)$. So g is an upper hemicontinuous correspondence. Define G: $\Omega \to \mathbf{R}^m_+$ by $G(a, b) = \{a\} - g(a, b)$. Then G is an upper hemicontinuous correspondence from Ω to $\mathbf{R}^m_+$, and w is upper semicontinuous on $\mathbf{R}^m_+$, so u is upper semicontinuous on Ω (Berge (1963), p. 116). To establish (A.5)(ii), simply note that since Ω is convex, and w is concave so u is concave on Ω.

To verify (A.4)(ii), let $(a,b) \in \Omega$ and $a' \geqslant a$, $0 \leqslant b' \leqslant b$. Let $x \in h(a, b)$. Then $(x, b) \in \Omega$, $x \leqslant a$, and $w(a-x) = u(a, b)$. Define $x' = x + (a'-a)$. Then $x' = a' + (x-a) \leqslant a'$, so $(x', b') \in \Omega$. Also $x' = a' + (x-a) \leqslant a'$, so $x' \in g(a',b')$. Thus, $u(a',b') \geqslant w(a'-x') = w(a-x) = u(a,b)$.

Finally, to verify (A.6), let $(a, b) \in \Omega$. Then since $a \in g(a, b)$, so $u(a, b) \geqslant w(a-a) = w(0)$, for all $(a, b) \in \Omega$.

We have now established that the model described by (Ω, u, δ) satisfies (A.1)–(A.6).

Next, we want to consider plans in terms of the general framework of section 2. Note that $\{x(t), y(t)\}_0^\infty$ is a plan from y if and only if $\{y(t)\}_0^\infty$ is a programme from y, and $x(t) \in g(y(t), y(t+1))$ for $t \geqslant 0$. Furthermore, if $\{\bar{x}(t), \bar{y}(t)\}_0^\infty$ is an optimal plan from y, then clearly $\bar{x}(t) \in h(\bar{y}(t), \bar{y}(t+1))$ and so $u(\bar{y}(t), \bar{y}(t+1)) = w(\bar{c}(t))$ for $t \geqslant 0$. Also, if $\{x(t),y(t)\}_0^\infty$ is a plan from y, then $w(c(t)) = w(y(t) - x(t)) \leqslant u(y(t),y(t+1))$. Using these facts, the inequality in (3.1) can be rewritten as

$$\sum_0^\infty \delta^t u(\bar{y}(t), \bar{y}(t+1)) \geqslant \sum_0^\infty \delta^t u(y(t), y(t+1))$$

for every plan $\{x(t), y(t)\}_0^\infty$ from y. In other words, $\{\bar{y}(t)\}_0^\infty$ is an optimal programme from y. Conversely, if $\{\bar{y}(t)\}_0^\infty$ is an optimal programme from y, then defining $\bar{x}(t) \in h(\bar{y}(t), \bar{y}(t+1))$ for $t \geqslant 0$, $\{\bar{x}(t), \bar{y}(t)\}_0^\infty$ is clearly an optimal plan from y.

3c. *Characterization of optimal plans in terms of dual variables*

An optimal plan can be characterized as a competitive plan satisfying a transversality condition. The standard references of this result are Peleg (1970) and Peleg and Ryder (1972). The (common) technique of proof of these two papers differs from both the approach of Weitzman (1973) and that of Dasgupta and Mitra (1987), that we have referred to in section 2; it consists of applying a separation theorem in the space of all bounded infinite sequences (of vectors in $\mathbf{R}^m$). Our main objective in presenting this result again is to draw attention to the fact that it can readily be derived as a special case of the results (R.3) and (R.4) which we have noted for the general model. It appears to us that this observation has been overlooked in the literature.

There is a secondary objective in presenting this result, and this has to do with the difference in the assumptions used by us from those used in the literature. Specifically, Peleg (1970) and Peleg-Ryder (1972) assume that the welfare function, w, is strictly increasing in each component; that is, if c, c' are in $\mathbf{R}^m_+$, and $c' \geqslant c$, $c' \neq c$, then $w(c') > w(c)$. We do not make this assumption (see (B.6) above) because commonly used welfare functions of the Gobb-Douglas type will not satisfy this assumption at the boundary of $\mathbf{R}^m_+$. It turns out that this difference in the assumption on w is not minor. To be precise, the technique of proof of Peleg (1970) and Peleg-Ryder (1972) depends *in an essential way* on w having this strong property, so the characterization result we report in Dasgupta-Mitra (1987) and also below cannot be readily obtained from their result.

We now formally state and prove our characterization results, by using the corresponding results for the general model.

Proposition 1: *If* $\{x(t), y(t), p(t)\}_0^\infty$ *is a competitive plan from* y, *and*

$$\lim_{t\to\infty} p(t)\, y(t) = 0 \tag{3.4}$$

then $\{x(t), y(t)\}_0^\infty$ *is an optimal plan from* y.

Proof: If $\{x(t), y(t), p(t)\}_0^\infty$ is a competitive plan from y, then using (3.2), (3.3), and $x_t = y_t - c_t$, one gets

$$\delta^t w(c(t)) + p(t+1)\, y(t+1) - p(t)y(t) \geqslant \delta^t w(c) + p(t+1)y$$
$$-p(t)\,(c+x) \text{ for all } (x, y) \in \Omega \text{ and } c \in \mathbf{R}^m_+ \tag{3.5}$$

Note that $x(t) \in g(y(t+1),\ y(t))$, since $(x(t),\ y(t+1)) \in \Omega$, and $x(t) \leqslant y(t)$. For any $x \in g(y(t),\ y(t+1))$, $(x,\ y(t+1)) \in \Omega$ and $x \leqslant y(t)$, so defining $c = y(t) - x \geqslant 0$, and using (3.5), $w(c(t)) \geqslant w(c)$. Thus, $x(t) \in h(y(t),\ y(t+1))$, and $w(c(t)) = u(y(t), y(t+1))$.

Let $(a, b) \in \Omega$. Then defining $x \in h(a, b)$, and $c = a - x$, we have $(x, b) \in \Omega$ and $c \geqslant 0$, so by (3.5),

$$\delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t)$$
$$\geqslant \delta^t u(a, b) + p(t+1)b - p(t)a \text{ for all } (a, b) \in \Omega.$$

Thus, $\{y(t), p(t)\}_0^\infty$ is a competitive programme from y, satisfying the transversality condition. Hence, by (R.3), $\{y(t)\}_0^\infty$ is an optimal programme from y. Since we have already checked that $x(t) \in h(y(t), y(t+1))$, so $\{x(t), y(t)\}_0^\infty$ is an optimai plan from y.

Proposition 2: Suppose $\{x(t), y(t)\}_0^\infty$ is an optimal plan from $y \in \mathbf{R}_{++}^m$. Suppose, also, that there is some sufficient vector in $\mathbf{R}_+^m$. Then, there is a sequence $\{p(t)\}_0^\infty$ with $p(t) \in \mathbf{R}_+^m$ for $t \geqslant 0$, such that

(i) $\{x(t), y(t), p(t)\}_0^\infty$ *is a competitive plan*;

(ii) $\delta^t V(y(t)) - p(t)y(t) \geqslant \delta^t V(y) - p(t)y$ *for all* $y \in \mathbf{R}_+^m$, and $t \geqslant 0$; (3.6)

and (iii) $\lim_{t\to\infty} p(t)\, y(t) = 0.$ (3.7)

Proof: Since $\{x(t), y(t)\}_0^\infty$ is an optimal plan from y, so $x(t) \in h(y(t), y(t+1))$, and $\{y(t)\}_0^\infty$ is an optimal programme from y. Hence, by (R.4), there is a sequence $(p(t)\}_0^\infty$ such that $p(t) \in \mathbf{R}_+^m$ for $t \geqslant 0$, $\{y(t), p(t)\}_0^\infty$ is a competitive programme from y, and (3.6), (3.7) hold. It remains to verify (i).

Given any t, define $\theta_t(c) = \delta^t w(c) - p(t)c$ for all $c \in \mathbf{R}_+^m$, and $\pi_t(x, y) = p(t+1)y - p(t)x$ for all $(x, y) \in \Omega$

Next, given t, we define the following two sets:

$A(t) = \{\alpha$: there exists $c \geqslant 0$, satisfying $\theta_t(c) - \theta_t(c(t)) > \alpha\}$

$B(t) = \{\alpha$: there exists $(x, y) \in \Omega$ satisfying $\pi_t(x, y) - \pi_t(x(t), y(t+1)) > -\alpha\}$

We claim that (for each t),

$$A(t) \text{ and } B(t) \text{ are disjoint} \tag{3.8}$$

If (3.8) does not hold (for some t), there is some α which belongs to both $A(t)$ and $B(t)$. Then, there is $(x, y) \in \Omega$ and $c \geqslant 0$, such that $\theta_t(c) - \theta_t(c(t)) > \alpha$, and $\pi_t(x, v) - \pi_t(x(t), y(t+1)) > -\alpha$. Thus, $\delta^t(w(c) + p(t+1)y - p(t)(x + c) > \delta^t w(c(t)) + p(t+1)\, y(t+1) - p(t)y(t)$. Defining $a = (x+c)$, we have $(a, y) \in \Omega$. and $u(a, y) \geqslant w(a-x) = w(c)$. Thus, $\delta^t u(a, y) + p(t+1)y - p(t)a \geqslant \delta^t w(c) + p(t+1)y - p(t)(x+c)$. Also, since $x(t) \in h(y(t), y(t+1))$, so $w(c(t)) = w(y(t) - x(t)) = u(y(t), y(t+1))$. Hence,

$$\delta^t u(a, y) + p(t+1)y - p(t)a > \delta^t u(y(t), y(t+1)) + p(t+1)y(t+1) - p(t)y(t)$$

which contradicts the fact that $\{y(t), p(t)\}_0^\infty$ is a competitive programme from y. This establishes our claim (3.8).

Next, we note that, by definition of the sets $A(t)$ and $B(t)$,

(a) if $\alpha < 0$, then $\alpha \epsilon A(t)$; (b) if $\alpha > 0$, then $\alpha \epsilon B(t)$ (3.9)

Now suppose there is some $c \, \varepsilon \, \mathrm{R}^m_+$, such that $\theta_t(c) > \theta_t(c(t))$. Then by defining $\alpha = ½ [\theta_t(c) - \theta_t(c(t))]$, we have $\alpha > 0$, and $\alpha \in A(t)$. By (3.9), $\alpha \in B(t)$, which contradicts (3.8). Hence $\theta_t(c) \leqslant \theta_t(c(t))$ for all $c \in \mathrm{R}^m_+$, which is (3.2).

Suppose there is some $(x, y) \in \Omega$ such that $\pi_t(x, y) > \pi_t(x(t), y(t+1))$. Then by defining $\alpha = - ½ [\pi_t(x, y) - \pi_t(x(t), y(t+1))]$, we note that $(-\alpha) = ½ [\pi_t(x(t), y(t+1))]$, so $\alpha \epsilon B(t)$, and $\alpha < 0$. By (3.9), $\alpha \in A(t)$, which contradicts (3.8). Thus $\pi_t(x, y) \leqslant \pi_t(x(t), y(t+1))$ for all $(x, y) \in \Omega$, which is (3.3).

We have now shown that $\{x(t), y(t), p(t)\}_0^\infty$ is a competitive plan from y so that (i) holds. This completes the proof of the proposition.

3d. *Existence of a stationary optimal stock*

The existence of a non-trivial stationary optimal stock has been obtained in the literature by Peleg and Ryder (1974) by using duality theory. Our main purpose in presenting this result again is to point out that it can be obtained as a special case of the result (R.6) which we have noted for the general model. We believe this observation is new. As in section 3c, we note again that the result of Peleg-Ryder (1974) is obtained when the welfare function, w, is strictly increasing in each component. This assumption turns out, again, to be essential to their method of proof. Thus, our existence result, noted below and in Dasgupta-Mitra (1987), cannot be readily obtained from their result.

Call the technology set Ω *δ-productive* if there exists $(\bar{x}, \bar{y})$ in Ω such that $\delta\bar{y} >> \bar{x}$. Note that if Ω is δ-productive, then with the definition of u given in section 3b, and assumptions (B.4) and (B.6), Ω is δu-productive. For $(\delta\bar{y}, \bar{y})$ is clearly in Ω by (B.4), and $\bar{x}$ is in $g(\delta\bar{y}, \bar{y})$. So $u(\delta\bar{y}, \bar{y}) \geqslant w(\delta\bar{y} - \bar{x}) > w(0)$. Noting that $u(0,0) = w(0)$, we obtain $u(\delta\bar{y}, \bar{y}) > u(0,0)$.

Proposition 3: If Ω is δ-productive, there is a triple $(\hat{x}, \hat{y}, \hat{p})$ such that $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium. Furthermore, $\hat{y}$ is a non-trivial stationary optimal stock.

Proof: Since Ω is δ-productive, so it is also δu-productive. So, using (R.6), there is a pair $(\hat{y}, \hat{p})$ such that $(\hat{y}, \hat{p})$ is a discounted golden-rule equilibrium and $\hat{y}$ is a non-trivial stationary optimal state. That is, $(\hat{y}, \hat{y}) \in \Omega$, $\hat{p} \in \mathbf{R}_{+}^{m}$, and for all $(a, b) \in \Omega$,

$$u(\hat{y}, \hat{y}) + \delta\hat{p}\hat{y} - \hat{p}\hat{y} \geq u(a, b) + \delta\hat{p}b - \hat{p}a \qquad (3.10)$$

Let $\hat{x}$ be an element of $h(\hat{y}, \hat{y})$. Then, $(\hat{x}, \hat{y}) \in \Omega$, and denoting $(\hat{y}-\hat{x})$ *by* $\hat{c}$, we have $\hat{c} \geq 0$ and $w(\hat{c}) = u(\hat{y}, \hat{y})$.

Define $\theta(c) \equiv w(c) - \hat{p}c$ for all $c \in \mathbf{R}_{+}^{m}$, and $\pi(x, y) \equiv \delta\hat{p}y - \hat{p}x$ for all $(x, y) \in \Omega$. Now, following the method of proof in Proposition 2, one can establish that $\theta(c) \leq \theta(\hat{c})$ for all $c \in \mathbf{R}_{+}^{m}$, and $\pi(x, y) \leq \pi(\hat{x}, \hat{y})$ for all $(x, y) \in \Omega$. Hence, $(\hat{x}, \hat{y}, \hat{p})$ is a modified golden-rule equilibrium.

Using Proposition 1, $\{\hat{x}, \hat{y}\}_0^{\infty}$ is a stationary optimal programme from $\hat{y}$. Since $\hat{y}$ is a non-trivial stationary optimal state, it is also a non-trivial stationary optimal stock.

REFERENCES

1. Berge, C. (1963): 'Topological Spaces', Oliver & Boyd, Edinburgh and London.
2. Dasgupta, S. and T. Mitra (1987): 'On Price Characterization of Optimal Plans in a Multi-Sector Economy', mimeograph, Department of Economics, Cornell University.
3. Dutta, P. and T. Mitra (1986): 'On Continuity of the Utility Function in Intertemporal Allocation Models: An Example', Cornell University Working Paper #364, *Int. Econ. Rev.*, forthcoming.
4. Flynn, J. (1980): 'The Existence of Optimal Invariant Stocks in a Multi-Sector Economy', *Rev. Econ. Studies*, 47, pp. 809–11.
5. Khan, M. A. and T. Mitra (1986): 'On the Existence of a Stationary Optimal Stock for a Multi-Sector Economy: A Primal Approach', *J. Econ. Theory*, 40, pp. 319–28.
6. McKenzie, L. (1982): 'A Primal Route to the Turnpike and Lyapounov Stability', *J. Econ. Theory*, 27, pp. 194–209.
7. ——— (1986): 'Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics', in *Handbook of Mathematical Economics*, Vol. III, K. J. Arrow and M. Intrilligator (eds.), North Holland, New York.
8. Peleg, B. (1970): 'Efficiency Prices for Optimal Consumption Plans, III', *J. Math. Anal. Appl.*, 32, pp. 630–8.

9. ——— (1973): 'A Weakly Maximal Golden-rule Program for a Multi-Sector Economy', *Int. Econ. Rev.*, 14, pp. 574–9.
10. Peleg, B. and H. E. Ryder, Jr. (1972): 'On Optimal Consumption Plans in a Multi-Sector Economy', *Rev. Econ. Studies*, 39, pp. 159–69.
11. ——— (1974): 'The Modified Golden-Rule of a Multi-Sector Economy', *J. Math. Econ.*, 1, pp. 193–8.
12. Sutherland, W. R. S. (1970): 'On Optimal Development in a Multi-Sectoral Economy: The Discounted Case', *Rev. Econ. Studies*, 37, pp. 585–9.
13. Weitzman, M. L. (1973): 'Duality Theory for Infinite Horizon Convex Models', *Management Sci.*, 19, pp. 783–9.