Discounting and the Growth of Net National Product*

Amitava Bose§

Swapan Dasgupta†

and

Tapan Mitra‡

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* This paper is dedicated, with respect, to Professor Emmanuel Drandakis.

§ Indian Institute of Management, Joka, Calcutta 700027, India; Phone: (91) 33 467-8310; Fax: (91) 33 467-8307; E-mail address: abose@iimcal.ac.in

† Department of Economics, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5; Phone: 902-494-2026; Fax: 902-494-6917; E-mail address: dasgupta@is.dal.ca

‡ Department of Economics, Cornell University, Ithaca, NY 14853, USA; Phone:(607) 255-4062; Fax:(607) 255-2818; E-mail address: tm19@cornell.edu

1. Introduction

National Income is a normative concept. Moreover, the concept belongs to economic dynamics and not statics, for it evaluates the potential of current production to contribute to social welfare over time.[1] While this much is taken for granted, there have been relatively few attempts to establish rigorously the precise relation between current production and future welfare. The notable exception to this is the work of Weitzman (1976). Using a dynamic optimization framework, Weitzman showed that when future utilities are discounted, Net National Product at any date along an optimal path, measures the annuity equivalent of the social welfare of the economy, starting from that date.[2]

The purpose of this paper is to extend the scope of Weitzman's analysis to cover the case of undiscounted utilities. We then demonstrate the following: discounting future utilities is a *necessary* condition for growth of Net National Product, as defined by Weitzman (1976). We establish this by showing that when future utilities are *not* discounted, Net National Product is constant over time along every competitive path.[3]

This observation can be seen as an interpretation of the *Keynes-Ramsey rule* of optimal saving (see Ramsey (1928), Koopmans (1965)), extended from the set of "optimal paths" for which it was originally derived, to the larger set of "competitive paths". Equivalently, our observation is a reinterpretation of the result that the *Hamiltonian* is constant along a competitive path when future utilities are not discounted (see Samuelson (1965), Dasgupta (1969)):

$$H(k(t), p(t)) = u(k(t), \dot{k}(t)) + p(t)\dot{k}(t) = \text{constant} \tag{1.1}$$

where k is the vector of capital goods and p the vector of prices of investment goods (in terms of current utility). The original Keynes-Ramsey rule follows from (1.1) by noting that (a) on the "bliss" path ("golden rule" path), we have $\dot{k}(t) = 0$, so on paths converging to the bliss point, the Hamiltonian is constant at the bliss level of utility; and (b) the utility price of an investment good is the current marginal disutility of that investment. In what follows, by the Keynes-Ramsey rule, we shall mean the constancy over time of the Hamiltonian.

In Weitzman (1976), Net National Product (or NNP) is defined as the value of the Hamiltonian along a competitive path.[4] We accept this definition since it is our view also that NNP seeks to capture welfare *potential* and competitive conditions, being necessary conditions for optimality, are indicative of potentiality. However, Weitzman uses a positive discount rate and so in his framework NNP is not constant. As noted in Dasgupta and Mitra (2001), Weitzman's result is actually a reinterpretation of the Bellman equation of optimality in dynamic programming:

$$V'(k(t))\dot{k}(t) = u(k(t),\dot{k}(t)) - rV(k(t)) \tag{1.2}$$

where V is the value function associated with the dynamic optimization problem, and therefore satisfies:

$$V(k(t)) = \int_t^\infty e^{-r(s-t)} u(k(s),\dot{k}(s))ds = Max\int_t^\infty e^{-r(s-t)} u(k'(s),\dot{k}'(s))ds \tag{1.3}$$

and the maximization in (1.3) is understood to be over all feasible paths $(k'(s),\dot{k}'(s))$ with $k'(t) = k(t)$. Weitzman (1976) established that for each t:

$$Y(t) = r\int_t^\infty e^{-r(s-t)} u(k(s),\dot{k}(s))ds \tag{1.4}$$

where $Y(t)$ is NNP at time t, defined as $H(k(t), p(t))$. His result (1.4), which we will refer to as *Weitzman's Rule*, can be seen as following from (1.2) and (1.3), by recognizing that along optimal paths, the derivative of the value function, $V'(k(t))$, equals the (shadow) price of investment, $p(t)$.

The validity of the Keynes-Ramsey Rule is demonstrated in the literature, in closely related one-sector models of optimal growth, by Ramsey (1928) and Koopmans (1965). The result that the Hamiltonian is constant along an optimal path, is established by Samuelson (1965) in a one-sector model, and by Dasgupta (1969) in a two-sector model with non-shiftable capital. On the other hand, the result of Weitzman, referred to above, is a perfectly general one, holding for a model of capital accumulation with heterogeneous capital goods (some of which can be non-renewable resources). Thus, it is important to demonstrate the Keynes-Ramsey Rule (under a zero discount rate) for a comparably general model of inter-temporal allocation. This is a fairly straightforward task.

Ensuring the existence of an optimal path (in the zero discount rate case) is a well-known difficult problem. But, it does not have a direct bearing on the point we wish to make. To emphasize this, we will find it more convenient to establish that the Keynes-Ramsey rule holds for any "competitive path", whether optimal or not. In this respect, our result for the undiscounted case is actually simpler than the corresponding result for the discounted case, since the validity of Weitzman's Rule along a competitive path requires that (see Dasgupta and Mitra (1999)) an *investment value transversality condition* holds on the path.

The demonstration of the Keynes-Ramsey Rule for optimal paths, in more specific models, as discussed above, relies on the (myopic) competitive properties of an optimal path, and not on its asymptotic (transversality) behavior. Thus, our approach is an extension of the method already employed in the literature, but it clarifies the essential argument involved in arriving at the result. If an optimal path does exist, then it will turn out to be competitive in our sense (see, for example, Takekuma (1982)) and so it will satisfy the Keynes-Ramsey Rule.

2. The Framework

Consider a general framework of capital accumulation along the lines of Cass and Shell (1976). We assume that population and technology are stationary, and individuals at each date are identical in all respects (so one can think in terms of a representative agent and ignore distribution considerations).

Denote by $k_i \geq 0$ the stock of the i th capital good, where $i=1,\ldots,n$ and by z_i the investment flow, net of depreciation, of the i th capital good. Denote the vectors $(k_1,\ldots,k_n)$ and $(z_1,\ldots,z_n)$ by k and z respectively. The *technology set*, denoted by Λ, is a set of pairs (z,k) in $\mathbb{R}^n \times \mathbb{R}^n_+$. By a typical point (z,k) of Λ we understand that from capital input stock k, it is technologically feasible to obtain the flows of net investment, z. The *utility function* is denoted by a function $u : \Lambda \rightarrow \mathbb{R}$. We will make the following assumptions[5] on Λ and u:

(A.1) Λ is closed and convex; for each $k \geq 0$, there is a z in $\mathbb{R}^n$ such that $(z,k) \in \Lambda$.

(A.2) Given any number $\xi > 0$, there is a number $\eta > 0$ such that $(z,k) \in \Lambda$, and $||k|| \leq \xi$ implies $|u(z,k)| \leq \eta$ and $||z|| \leq \eta$.

(A.3) u is continuous on Λ and twice continuously differentiable in the interior of Λ.

(A.4) $u(z,k) \geq 0$ for $(z,k) \in \Lambda$; u is non-increasing in z.

(A.5) u is a concave function on Λ ; for each $k >> 0$, $u(z,k)$ is a strictly concave function of z ; in the interior of Λ, the matrix of second-order partials of u with respect to z, $[\partial^2 u(z,k)/\partial z^2]$ is negative definite.

For each $k \geq 0$, defining the set $\Lambda(k) \equiv \{z : (z,k) \in \Lambda\}$, we note that $\Lambda(k)$ is a non-empty, compact and convex subset of $\mathbb{R}^n$.

A *path* from initial stock K in $\mathbb{R}^n_+$ is a pair of functions $(z(\cdot), k(\cdot))$, where $z(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$ and $k(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n_+$, such that $k(\cdot)$ is absolutely continuous and[6]

$$(z(t), k(t)) \in \Lambda \text{ for } t \geq 0, \text{ a.e.; } \dot{k}(t) = z(t) \text{ for } t \geq 0, \text{ a.e.; and, } k(0) = K \tag{2.1}$$

Denote by $\Im(K)$ the set of paths from initial stock K. We will assume:

(A.6) For each K in $\mathbb{R}^n_+$, the set $\Im(K)$ is non-empty.

A path $(z(t), k(t))$ from K is called *optimal*[7] if for every path $(z'(t), k'(t))$ from K, we have :

$$\liminf_{T\to\infty} \int_0^T [u(z'(t),k'(t)) - u(z(t),k(t))]dt \leq 0 \tag{2.2}$$

3. Competitive Paths and the Keynes-Ramsey Rule

We will first elaborate on what we mean by a time path of quantities and prices, which evolve along a competitive path. Then, we will show that along such a path, the Keynes-Ramsey rule must hold.

Let $p = (p_1, ..., p_n)$ denote the prices of the investment goods. Define a function, $H : \mathbb{R}^n_+ \times \mathbb{R}^n$ by:

$$H(k,p) = \left.\begin{array}{cc} Max & [u(z,k)+pz] \\ subject\ to & (z,k)\in\Lambda \end{array}\right\} \qquad (3.1)$$

As noted above in Section 2, for each k in $\mathbb{R}^n_+$, $\Lambda(k)$ is non-empty and compact and, therefore, $H(k, p)$ is well-defined. Further, H is convex in p, and (since Λ is convex) concave in k.

By (A.5), for $k >> 0$, $u(z, k)$ is strictly concave in z and, therefore, there is a unique maximizing choice of investment, which solves (3.1). We can write this maximizing choice of z as a function $g(k, p)$, where g: $\mathbb{R}^n_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies:

$$H(k,p) = u(g(k, p), k) + p\, g(k, p) \text{ and } (g(k, p), k) \in \Lambda \qquad (3.2)$$

Remark 3.1: For (k^0, p^0) such that $k^0 >> 0$ and $(g(k^0, p^0), k^0)$ is in the interior of Λ,

(i) $p^0 + [\partial u(g(k^0, p^0), k^0) / \partial z] = 0$

(ii) By (A.3), the function $F(k,p,z) \equiv p + [\partial u(z,k) / \partial z]$ is defined in an open neighborhood around $(k^0, p^0, g(k^0, p^0))$, is continuously differentiable, and its derivative matrix with respect to z is non-singular. Thus, by the implicit function theorem, $g(k, p)$ is continuously differentiable with respect to (k, p) in an open neighborhood N of (k^0, p^0) and the range of $(g(k, p), k)$ for (k, p) in N , is an open subset of Λ. It follows that in this neighborhood N of (k^0, p^0), H is continuously differentiable, and by the envelope theorem,

$$[\partial H(k, p) / \partial p] = g(k, p) \text{ and } [\partial H(k, p) / \partial k] = [\partial u(g(k, p), k) / \partial k]$$

A *competitive path* is a path $(z(t), k(t))$ with associated prices, denoted by absolutely continuous functions of time $(p_1(t), \ldots , p_n(t)) \equiv (p(t))$, with $p(t) \geq 0$ for $t \geq 0$, a.e., satisfying the following two conditions:

$$u(z(t), k(t)) + p(t)\, z(t) \;=\; H(k(t), p(t)) \quad \text{for } t \geq 0, \text{ a.e.} \tag{3.3}$$

$$\dot{p}(t) = -\,[\partial H(k(t), p(t)) \,/\, \partial k] \quad \text{for } t \geq 0, \text{ a.e.} \tag{3.4}$$

Here, $p(t)$ is the vector of (present value = current value) prices of the investment goods, prevailing along a competitive path at date t. Use the notation $(z(t), k(t), p(t))$ to denote a competitive path with its associated prices. Along a competitive path, for $t \geq 0$, denote $H(k(t), p(t))$ by $Y(t)$; that is:

$$Y(t) \;\equiv\; H(k(t), p(t)) \quad \text{for } t \geq 0 \tag{3.5}$$

Interpreting utility as output with present value price of unity, (3.3) says that the maximum value of output achievable from capital stocks $k(t)$ at the prices $p(t)$, [that is, $H(k(t), p(t))$] is realized along a competitive path:

$$Y(t) \;=\; u(z(t), k(t)) + p(t)\, z(t) \quad \text{for } t \geq 0, \text{ a.e.} \tag{3.6}$$

Equation (3.4) says that asset markets are in equilibrium; that is, no gains can be made by pure arbitrage [see Dorfman, Samuelson and Solow (1958), Weitzman (1976)].

We are now in a position to state and prove the principal result of the paper: competitive paths satisfy the Keynes-Ramsey Rule in the sense that the Hamiltonian is constant over time along the path.

If $(z(\cdot), k(\cdot))$ is a path from K in $\mathbb{R}^n_+$, we shall say that it is *interior* if (i) $(z(t), k(t))$ is in the interior of Λ for $t \geq 0$, a.e., and (ii) $k(t) >> 0$ for $t \geq 0$.

Theorem 1: If $(z(t), k(t), p(t))$ is an interior competitive path from K in $\mathbb{R}^n_{++}$, then

(i) the function $Y(t)$, defined in (3.5), is an absolutely continuous function of t; and,

(ii) $\dot{Y}(t) = 0$ for $t \geq 0$ a.e.

Proof: Before coming to the proof, note that assertion (ii) of the theorem is the Keynes-Ramsey Rule. Assertion (i) is a technical result which is stated because it ensures that the function $Y(t)$ is differentiable for $t \geq 0$, a.e., and enables us to *state* assertion (ii).

economy. For this purpose, our concept of net national product will be the same as in Weitzman (1976), and we will be concerned about this magnitude along an *optimal* path, when future utilities are *not* discounted.

In order to make our discussion precise, we will conduct our analysis entirely in terms of a one sector neoclassical model. Briefly, the ingredients of this well-known model are a gross production function, $G: \mathbb{R}_+ \to \mathbb{R}_+$, a constant exponential rate of depreciation, $\delta \in (0, \infty)$ and a welfare function, $w: \mathbb{R}_+ \to \mathbb{R}_+$. A net output function, f, can be defined as $f(k) = G(k) - \delta k$ for $k \in \mathbb{R}_+$. Along a path $(z(t), k(t))$, we have consumption at date t, denoted by $c(t)$, and defined by $c(t) = f(k(t)) - z(t)$ for $t \in [0, \infty)$. Utility is obtained from consumption, so that $u(z(t), k(t)) = w(f(k(t)) - z(t))$ for $t \in [0, \infty)$. We assume that G, w, and δ satisfy the following standard properties:

(N.1) $G(0) = 0$; G *is continuous on* $\mathbb{R}_+$ and twice continuously differentiable on $\mathbb{R}_{++}$; for $k > 0$, $G'(k) > 0$ and $G''(k) \leq 0$; there is $k' > 0$ such that for $k \in (0, k']$, $G'(k) > \delta$; there is $k'' > 0$ such that for $k \in [k'', \infty)$, $G'(k) < \delta$.

(N.2) $w(0) = 0$; w is continuous and concave on $\mathbb{R}_+$; w is twice continuously differentiable on $\mathbb{R}_{++}$; $w'(c) > 0$ and $w''(c) < 0$ for all $c > 0$; $w'(c) \to \infty$ as $c \to 0$.

Under these assumptions, it can be verified easily that this one-sector neoclassical model is a special case of the general framework described in Section 2.

In this one-sector model, it is known that an optimal path $(z(t), k(t))$ exists from every initial stock, $K > 0$, and is competitive (see Koopmans (1965)). Thus, by Theorem 1 of Section 3, it must satisfy the property that the Hamiltonian is constant along the path, as noted in (1.1). But, because we are now dealing with an *optimal* path, not just a competitive one, we can say more. It is known that $k(t) \to k^*$ as $t \to \infty$, so that $z(t) \to 0$

and $c(t) \to c^* \equiv f(k^*)$, where k^* is the golden-rule capital stock (and c^* is the corresponding golden-rule consumption). The prices $p(t)$ associated with the program satisfy $p(t) = w'(c(t))$, so that $p(t) \to w'(c^*)$ as $t \to \infty$. Using this in (1.1), we see that the *constant value of the Hamiltonian on an optimal path must be equal to the golden-rule utility level*:

$$H(k(t), p(t)) = u(k(t), \dot{k}(t)) + p(t)\dot{k}(t) = u(0, k^*) = w(c^*) \qquad (4.1)$$

This is, in fact, the original form of the Keynes-Ramsey Rule.

We now turn to a discussion of net national product along an optimal path of this one-sector economy, and its relation to the maximum utility level that it can sustain. Weitzman (1976, p.159) observes that "a standard welfare interpretation of NNP is that it is the largest permanently maintainable value of consumption". [Note that Weitzman's "consumption" is "utility" in our terminology; see endnote 4]. He explains that this conventional wisdom is, in general, flawed and provides a diagram (due to Samuelson (1961)) to illustrate why this is so.

In the context of the one-sector model discussed above, we would like to point out that for a class of initial capital stocks (namely, those at or above the golden-rule capital stock), NNP along an optimal path (in the undiscounted case) measures *precisely* the maximum sustainable utility of the economy. This is because NNP along any optimal path equals golden-rule utility [by (4.1) above]. And, from every initial stock at or above the golden-rule stock, the maximum sustainable utility level is also the golden-rule utility level. This last statement can be verified by checking two elementary facts: (a) any higher utility level than the golden-rule utility level *cannot* be maintained forever, and (b) the constant golden-rule utility level *can* be maintained forever.

Our observation should not be taken as defending conventional wisdom, which is clearly incorrect in general, but as a clarification of its relation to Weitzman's contribution. To see the flaw in conventional wisdom in terms of our one-sector model, note that NNP along an optimal path, starting from an initial stock, K, (strictly) below the golden-rule stock, equals golden-rule utility, by (4.1) above. But, the maximum sustainable utility level starting from such an initial stock, K, is precisely $w(f(K))$, which is (strictly) less than the golden-rule utility level of $u(0, k^*) = w(f(k^*)) = w(c^*)$.

The above comments prompt us to examine a bit more closely Weitzman's (1976, p.160) observation that conventional wisdom is valid only if the transformation between z and $u(z, k)$ is linear. Clearly, there is no such linearity in our one-sector model, and yet conventional wisdom does give the right answer for a class of optimal paths. The mystery disappears when one recognizes that Weitzman identifies paths of maximum sustainable utility with paths having stationary capital stocks ($z = 0$). While this is correct for initial stocks below the golden-rule stock, it is not so for initial stocks strictly above the golden-rule stock. In the latter case, paths of maximum sustainable utility will necessarily disinvest ($z(t) < 0$) and approach (or reach) the golden-rule capital stock over time.

It appears, then, that for a complete analysis of the difference between NNP and maximum sustainable utility in our one-sector model, one needs to extend the Samuelson-Weitzman diagram appropriately. We provide this extension in Figure 1. Following Samuelson-Weitzman, the diagram depicts transformation curves between investment (z) and utility ($u(z, k)$), given k.

First consider the transformation curve when the capital stock equals the golden-rule capital stock, k^*. Then, corresponding to $z^* = 0$, we have golden-rule utility, $u(0, k^*)$, depicted by OA. The (negative of the) slope of the tangent to the transformation curve at A denotes the price, p^*, of the investment good (in terms of utility), which equals $w'(c^*)$, where c^* is golden-rule consumption. Note that NNP = $u(0,k^*) + p^*z^* = u(0,k^*)$, so Weitzman's measure of NNP coincides with the maximum sustainable utility.

Now, consider a transformation curve corresponding to $k > k^*$. Note that $f(k) < f(k^*)$, recalling that f is the net output function. Thus, the new transformation curve lies wholly *below* the old one. Along an optimal path, one chooses $z < 0$ and $c > c^*$; so B represents a typical optimal point. By the Keynes-Ramsey Rule, we have $u(z, k) + p\ z = u(0, k^*)$, so the tangent to the new transformation curve must pass through the point A. Once again, NNP measures maximum sustainable utility, namely $u(0,k^*)$. Along the optimal path, as k falls, the transformation curve rises (while always remaining below the

k

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Endnotes

1. Since the focus is on potentials, it is appropriate to consider social welfare on the optimal path.
2. In this framework, social welfare is seen as the present discounted value of the stream of current utilities over time.
3. Optimality requires satisfying both the competitive (myopic) and the transversality (asymptotic) conditions. Our result relies on the former only and is valid for all competitive paths, whether optimal or not.
4. In Weitzman's set up, the consumption level can be represented by a single number, which "might be calculated as an index number with given price weights, or as a multiple of some fixed basket of goods, or more generally as any cardinal utility function" [see Weitzman(1976), pp.156-157]. Thus, the consumption level is measured at date t by $u(z(t), k(t))$. Adding this to the total valuation of investment at date t, $p(t)\dot{k}(t)$, including possible dis-investments (using up) of non-renewable natural resources, we get Weitzman's notion of Net National Product.
5. For x, y in $\mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$ for $i=1,...,n$; $x > y$ means $x \geq y$ and $x \neq y$; $x >> y$ means $x_i > y_i$ for $i=1,...,n$. For x in $\mathbb{R}^n$, the sum norm of x, denoted by $||x||$ is defined by $||x|| = \sum_{i=1}^{n} | x_i |$.
6. The notation "a.e." stands for "almost everywhere"; more precisely, if A is a subset of $\mathbb{R}$, then by the expression "for $t \in$ A, a.e." we mean "for $t \in$ B, where B is a subset of A, such that the complement of B in A is a set of Lebesgue measure zero"; if the set A is an interval $[a, \infty)$, we often use the expression "for $t \geq a$, a.e." instead of "for $t \in [a, \infty)$, a.e.".
7. Our concept of *optimality* is due to Brock (1970), who calls it "weak maximality" in his paper. Gale (1967) calls the path $(z(t), k(t))$ "optimal" if (2.2) holds with "lim inf" replaced by "lim sup". Since we are concerned with only one concept of optimality, our departure from historical practice is not likely to be a source of confusion. With this notion of optimality, the existence of optimal paths can be shown for the widest class of growth models.

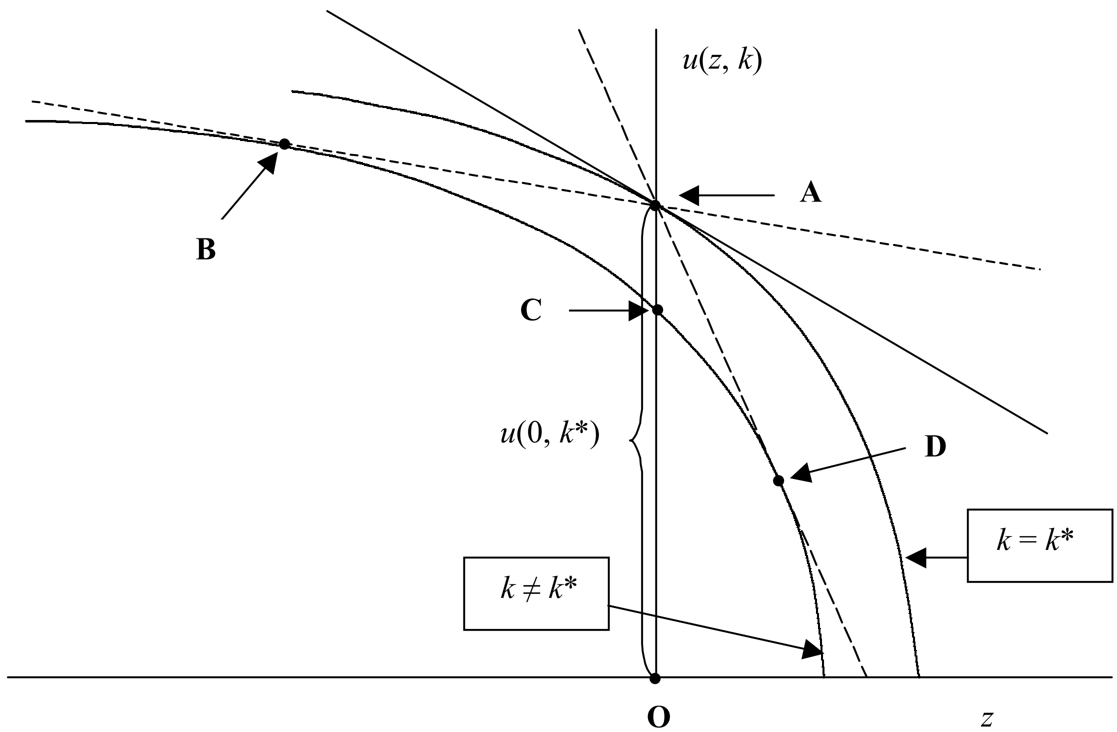

FIGURE 1