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Decentralized Evolutionary Mechanisms for Intertemporal Economies:

A Possibility Result

By

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We consider a stationary, infinite horizon aggregative model with one consumer and one producer living in each period. A decentralized intertemporal mechanism, satisfying the following "evolutionary" property, is constructed: if the current period's producer and consumer verify their equilibrium conditions, then the allocation is actually executed, without further verification by future agents. The mechanism is based on the idea of continual planning revision. It is shown that the outcome is an intertemporally efficient allocation which maximizes the long run average of one period utilities from consumption.

1. Introduction

From Adam Smith onwards, a long line of economists "... have sought to show that a decentralized economy motivated by self-interest and guided by price signals would be compatible with a coherent disposition of economic resources that could be regarded, in a well-defined sense, as superior to a large class of possible alternative dispositions." (Arrow and Hahn, 1971, p. vii).

In a static world with finitely many commodities and consumers, economists have largely succeeded in establishing that a competitive

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equilibrium in a convex economy without externalities (that is, in a "classical environment") achieves a Pareto optimal outcome. However, once we introduce time and look at an economy over an infinite horizon, the welfare theorems break down, even in a classical environment. Thus, Malinvaud (1953) provides an example of an infinite horizon economy where even though producers are maximizing intertemporal profits in every period the (inefficient) outcome is one with zero consumption in all periods! Going beyond efficiency, Samuelson's (1958) overlapping generations model shows that a competitive equilibrium need not be Pareto optimal once an infinity of commodities and agents are admitted.

These examples suggest that a decentralized resource allocation mechanism using a "competitive" price system to guide allocation decisions can lead to sub-optimal outcomes in an infinite horizon classical economy. One is then led to ask: is it possible to design any (not necessarily price-guided) decentralized infinite-horizon mechanism which realizes efficient/optimal outcomes for an interesting class of environments?

Noting Malinvaud's result that in addition to intertemporal profit maximization, a "transversality" condition (that is, the present value of the input sequence converges to zero) guarantees efficiency, Koopmans (1957) conjectured that the answer to the above question is "No," since no finitely-lived agent can actually verify such a requirement.

Koopmans' conjecture, strictly speaking, turns out to be untrue, as Dasgupta and Mitra (1988a, 1988b), Brock and Majumdar (1988) and Hurwicz and Weinberger (1990) have demonstrated in the context of alternative models. For example, condition (S) in Majumdar (1988), along with the competitive conditions for intertemporal economies, provides a rule by which optimal plans can be attained via period-by-period verifications by agents living in each period. As a consequence it is possible, at least in principle, to obtain intertemporal optimality through a decentralized system.

However, a new issue arises here. Paraphrasing from Hurwicz (1986, p. 244), we can imagine that allocations in a static Arrow-Debreu economy are made in the following way: the economic agents are presented with a proposed message (prices and allocation); if all agents accept it as an equilibrium (that is, say "yes") then the allocation is carried out. If someone says "no," a new message must be proposed and verified by the agents, the process continuing until an equilibrium message is found, to which everyone agrees. The difficulties in finding an equilibrium message in this framework starting from a disequilibrium initial position have already been noted (see, for instance, Arrow and Hahn, 1971). First, if no trade is allowed out of equilibrium, then no trade can ever take place if the tatonnement fails to converge (as in Scarf's example, 1960), or, if it does not converge in finite time. Secondly, some related adjustment processes have their difficulties from the point of view of "decentralization" or of mimicking the "invisible hand" (see the assessments of Hahn, 1982, and Smale, 1986). However, if the auctioneer happens to present an equilibrium message (price and allocation) to begin with, the above verification scenario can certainly be envisaged.

In the intertemporal framework, however, a conceptual problem arises when we try to apply Hurwicz's paradigm directly: at any given time not all agents are present — many are yet to be born. But then, as Hurwicz and Weinberger (1990) (H-W henceforth) point out, this means that the auctioneer/planner has to wait until eternity before *all* agents can verify their conditions for equilibrium, so that no allocation decision can actually be carried out in *any* period. It should be emphasized that *this is a difficulty that persists even when the auctioneer chooses an equilibrium message to begin with.* Hence, a process that requires (in the spirit of tatonnement) waiting till all the verification is complete is really ". . . a prescription for economic paralysis rather than a realistic model for economic behaviour." (H-W, p. 317).

To resolve this issue, H-W define an *evolutionary* process. This is a mechanism in which, if agents up to any finite time respond positively to the verification rules for their part of a proposed plan, then the designer actually carries out that part of the plan. This is done *irrespective of whether future agents verify or fail to verify the rules of the mechanism* for their portion of the designer's proposal. (This appears to be somewhat in contrast to the usual non-tatonnement models, in which actual consumption or production of goods does not take place. Typically, trading means moving to a new position in the Edgeworth box, not a change in the size of the box.)

The main result of H-W is negative; they show the impossibility of achieving optimal outcomes for an aggregative growth model using evolutionary processes, when the optimality criterion is the maximization of the discounted sum of one-period utilities.

Our approach in this paper is somewhat different. Instead of focusing on the possibility (or impossibility) of realizing an optimal plan through a decentralized process, we actually *construct* a decentralized mechanism where decisions are carried out period after period in the evolutionary manner of H-W. (The relevant concepts are formally defined in Section 5.) It is shown that this process has interesting normative properties: the allocation sequence generated by it is intertemporally efficient in the sense of Malinvaud (1953), and also maximizes the long run average utility from consumption. (See

Section 2a for precise definitions of these concepts and Section 6 for the stated result.)

Speaking informally, perhaps the reason why our mechanism succeeds despite the H-W impossibility result is this: the optimality criterion we use leads to many (more precisely, an infinite number of) optimal plans. In contrast, the criterion of H-W typically leads to a unique plan which maximizes the discounted sum of utilities (for instance, when the one-period utility function is strictly concave). Intuitively, one expects that it is easier for an evolutionary mechanism, in which plans are continually revised and updated as new information comes in, to achieve one out of an infinite set of outcomes, as opposed to attaining a unique outcome.

A few remarks on the optimality criteria explored in the context of intertemporal economics might be useful to put our result in proper perspective. One can argue on both philosophical grounds (Rawls, 1971) and economic ones (Pigou, 1928; Ramsey, 1928) that when making decisions, the designer should not favor nearby generations to the detriment of generations far in the future. If we accept these arguments, then we must look for some "undiscounted" optimality criterion for making welfare judgements.

An optimality notion that has appeared in the undiscounted case is the maximization of the "long-run average reward." This criterion was first explored in the operations research and statistics literature (see Howard, 1960, or Blackwell, 1962; later references include Veinott, 1966, and Ross, 1968). There are actually several versions of this criterion (for a discussion, see Flynn, 1976). The one we use requires a program to maximize $\liminf_{T\to\infty} [T^{-1}\sum_{t=1}^{T} u(c_t)]$, where $u(\cdot)$ is the one period utility function. Bhattacharya and Majumdar (1989) have shown the existence of stationary optimal policies under such a criterion, for a very general class of semi-Markov models. In economics, a version of this criterion was suggested by Dasgupta (1964), and was subsequently studied by Jeanjean (1974) and Dutta (1986, 1989). Dutta, in particular, has analyzed it in detail, and outlines its relationships with alternative undiscounted optimality notions. We should stress that this criterion has its drawbacks, a major one being that there is no "weight" on the consumption sequence for *any* finite length of time. Consequently, we require an optimal program to maximize the long-run average utility and — in addition — to be efficient.

The mechanism that we construct works via a continual planning revision process (see Goldman, 1968). Roughly speaking this process can be described as follows: given an initial stock $\mathbf{x} > 0$, a 2-period utility maximization problem is considered where the terminal stock is set equal to x. The maximal first period consumption (say c_1) takes place and the economy moves to the stock level $x_1 = f(\mathrm{x}) - c_1$, where $f(\cdot)$ is the production function. Next period another 2-period utility maximization problem is contemplated with initial stock and terminal stock set equal to x_1. Again, the maximal consumption takes place and the process is repeated. The sequence of stocks generated in this way is called a "rolling plan." In Section 3, monotonicity and asymptotic properties of such plans are studied. In Section 4, these properties are used to establish that rolling plans are optimal.

In Section 6 we formally verify that a rolling plan can be achieved by a decentralized evolutionary mechanism. Decentralization of decision-making is achieved by introducing some accounting prices. The consumer is required to equate a specific marginal rate of intertemporal substitution with a price ratio and the producer is required to verify feasibility and a condition of intertemporal profit maximization. We note that the above pricing scheme leads to the intertemporal profit maximizing shadow prices that Malinvaud (1953) was concerned with. However, one should observe that it is different from the dual prices in "optimal growth theory," used by Gale (1967) and Gale and Sutherland (1968). The difference is due to the fact that even though the consumer in any period equates his marginal rate of substitution with the accounting price-ratio, only the current part of his resulting 2-period plan is carried out. This difference is perhaps the most significant economic feature of our scheme when contrasted with the earlier duality theory.

2. The Framework

2a. Plans

We consider a one-good model, with a technology given by a production function f from $\Re_+$ to itself.

We define a *plan* from $\mathrm{x} \geq 0$ as a sequence (x_t) satisfying

$$x_0 = \mathrm{x},\ 0 \leq x_t \leq f(x_{t-1}) \quad \text{for } t \geq 1 \ .$$

The consumption sequence (c_t) generated by a plan (x_t) is given by

$$c_t = f(x_{t-1}) - x_t \quad \text{for } t \geq 1 \ .$$

A plan (x_t) from $\mathrm{x} \geq 0$ is called *interior* if $x_t > 0$ and $c_{t+1} > 0$ for $t \geq 0$.

A plan (x_t) from $\mathrm{x} \geq 0$ is *inefficient* if there is a plan (x'_t) from

$\mathrm{x} \geq 0$ satisfying $c'_t \geq c_t$ for all $t \geq 1$ and $c'_t > c_t$ for some t. It is *efficient* if it is not inefficient.

The following assumptions on f are used:

(A.1) $f(0) = 0$.
(A.2) f is continuous on $\Re_+$ and twice continuously differentiable on $\Re_{++}$.
(A.3) f is strictly increasing on $\Re_+$, with $f'(x) > 0$ for $x > 0$.
(A.4) f is strictly concave on $\Re_+$, with $f''(x) < 0$ for $x > 0$.
(A.5) f satisfies the end-point conditions: $f'(x) \to d < 1$ as $x \to \infty$; $f'(x) \to \infty$ as $x \to 0$.

Under (A.1)–(A.5), there exist unique numbers $\hat{k}$, k satisfying $0 < \hat{k} < k < \infty$, $f'(\hat{k}) = 1$, $f(k) = k$. We note that $\hat{k}$ satisfies the inequality $f(\hat{k}) > \hat{k}$. We denote $[f(\hat{k}) - \hat{k}]$ by $\hat{c}$. Thus the sequence (x_t) defined by $x_t = \hat{k}$ for $t \geq 0$ is a plan from $\hat{k}$. We refer to $\hat{k}$ as the *golden-rule input stock* and to $(\hat{k})$ as the *golden-rule plan*. The consumption sequence associated with the golden-rule plan is $(\hat{c})$. We refer to $\hat{c}$ as the *golden-rule consumption*. For any plan (x_t) from $\mathrm{x} \in (0, k)$, it can be shown that $(x_t, c_t) \ll (k, k)$ for $t \geq 1$. We refer to k as the *maximum sustainable input stock*.

Preferences are represented by a utility function u, from $\Re_+$ to $\Re$.

We will say that a plan (x_t) from $\mathrm{x} \geq 0$ *maximizes the long-run average utility* if

$$\liminf_{T\to\infty} \left[\sum_{t=1}^{T} u(c_t)/T\right] \geq \liminf_{T\to\infty} \left[\sum_{t=1}^{T} u(c'_t)/T\right]$$

for every plan (x'_t) from x. A plan (x_t) from $\mathrm{x} \geq 0$ is *optimal* if it maximizes the long-run average utility and, in addition, is efficient.

The following assumptions on u will be used:

(A.6) u is continuous on $\Re_+$ and twice continuously differentiable on $\Re_{++}$.
(A.7) u is strictly increasing on $\Re_+$, with $u'(c) > 0$ for $c > 0$.
(A.8) u is strictly concave on $\Re_+$, with $u''(c) < 0$ for $c > 0$.
(A.9) u satisfies the end-point condition: $u'(c) \to 0$ as $c \to \infty$.

2b. Finite-Horizon Plans

In this sub-section, we will describe what are commonly referred to as "finite-horizon plans." We will focus on those finite-horizon plans for which the terminal (end of horizon) input stock is the same as the initial (beginning of horizon) input stock. This provides the motivation for the formal definition that follows.

Let T be a positive integer greater than one. A *T-plan* from $\mathrm{x} \in (0, k)$ is a finite sequence $(x_t)_{t=0}^{T}$ satisfying

$$x_T = x_0 = \mathrm{x},\ 0 \leq x_t \leq f(x_{t-1}) \quad \text{for } 1 \leq t \leq T\ .$$

The finite consumption sequence $(c_t)_{t=1}^{T}$ generated by a T-plan $(x_t)_{t=0}^{T}$ is given by

$$c_t = f(x_{t-1}) - x_t \quad \text{for } 1 \leq t \leq T\ .$$

If $(x_t)_{t=0}^{T}$ is a T-plan from $\mathrm{x} \in (0, k)$, it can be checked that $(x_t, c_t) \ll (k, k)$ for $1 \leq t \leq T$.

A *maximal T-plan* from $\mathrm{x} \in (0, k)$ is a T-plan $(x_t^*)_{t=0}^{T}$ from x satisfying

$$\sum_{t=1}^{T} u(c_t^*) \geq \sum_{t=1}^{T} u(c_t)$$

for every T-plan $(x_t)_{t=0}^{T}$ from x.

Given $\mathrm{x} \in (0, k)$, we can define

$$C(\mathrm{x}) = \{(c_t)_{t=1}^{T} \colon (c_t)_{t=1}^{T} \text{ is a consumption sequence generated by a } T\text{-plan } (x_t)_{t=0}^{T} \text{ from } \mathrm{x}\}\ .$$

Then it can be checked that $C(\mathrm{x})$ is a non-empty, closed and bounded subset of $\Re_+^T$. We can define $U\colon \Re_+^T \to \Re$ by

$$U(c_1, \ldots, c_t) = \sum_{t=1}^{T} u(c_t)\ .$$

Then U is continuous on $\Re_+^T$ and therefore on $C(\mathrm{x})$. Using the Weierstrass theorem, there is $(c_t^*)_{t=1}^{T}$ in $C(\mathrm{x})$ which maximizes U among all $(c_t)_{t=1}^{T}$ in $C(\mathrm{x})$. That is, there exists a maximal T-plan $(x_t^*)_{t=0}^{T}$ from x. Using the concavity of f, $C(\mathrm{x})$ is a convex set, and using the strict concavity of u, U is strictly concave on $C(\mathrm{x})$. Thus U is maximized on $C(\mathrm{x})$ at a unique point, $(c_t^*)_{t=1}^{T}$. Given the definition of a T-plan, this also means that there is a unique maximal T-plan $(x_t^*)_{t=0}^{T}$ from x.

If $(x_t^*)_{t=0}^{T}$ is the maximal T-plan from $\mathrm{x} \in (0, k)$, then using the end-point condition on u, it can be checked that $c_t^* > 0$ for

$t = 1, \ldots, T$. Using $f(0) = 0$, it then follows that $x_t^* > 0$ for $t = 0, \ldots, T$. For $1 \leq t \leq T-1$, the expression

$$V(x) \equiv u(f(x_{t-1}^*) - x) + u(f(x) - x_{t+1}^*)$$

must be maximized at $x = x_t^*$ among all $x \geq 0$ which satisfy $f(x_{t-1}^*) - x \geq 0$ and $f(x) - x_{t+1}^* \geq 0$. Since $x_t^* > 0$, $c_t^* > 0$ and $c_{t+1}^* > 0$, the maximum is attained at an interior point, so that

$$V'(x_t^*) = u'(c_t^*)(-1) + u'(c_{t+1}^*)f'(x_t^*) = 0 \ .$$

This yields the well-known *Ramsey-Euler equations*:

$$u'(c_t^*) = u'(c_{t+1}^*)f'(x_t^*) \quad \text{for } 1 \leq t \leq T-1 \ .$$

In Sections 4 and 6 of this paper, we will be concerned with T-plans for the special case of $T = 2$; we refer to these naturally as 2-plans. A 2-plan from $\mathrm{x} \in (0, k)$ can be described by the input sequence $(\mathrm{x}, x_1, \mathrm{x})$, with associated consumption sequence $(c_1, c_2) \geq 0$ given by $c_1 = f(\mathrm{x}) - x_1$ and $c_2 = f(x_1) - \mathrm{x}$. We note here, for ready reference, a convenient characterization of maximal 2-plans.

Proposition 2.1: Let $(\mathrm{x}, x_1, \mathrm{x})$ be a 2-plan from $\mathrm{x} \in (0, k)$, with $c_1 \equiv f(\mathrm{x}) - x_1 > 0$ and $c_2 \equiv f(x_1) - \mathrm{x} > 0$. Then $(\mathrm{x}, x_1, \mathrm{x})$ is a maximal 2-plan if and only if

$$u'(f(\mathrm{x}) - x_1) = u'(f(x_1) - \mathrm{x})\, f'(x_1) \ .$$

Proof: Clearly, necessity follows from our above discussion showing that maximal T-plans satisfy the Ramsey-Euler equations.

For the sufficiency part, let $(\mathrm{x}, x_1, \mathrm{x})$ be any 2-plan from x, with associated consumption sequence (c_1', c_2') defined by $c_1' = f(\mathrm{x}) - x_1'$, $c_2' = f(x_1') - \mathrm{x}$. Then $[u(c_1') + u(c_2')] - [u(c_1) + u(c_2)] \leq u'(c_1)(c_1' - c_1) + u'(c_2)(c_2' - c_2) = u'(c_2)[f'(x_1)(c_1' - c_1) + (c_2' - c_2)] = u'(c_2)[f'(x_1)(x_1 - x_1') + (f(x_1') - f(x_1))] \leq u'(c_2)[f'(x_1)(x_1 - x_1') + f'(x_1)(x_1' - x_1)] = 0$. This shows that $(\mathrm{x}, x_1, \mathrm{x})$ is a maximal 2-plan from x.

2c. Rolling Plans

Rolling plans are defined in terms of "finite-horizon plans." We will focus on rolling plans "generated by" those finite-horizon plans for which the terminal input stock is the same as the initial input stock; that is by those finite horizon plans which we referred to as T-plans in the previous sub-section.

Using the results of the previous sub-section, we can conclude that there is a function h from $(0, k)$ to $\Re_{+}$, such that if $x \in (0, k)$, and $(x_t)_{t=1}^{T}$ is the unique maximal T-plan from x, then $x_1 = h(x)$. We might further observe that given any $x \in (0, k)$, $0 < h(x) < f(x) < k$, so that h is a function from $(0, k)$ to $(0, k)$.

A *rolling plan* from x $\in (0, k)$ is a sequence (x_t) satisfying

$$x_0 = x, \ x_{t+1} = h(x_t) \quad \text{for } t \geq 0 \ .$$

Notice that the sequence (x_t) is well defined, since h maps $(0, k)$ to $(0, k)$. Furthermore (x_t) is a plan from x, since $h(x_t) \leq f(x_t)$ for $t \geq 0$. Thus it has an associated consumption sequence (c_t) defined by

$$c_t = f(x_{t-1}) - h(x_{t-1}) \quad \text{for } t \geq 1 \ .$$

We refer to h as the "generating function" of rolling plans.

In view of Proposition 2.1, if $T = 2$, and h is the generating function of rolling plans then for every $x \in (0, k)$, $u'(f(x) - h(x)) = u'(f(h(x)) - x)f'(h(x))$.

3. Monotone Convergence Properties of Rolling Plans

Rolling plans can be shown to be monotone in input stocks over time. If a rolling plan starts from an initial input stock below the golden-rule input stock ($\hat{k}$), then the input stocks monotonically increase and converge to the golden-rule input stock. (An analogous statement can be made if the initial input stock is above the golden-rule input stock.) Such properties were established by Goldman (1968) in the context of a continuous-time aggregative model (with discounting of future utilities). In discrete-time aggregative models, similar properties can be established by focusing on the properties of what we have called the "generating function" of rolling plans. We follow this method in this section. Specifically, Lemma 3.1 shows that the generating function is above (below) the 45-degree line at input stocks below (above) the golden-rule level; Lemma 3.2 shows that the generating function is also increasing on its domain. These properties are then summarized in Proposition 3.1. Proposition 3.2 establishes that these properties imply monotone convergence of rolling plans to the golden-rule.

Lemma 3.1: If $(x_t^*)_{t=0}^{T}$ is the maximal T-plan from x $\in (0, k)$, then (a) x $< \hat{k}$ implies x $< x_1^* < \hat{k}$; (b) x $= \hat{k}$ implies $x_1^* = \hat{k}$; (c) x $> \hat{k}$ implies x $> x_1^* > \hat{x}$.

Proof: We will prove (a); the proofs of (b) and (c) can be worked out analogously. We first establish that $x_1^* < \hat{k}$. Suppose instead that $x_1^* \geq \hat{k}$. Then $f'(x_1^*) \leq 1$, so by the Ramsey-Euler equation, $u'(c_1^*) = f'(x_1^*)u'(c_2^*) \leq u'(c_2^*)$. Since $u''(c) < 0$ for $c > 0$, we get $c_1^* \geq c_2^*$. Thus

$$f(\mathbf{x}) - x_1^* \geq f(x_1^*) - x_2^* > f(\mathbf{x}) - x_2^* ,$$

the last inequality following from $x_1^* \geq \hat{k} > \mathbf{x}$, and the fact that f is increasing. Thus, $x_2^* > x_1^* \geq \hat{k}$. We can then repeat the steps to obtain

$$x_T^* > x_{T-1}^* > \ldots > x_2^* > x_1^* > \mathbf{x} ,$$

so that $x_T^* > \mathbf{x}$, which contradicts the fact that $x_T^* = \mathbf{x}$ by definition of a T-plan. Thus $x_1^* < \hat{k}$.

Next, we establish that $x_1^* > \mathbf{x}$. Suppose instead that $x_1^* \leq \mathbf{x}$. Since $x_1^* < \hat{k}$, we have $f'(x_1^*) > 1$, so by the Ramsey-Euler equation, $u'(c_1^*) = f'(x_1^*)u'(c_2^*) > u'(c_2^*)$. Thus, $c_1^* < c_2^*$, and

$$f(\mathbf{x}) - x_1^* < f(x_1^*) - x_2^* \leq f(\mathbf{x}) - x_2^* .$$

Hence, $x_2^* < x_1^*$. We can then repeat the steps to obtain

$$x_T^* < x_{T-1}^* < \ldots < x_2^* < x_1^* \leq \mathbf{x} ,$$

so that $x_T^* < \mathbf{x}$, which contradicts the fact that $x_T^* = \mathbf{x}$. Thus $x_1^* > \mathbf{x}$. Q.E.D.

Lemma 3.2: If $(x_t^*)_{t=0}^T$ is the maximal T-plan from $x^* \in (0, k)$, and $(\bar{x}_t)_{t=0}^T$ is the maximal T-plan from $\bar{x} \in (0, k)$, and $x^* > \bar{x}$, then $x_1^* > \bar{x}_1$.

Proof: Suppose the hypotheses of Lemma 3.2 are valid, but $x_1^* \leq \bar{x}_1$. Then, we have $c_1^* = f(x^*) - x_1^* > f(\bar{x}) - \bar{x}_1 = \bar{c}_1$. Thus, we must have $f'(x_1^*) \geq f'(\bar{x}_1)$ and $u'(c_1^*) < u'(\bar{c}_1)$. Using the Ramsey-Euler equations for the two maximal T-plans, we obtain

$$1 > \frac{u'(c_1^*)}{u'(\bar{c}_1)} = \frac{f'(x_1^*)u'(c_2^*)}{f'(\bar{x}_1)u'(\bar{c}_2)} \geq \frac{u'(c_2^*)}{u'(\bar{c}_2)} .$$

This means $u'(c_2^*) < u'(\bar{c}_2)$, and so $c_2^* > \bar{c}_2$. Thus, we obtain

$$f(x_1^*) - x_2^* > f(\bar{x}_1) - \bar{x}_2 \geq f(x_1^*) - \bar{x}_2 ,$$

so that $\bar{x}_2 > x_2^*$. The above argument can then be repeated to get $\bar{x}_t > x_t^*$ for $t = 2, \ldots, T$. Thus, by definition of T-plans, we obtain $x^* = x_T^* < \bar{x}_T = \bar{x}$, which contradicts the hypothesis that $x^* > \bar{x}$.

Q.E.D.

Proposition 3.1: The generating function, h, has the following properties: (a) for $x \in (0, k)$, $0 < h(x) < f(x) < k$; (b) for $x \in (0, k)$, $\hat{k} \gtreqless h(x) \gtreqless x$ as $\hat{k} \gtreqless x$; (c) h is increasing on $(0, k)$; (d) h is continuous on $(0, k)$; (e) $\lim_{x \to 0} h(x) = 0$; (f) $\lim_{x \to k} h(x) = k$.

Proof: Clearly, (a) follows from our discussion in Section 2c. Also (b) follows from Lemma 3.1 and (c) follows from Lemma 3.2.

To establish (d), we proceed to apply the Maximum Theorem (Berge, 1963, p. 116). Define $D = (0, k)$, $\bar{D} = [0, k]$; then $\bar{D}^T$ is a compact subset of $\Re^T$, and D is a subset of $\Re$. Note that $U(c_1, \ldots, c_T)$ is a continuous function from $\bar{D}^T$ to $\Re$ (by continuity of u). Also $C(x)$ is a continuous correspondence from D to $\bar{D}^T$ (for definitions of $U(c_1, \ldots, c_T)$ and $C(x)$, see Section 2b). To see this last assertion, note that $C(x)$ is clearly an upper semicontinuous correspondence from D to $\bar{D}^T$, by continuity of f. To check lower semicontinuity of $C(x)$, let $x^s \in D$ for $s = 1, 2, \ldots$, with $x^s \to \bar{x} \in D$, and let $(\bar{c}_1, \ldots, \bar{c}_T) \in C(\bar{x})$. Then there is a T-plan $(\bar{x}_t)_{t=0}^T$ from $\bar{x}$, such that $(\bar{c}_t)_{t=1}^T$ is the consumption sequence generated by it. From our discussion of Section 2b, we know that $(\bar{x}_t, \bar{c}_t) \gg 0$ for $t = 1, \ldots, T$. Thus (using the continuity of f) we can pick $\epsilon > 0$ such that $1 - \epsilon > 0$, $\bar{x}(1 + \epsilon) < k$, and for all $\lambda \in [1 - \epsilon,\ 1 + \epsilon]$, $f(\lambda \bar{x}_t) - \lambda \bar{x}_{t+1} > 0$ for $t = 0, \ldots, T - 1$. Since $x^s \to x$ as $s \to \infty$, there is some $\bar{s}$, such that for $s \geq \bar{s}$, $(x^s/\bar{x}) \in [1 - \epsilon,\ 1 + \epsilon]$. For $s \geq \bar{s}$, define $\lambda^s = (x^s/\bar{x})$, and a sequence $(x_t^s)_{t=0}^T$ by $x_t^s = \lambda^s \bar{x}_t$ for $t = 0, \ldots, T$. Then by construction, $(x_t^s)_{t=0}^T$ is a T-plan for each $s \geq \bar{s}$, and its associated consumption sequence $(c_t^s)_{t=1}^T$ satisfies $c_t^s = f(\lambda^s \bar{x}_t) - \lambda^s \bar{x}_{t+1}$ for $t = 1, \ldots, T$. As $s \to \infty$, we have $x^s \to \bar{x}$, $\lambda^s \to 1$ and $c_t^s \to \bar{c}_t$ for $t = 1, \ldots, T$, by continuity of f. This establishes lower semicontinuity of $C(x)$ on D. Applying the Maximum Theorem, and denoting by $(c_1(x), \ldots, c_T(x))$ the (unique) maximizer of U on $C(x)$ for each $x \in D$, we note that $(c_1(x), \ldots, c_T(x))$ is a continuous function from D to $\bar{D}_T$. Denoting by $(x_t(x))_{t=0}^T$ the T-plan from x, with associated consumption sequence $(c_t(x))_{t=1}^T$, we note that $(x_1(x), \ldots, x_T(x))$ is also a continuous function from D to $\bar{D}^T$. In particular then, $h(x) \equiv x_1(x)$ is a continuous function on D.

We can establish (e) as follows. For $0 < x < \hat{k}$, we have, by (a) and (b), $x < h(x) < f(x)$. Thus as $x \to 0$, $f(x) \to 0$ by continuity of f and $f(0) = 0$, so that $h(x) \to 0$.

If (f) were violated, then there would exist a sequence (x^s), $s = 1, 2, \ldots$, such that $x^s \to k$ as $s \to \infty$, and $h(x^s) \to k' < k$ (using (c), $k' \geq \hat{k}$). Clearly, $f^{T-1}(k') < k$ (where f^{T-1} is the $(T-1)$ iteration of the function, f), and so we must have, for s large, $f^{T-1}(h(x^s)) < x^s$, which contradicts the definition of h.

Q.E.D.

Proposition 3.2: If (x_t) is a rolling plan from $\mathrm{x} \in (0, k)$, then (a) $\mathrm{x} < \hat{k}$ implies that x_t monotonically increases to $\hat{k}$ as $t \to \infty$; (b) $\mathrm{x} > \hat{k}$ implies that x_t monotonically decreases to $\hat{k}$ as $t \to \infty$; (c) $\mathrm{x} = \hat{k}$ implies that $x_t = \hat{k}$ for all $t \geq 1$.

Proof: We will establish (a); (b) and (c) can be proved in a similar manner. If $\mathrm{x} < \hat{k}$, then $\mathrm{x} < h(\mathrm{x}) < \hat{k}$ by Proposition 3.1 (b), so that $\mathrm{x} < x_1 < \hat{k}$. Using Proposition 3.1 (b) again, $x_1 < h(x_1) < \hat{k}$, so that $x_1 < x_2 < \hat{k}$. Repeating this step, we see that x_t monotonically increases while remaining bounded above by $\hat{k}$. Hence, it converges to some k^*, satisfying $0 < k^* \leq \hat{k}$. Using the fact that $x_t \leq h(x_t) \leq x_{t+2}$ for $t \geq 2$, and Proposition 3.1 (d), $h(k^*) = k^*$, so that by Proposition 3.1 (b), $k^* = \hat{k}$.

Q.E.D.

We conclude this section by presenting an example, where the generating function can be numerically computed.

Example 3.1: Let the production function be given by

$$f(x) = 2x^{1/2} \quad \text{for } x \geq 0 .$$

Then f satisfies (A.1)–(A.5); the golden-rule input stock $\hat{k} = 1$, and the maximum sustainable input stock $k = 4$. Let the utility function be given by

$$u(c) = c^{1/2} \quad \text{for } c \geq 0 .$$

Then u satisfies (A.6)–(A.9). Let the planning horizon be fixed at $T = 2$.

A 2-plan from $x \in (0, 4)$ is then a vector (x, x_1, x), with $0 \leq x_1 \leq f(x)$ and $x \leq f(x_1)$. If (x, x_1, x) is a maximal 2-plan from $x \in (0, 4)$, then using the Ramsey-Euler equations, we get

$$2x^{1/2} - x_1 + xx_1 = 2x_1^{3/2} .$$

Denoting $x^{1/2}$ by β, and $x_1^{1/2}$ by α, we get

$$2\alpha^3 + (1 - \beta^2)\,\alpha^2 - 2\beta = 0 \ , \tag{1}$$

where $\beta \in (0, 2)$.

Given $\beta \in (0, 2)$, equation (1) is a cubic in α. Since it is of odd degree, with the last coefficient negative ($-2\beta < 0$) and first coefficient positive ($2 > 0$) it has *at least* one positive real root. On the other hand, by Descartes' rule of signs, it has *at most* one positive real root (since regardless of the sign of $(1 - \beta^2)$, there is exactly one change of sign in the equation). Thus, there is exactly one positive real root to equation (1). If we call this root $\phi(\beta)$, then the generating function, h, is given by

$$h(x) = [\phi(x^{1/2})]^2 \ . \tag{2}$$

While our interest is naturally in the unique positive root of equation (1), we note that *all* the roots of the equation can be found by the standard Cardan-Tartaglia method or the trigonometric method, depending on the sign of the discriminant (see Birkhoff and MacLane, 1977, chs. 4, 5 for details).

The graph of the generating function defined in (2) can be numerically computed and is shown in Figure 1.

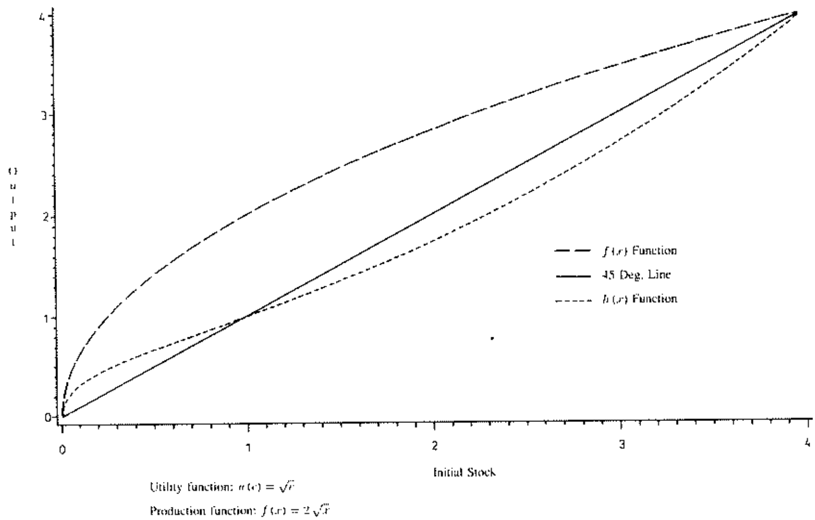

Fig. 1. Rolling plan generating function

4. Welfare Properties of Rolling Plans

The purpose of this section is to show that rolling plans are optimal according to the criterion given in Section 2a. To this end, we establish that rolling plans are efficient (Theorem 4.1), and "good" in the sense of Gale (1967) (Theorem 4.2). Both results are crucially dependent on the fact that the sequence of input stocks of a rolling plan converges to the golden-rule input stock at a geometric rate (Lemma 4.2). This result, in turn, is derived from the property of the generating function, that it has a derivative at the golden-rule input stock with a value strictly between zero and one (Lemma 4.1). Since good plans always maximize the long-run average utility, the above results can be combined to establish the optimality of rolling plans (Theorem 4.3).

In what follows, we fix the time horizon of T-plans at $T = 2$. In the more general case ($T \geq 2$), the result can be obtained by using monotone properties of maximal T-plans with respect to the initial stock and the length of the horizon.

We first introduce some notation which will ease the writing of our results and proofs. Recall that $\hat{c} \equiv f(\hat{k}) - \hat{k} > 0$. We write

$$\begin{aligned} \theta &= \min\left[\{\hat{c}/4f'(\hat{k}/2)\},\ (\hat{k}/2)\right] , \\ \Theta &= (\hat{k}-\theta,\ \hat{k}+\theta) . \end{aligned} \tag{3}$$

It follows from (3) that $\hat{k} - \theta > 0$, and $0 < \hat{c} - \theta = f(\hat{k}) - (\hat{k} + \theta) < k - (\hat{k} + \theta)$; thus, $\hat{k} + \theta < k$. Consequently the set Θ is an open sub-interval of $(0, k)$. Note that if $(x, z) \in \Theta^2$, then $f(x) - z \geq f(\hat{k} - \theta) - f(\hat{k}) + f(\hat{k}) - (\hat{k} + \theta) \geq f'(\hat{k} - \theta)(-\theta) + \hat{c} - \theta > (3\hat{c}/4) - \theta f'(\hat{k}/2) \geq (\hat{c}/2) > 0$.

Lemma 4.1: There is $\delta \in (0, \theta)$ (where θ is given by (3)) such that the generating function h is continuously differentiable on the set

$$\Delta \equiv [\hat{k} - \delta,\ \hat{k} + \delta] \tag{4}$$

and $0 < h'(x) < 1$ for all $x \in \Delta$.

Proof: Consider the function γ: $\Theta^2 \to \Re$ defined by

$$\gamma(x, z) \equiv u'(f(x) - z) + u'(f(z) - x)f'(z) .$$

Then γ is continuously differentiable on Θ^2. Furthermore, $\gamma(\hat{k}, \hat{k}) = 0$ and $\gamma_z(\hat{k}, \hat{k}) < 0$. So, by the implicit function theorem, there are open neighborhoods N_0 and N_1 of $\hat{k}$ (where both N_0 and N_1 are subsets of Θ), and a unique function L: $N_0 \to N_1$, such that $\hat{k} = L(\hat{k})$, and $\gamma(x, L(x)) = 0$ for all $x \in N_0$. Furthermore L is continuously differentiable on N_0.

Since N_0 and N_1 are open, we can find $\theta > \delta' > 0$ such that $N' \equiv (\hat{k} - \delta', \hat{k} + \delta')$ is a subset of N_0 and N_1. Since $h(\hat{k}) = \hat{k}$, and h is continuous on $(0, k)$ by Proposition 3.1, we can find $0 < \hat{\delta} < \delta'$, such that for $x \in \hat{N} \equiv (\hat{k} - \hat{\delta},\ \hat{k} + \hat{\delta})$, $h(x) \in N'$.

Now, we observe that by the definition of h, $\gamma(x,\ h(x)) = 0$ for all $x \in \hat{N}$ (see Section 2c). By the implicit function theorem (above), this is possible iff $h(x) = L(x)$ for all $x \in \hat{N}$. This proves that in the neighborhood $\hat{N}$ of $\hat{k}$, h is continuously differentiable.

Evaluating the derivative of h on $\hat{N}$,

$$h'(x) = \frac{u''(f(x) - z)f'(x) + u''(f(z) - x)f'(z)}{u''(f(x) - z) + u''(f(z) - x)f'(z)^2 + u'(f(z) - x)f''(z)} ,$$

where $z = h(x)$. Thus, evaluating the derivative of h at $\hat{k}$, we get

$$h'(\hat{k}) = 2u''(\hat{c})/[2u''(\hat{c}) + u'(\hat{c})f''(\hat{k})] .$$

Using the facts that $u'(\hat{c}) > 0$, $f''(\hat{k}) < 0$ and $u''(\hat{c}) < 0$, we obtain $0 < h'(\hat{k}) < 1$.

Since h' is continuous on $\hat{N}$, and $0 < h'(\hat{k}) < 1$, we can find $0 < \delta < \hat{\delta}$ such that on $\Delta \equiv [\hat{k} - \delta,\ \hat{k} + \delta]$, we have $0 < h'(x) < 1$. Clearly, $\delta \in (0, \theta)$.

Q.E.D.

Lemma 4.2: Suppose (x_t) is a rolling plan from $x \in (0, k)$. Then there is $A > 0$, a positive integer S, and $\rho \in (0, 1)$, such that $x_t \in \Delta$ for $t \geq S$ (where Δ is given by Lemma 4.1), and

$$|\, x_t - \hat{k} \,| \leq A\rho^t \quad \text{for } t \geq S .$$

Proof: Consider the interval Δ obtained in Lemma 4.1. We know that h' is continuous on Δ, and $0 < h'(x) < 1$ for all $x \in \Delta$. Let ρ be the maximum value of h' on Δ; then $0 < \rho < 1$.

By Proposition 3.2, there is a positive integer S, such that $x_t \in \Delta$ for all $t \geq S$. Since h is continuously differentiable on Δ, we can use the Mean Value theorem for each $t \geq S$ to obtain

$$|\, x_{t+1} - \hat{k} \,| = |\, h(x_t) - h(\hat{k}) \,| = |\, h'(z_t) \,|\, |\, x_t - \hat{k} \,| ,$$

where z_t is between x_t and $\hat{k}$. Since $z_t \in \Delta$, we obtain $0 < h'(z_t) \leq \rho < 1$. Thus, for each $t \geq S$,

$$|x_{t+1} - \hat{k}| \leq \rho |x_t - \hat{k}| .$$

Iterating this inequality we obtain for $t \geq S$

$$|x_t - \hat{k}| \leq \rho^{t-S} |x_s - \hat{k}| \leq \rho^{t-S} \delta ,$$

where δ is given by Lemma 4.1. Defining $A = (\delta/\rho^S)$, we have $|x_t - \hat{k}| \leq A\rho^t$ for $t \geq S$, which proves the Lemma.

Q.E.D.

Theorem 4.1: Suppose (x_t) is a rolling plan from $x \in (0, k)$. Then (x_t) is efficient.

Proof: If $x \leq \hat{k}$, then by Proposition 3.2, $x \leq x_t \leq \hat{k}$ for all $t \geq 0$. Thus, using the characterization of efficiency given in Cass (1972), (x_t) is efficient.

If $x > \hat{k}$, we proceed as follows. Consider the set $\Delta = [\hat{k} - \delta,\ \hat{k} + \delta]$ obtained in Lemma 4.1. Using Lemma 4.2, we can find $A > 0$, a positive integer S, and $\rho \in (0, 1)$, such that $x_t \in \Delta$ for $t \geq S$, and

$$|x_t - \hat{k}| \leq A\rho^t \quad \text{for } t \geq S .$$

Let m be the maximum value of $[-f''(x)]$ on Δ. Choose $s \geq S$, such that $[mA\rho^s/(1-\rho)] \leq (1/2)$. Now, for $t \geq s$, we have $x_t \in \Delta$, and so by the Mean Value theorem, $f'(x_t) - f'(\hat{k}) = f''(z_t)(x_t - \hat{k})$, where $x_t \geq z_t \geq \hat{k}$. Since $z_t \in \Delta$, we have $[-f''(z_t)] \leq m$. Thus, for $t \geq s$, $f'(x_t) \geq 1 - m(x_t - \hat{k}) \geq 1 - mA\rho^t = 1 - (mA\rho^s)\rho^{t-s} \geq 1 - [(1-\rho)\rho^{t-s}/2]$. Using this information, we obtain for $t \geq s$,

$$\begin{aligned}
\pi_{t+1} = \prod_{n=0}^{t} f'(x_n) &\geq \pi_s \prod_{n=s}^{t} \{1 - [(1-\rho)\rho^{n-s}/2]\} \\
&\geq \pi_s \{1 - \sum_{n=s}^{t} [(1-\rho)\rho^{n-s}/2]\} \\
&\geq \pi_s \{1 - \sum_{n=0}^{\infty} [(1-\rho)\rho^{n}/2]\} \\
&= \pi_s \{1 - (1/2)\} = (1/2)\pi_s .
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} \pi_{t+1}$ is divergent. Since $x_t \geq \hat{k}$ for $t \geq 0$, we can again use the characterization of efficiency in Cass (1972) to conclude that (x_t) is efficient.

Q.E.D.

Following Gale (1967), we define a plan (x_t) to be *good* if there is a real number B such that

$$\sum_{t=1}^{\tau}[u(c_t) - u(\hat{c})] \geq B \quad \text{for all } \tau \geq 1 \ .$$

Theorem 4.2: Suppose (x_t) is a rolling plan from $x \in (0, k)$. Then (x_t) is good.

Proof: Using Lemma 4.2, there is $A > 0$, a positive integer S and $\rho \in (0, 1)$, such that $x_t \in \Delta$ (where Δ is given by Lemma 4.1) and $|x_t - \hat{k}| \leq A\rho^t$ for $t \geq S$. Let b be the maximum value of f' on Δ. For $t \geq S$, $(x_t, \ x_{t+1})$ is in Δ^2 and so in Θ^2, where Θ is given by (3). Thus $f(x_t) - x_{t+1} \geq (\hat{c}/2)$ for $t \geq S$.

Consider first the case where $x < \hat{k}$. In this case, by Proposition 3.2, $0 \leq (\hat{x} - x_t) \leq A\rho^t$ for $t \geq S$. Then, for $t \geq S$, we obtain $\hat{c} - c_{t+1} = [f(\hat{k}) - \hat{k}] - [f(x_t) - x_{t+1}] \leq [f(\hat{k}) - f(x_t)] \leq f'(x_t)[\hat{k} - x_t] \leq bA\rho^t$. This information yields for $t \geq S$, $u(\hat{c}) - u(c_{t+1}) \leq u'(c_{t+1})(\hat{c} - c_{t+1}) \leq u'(\hat{c}/2)bA\rho^t$. Summing this inequality from S to ∞, we get

$$\sum_{t=S}^{\infty}[u(\hat{c}) - u(c_{t+1})] \leq u'(\hat{c}/2)bA\rho^S/(1 - \rho) \ .$$

It follows immediately that (x_t) is good.

Consider next the case where $x \geq \hat{k}$. By Proposition 3.2, $0 \leq (x_t - \hat{x}) \leq A\rho^t$ for $t \geq S$. Then for $t \geq S$, we obtain $\hat{c} - c_{t+1} = [f(\hat{k}) - \hat{k}] - [f(x_t) - x_{t+1}] \leq (x_{t+1} - \hat{k}) \leq A\rho^{t+1}$. This yields for $t \geq S$

$$u(\hat{c}) - u(c_{t+1}) \leq u'(c_{t+1})(\hat{c} - c_{t+1}) \leq u'(\hat{c}/2)A\rho^{t+1} \ .$$

Then, following the above procedure, (x_t) is good.

Q.E.D.

Theorem 4.3: Suppose (x_t) is a rolling plan from $x \in (0, k)$. Then (x_t) is optimal.

Proof: Recall that $\hat{c} = f(\hat{k}) - \hat{k}$ is the golden-rule consumption. Denote $u'(\hat{c})$ by $\hat{p}$. Let (x'_t) be any plan from x. Then, for $t \geq 1$, $[u(c'_t) - u(\hat{c})] \leq \hat{p}[c'_t - \hat{c}] = \hat{p}[\{f(x'_{t-1}) - x'_t\} - \{f(\hat{k}) - \hat{k}\}] =$

$\hat{p}[f(x'_{t-1}) - f(\hat{k})] - \hat{p}[x'_t - \hat{k}] \leq \hat{p}[x'_{t-1} - \hat{k}] - \hat{p}[x'_t - \hat{k}]$, using the concavity of f and $f'(\hat{k}) = 1$. Thus, summing this inequality from $t = 1$ to $t = \tau$ we get

$$\sum_{t=1}^{\tau}[u(c'_t) - u(\hat{c})] \leq \hat{p}x \ .$$

This yields for all $\tau \geq 1$

$$(1/\tau)\sum_{t=1}^{\tau} u(c'_t) \leq u(\hat{c}) + (\hat{p}x/\tau) \ .$$

Thus, taking the inferior limit of both sides,

$$\liminf_{\tau\to\infty} (1/\tau)\sum_{t=1}^{\tau} u(c'_t) \leq u(\hat{c}) \ . \tag{5}$$

Using Theorem 4.2 above, we know that (x_t) is good. Thus, there is some real number B such that for all $\tau \geq 1$,

$$\sum_{t=1}^{\tau}[u(c_t) - u(\hat{c})] \geq B \ .$$

This yields for all $\tau \geq 1$

$$(1/\tau)\sum_{t=1}^{\tau} u(c_t) \geq u(\hat{c}) + (B/\tau) \ .$$

Again, taking the inferior limit of both sides,

$$\liminf_{\tau\to\infty} (1/\tau)\sum_{t=1}^{\tau} u(c_t) \geq u(\hat{c}) \ . \tag{6}$$

Inequalities (5) and (6) clearly imply that the rolling plan (x_t) maximizes long-run average utility. Using Theorem 4.1 above, (x_t) is also efficient. Hence, (x_t) is optimal.

Q.E.D.

Remarks:

(i) The fact that a good plan maximizes long-run average utility among all plans has already been noted in the literature (see, for example, Jeanjean, 1974). We have given the proof here for the sake of completeness.

(ii) A plan which maximizes long-run average utility need not be good. Consider $0 < x < \hat{k}$ satisfying $f(x) > \hat{k}$, and a sequence (x_t) defined by $x_t = x$ for $t = 2^n$ $(n = 0, 1, 2, \ldots)$ and $x_t = \hat{k}$ for $t \neq 2^n$. It can be checked that (x_t) is a plan from x which maximizes long-run average utility and is also efficient but is not good. Thus, Theorems 4.1 and 4.2 actually establish a stronger welfare result about rolling plans than is reflected in Theorem 4.3.

(iii) The properties of efficiency and "goodness" of a plan are independent of each other. An efficient plan need not be good (see the example in remark (ii) above). Similarly, a good plan need not be efficient. Consider $\hat{k} < x < k$, and a sequence (x_t) defined by $x_0 = x$, $x_{t+1} = f(x_t) - \hat{c}$ for $t \geq 0$. It can be checked that (x_t) is a plan. It is clearly good, since $c_t = \hat{c}$ for all $t \geq 1$. It is also clearly inefficient since the sequence (x'_t) defined by $x'_0 = x$, $x'_t = \hat{k}$ for $t \geq 0$ is a plan from x with $c_1 > \hat{c}$ and $c_t = \hat{c}$ for all $t \geq 2$.

5. Decentralized Evolutionary Mechanisms

5a. Evolutionary Mechanisms

In the rest of the paper, we are primarily concerned with the problem of realizing optimal allocations in our intertemporal economy with the help of a suitably constructed "decentralized evolutionary mechanism." The notion of a mechanism is by now a familiar one (see, for instance, Mount and Reiter, 1974, and Hurwicz, 1986, for discussions), but we will have to be careful in defining an intertemporal version of it, if we want to capture the special structure of sequential decision-making that is involved. This section is entirely devoted to this task.

We first define formally what we mean by an "evolutionary mechanism." This is followed by an informal discussion of how this mechanism is supposed to operate.

We consider the following objects to be given: a *set of environments* $E \subset \prod_{t=1}^{\infty} E_t$, a *space of allocations* $A = \prod_{t=1}^{\infty} A_t$, and a *state space* $S = \prod_{t=0}^{\infty} S_t$. We consider A_t to be a subset of a finite dimensional real space $\Re^{\ell}$, and S_t to be a subset of a finite dimensional real space $\Re^{q}$, for all $t \geq 1$.

An *evolutionary mechanism* is a sequence (M_t, G_t, H_t) where:

(a) M_t, the *message space* in period t, is a subset of a finite dimensional real space, denoted by $\Re^{m}$;

(b) G_t, the *verification function* in period t, is a mapping from $E_t \times S_{t-1} \times M_t$ to a finite dimensional real space, denoted

by $\Re^n$. It is required to satisfy the following condition for each $(e_t, s_{t-1}) \in E_t \times S_{t-1}$:
There is a unique message $m_t \in M_t$ such that $G_t(e_t, s_{t-1}, m_t) = 0$;

(c) $H_t \equiv (H_t^1, H_t^2)$, the *outcome function* in period t, is a mapping from M_t to $\Re^q \times \Re^\ell$. It is required to satisfy the following condition for each $(e_t, s_{t-1}) \in E_t \times S_{t-1}$:
If $m_t \in M_t$ and $G_t(e_t, s_{t-1}, m_t) = 0$ then $H_t(m_t) \in S_t \times A_t$.

Given an evolutionary mechanism (M_t, G_t, H_t), we define the *equilibrium message function* in period t as

$$\mu_t(e_t, s_{t-1}) = \{m_t \in M_t\colon\ G_t(e_t, s_{t-1}, m_t) = 0\}$$

for each $(e_t, s_{t-1}) \in E_t \times S_{t-1}$. The *equilibrium outcome function* in period t is defined as

$$\nu_t(e_t, s_{t-1}) = \{H_t(m_t)\colon\ m_t \in \mu_t(e_t, s_{t-1})\}$$

for each $(e_t, s_{t-1}) \in E_t \times S_{t-1}$. We refer to ν_t^1 $[\equiv H_t^1(m_t)]$ as the *equilibrium state* in period t, and to ν_t^2 $[\equiv H_t^2(m_t)]$ as the *equilibrium allocation* in period t (where m_t is the equilibrium message in period t).

Several observations about the above definitions are worth making at this point:

(i) First, our definition of an evolutionary mechanism is in the same spirit as the notion of an "evolutionary process" introduced by Hurwicz and Weinberger (1990, p. 317) (see particularly the discussion of this concept on pp. 317–318 of their paper). Both the definitions require that the outcome H_t in period t be independent of verifications in the future (that is, verifications for $t{+}1$ onwards). However, the exact equivalence of the two notions is not a subject we wish to pursue here.

(ii) From the point of view of applications, we have found it important to bring in explicitly the notion of a state space, and to include the "state variable" as an argument in the verification functions, G_t.

(iii) The dimensionality constraints on the message space $[M_t \subset \Re^m]$ and on the range of the verification functions $[G_t(e_t, s_{t-1}, m_t) \in \Re^n$ for each $(e_t, s_{t-1}, m_t) \in E_t \times S_{t-1} \times M_t]$ reflect the notion that transmission and usage of information is costly and hence that agents can communicate or process only a finite amount of information in each period.

(iv) The condition which G_t is required to satisfy ensures that μ_t and ν_t are well-defined functions. The condition that there is a *unique* equilibrium message for each $(e_t, s_{t-1}) \in E_t \times S_{t-1}$ is surely restrictive. The more general case of an "equilibrium message correspondence" can, of course, be treated, but at the expense of considerable complication which would add little to the applications we have in mind. Thus, we have deliberately kept the strong restriction on the verification function. Since our main interest is in demonstrating a "possibility result," the more demanding are our requirements for an evolutionary mechanism, the stronger is our possibility result.

To see how this mechanism operates, consider that an environment for period 1, e_1, is given. This would typically describe the preferences of consumers and technological possibilities of producers in period 1.

Consider, also, that an initial state, s_0, is given. The initial state would typically be described by the various capital and resource stocks available at the end of period zero.

In period 1, the mechanism designer proposes a message m_1 to agents. The agents (knowing e_1 and s_0) would then verify whether $G_1(e_1, s_0, m_1) = 0$. Notice that agents in period 1 are being required to be informed about the environment in period 1, e_1, as well as the previous period's state, s_0; these appear to be plausible requirements. If $G_1(e_1, s_0, m_1) = 0$, then m_1 is the equilibrium message. (If $G_1(e_1, s_0, m_1) \neq 0$, another message has to be proposed and the process has to be repeated until the equilibrium message is found. How the equilibrium message is found is itself a topic of considerable interest; but we will not be concerned with it here.)

If m_1 is the equilibrium message in period 1, the outcome function H_1^1 specifies a state s_1 in S_1, consisting typically of capital and resource stocks available at the end of period 1 and the outcome function H_1^2 specifies an allocation a_1 in A_1, consisting typically of consumption and investment decisions in period 1. This state, allocation pair is the "equilibrium outcome" of period 1. It is to be understood that the allocation corresponding to this equilibrium outcome is actually carried out; similarly the state corresponding to this equilibrium outcome is actually attained.

In period 2, knowing the state that was actually attained in period 1 (s_1) and the environment in period 2 (e_2), the same procedure yields the equilibrium outcome of period 2, and hence the state at the end of period 2. This step is repeated indefinitely.

The above description of the operation of a mechanism (M_t, G_t, H_t) implies that if an environment $e \in E$ is given, and an initial state $s \in S_0$ is specified, the mechanism then defines uniquely the state and allocation sequence for all $t \geq 1$. Specifically, the *state sequence* (s_t), generated by the mechanism (M_t, G_t, H_t) is defined by

$$s_0 = s, \ s_t = \nu_t^1(e_t, s_{t-1}) \quad \text{for } t \geq 1 \ .$$

The *allocation sequence* (a_t) generated by the mechanism (M_t, G_t, H_t) is defined by

$$a_t = \nu_t^2(e_t, s_{t-1}) \quad \text{for } t \geq 1 .$$

5b. Decentralization

The evolutionary mechanism defined in the previous subsection need not be decentralized. This is because in verifying whether a message is an equilibrium message in some period t, we did not rule out the possibility of a single agent having the full information, e_t, about the environment. We do so now by formally introducing the notion of decentralization of information.

Assume that for each t, the set of environments at that date, E_t, is defined by two independent pieces of information, which are held by two separate agents. Specifically, assume that

$$E_t = U_t \times F_t \ ,$$

where U_t is to be interpreted as a set of utility functions (with typical element u_t), and F_t a set of production functions (with typical element f_t).

The evolutionary mechanism (M_t, G_t, H_t) is said to be *decentralized* if there exist sequences (A_t, B_t) such that
(i) A_t and B_t map $E_t \times S_{t-1} \times M_t$ into $\Re^n$,
(ii) $G_t(u_t, f_t, s_{t-1}, m_t) = A_t(u_t, s_{t-1}, m_t) + B_t(f_t, s_{t-1}, m_t)$.
This definition follows Hurwicz and Weinberger (1990) closely. Their paper also contains alternative but equivalent ways of defining the concept of decentralization. The definition basically conveys the idea that a consumer's "response" (A_t) to a message (m_t) can utilize only the information which consumers have, namely his utility function and the previous period's state (which is treated as "common knowledge"); a similar remark applies to a producer's "response" (B_t) to a message (m_t).

5c. Evaluation of the Performance of a Mechanism

The performance of a mechanism is evaluated by setting up a goal correspondence which, loosely speaking, specifies a set of allocations judged to be "socially desirable." Formally, a *goal correspondence* is a mapping Q from $E \times S_0$ to subsets of A.

For each specification of an environment $e \in E$, and an initial state $s \in S_0$, Q specifies a set of allocation sequences (a_t), the attainment of which should be the aim of a constructed mechanism.

The mechanism (M_t, G_t, H_t) is said to *realize the goal correspondence* Q if for each $(e; s) \in E \times S_0$, the sequence of allocations, (a_t), generated by the mechanism belongs to $Q(e, s)$.

6. Decentralized Evolutionary Realization of Optimality

In this final section, we show that a rolling plan can be obtained through a suitably designed decentralized evolutionary mechanism. This establishes, in particular, the following possibility result: there is a decentralized evolutionary mechanism which realizes optimality as defined in section 2a.

We begin by specifying the set of environments. Let $\xi > 0$ be a fixed real number, and $\bar{k}$ be another fixed real number satisfying $0 < \bar{k} < \xi$. Define

$$\begin{aligned} F &= \{f\colon \Re_+ \to \Re_+ \mid f \text{ satisfies (A.1)–(A.5) and} \\ &\qquad f(\bar{k}) \geq \bar{k},\ f(\xi) < \xi,\ f'(\bar{k}) < 1\}, \\ U &= \{u\colon \Re_+ \to \Re \mid u \text{ satisfies (A.6)–(A.9)}\} . \end{aligned}$$

We now specify $E_t = U \times F$ for all $t \geq 1$ and $E = \{(u, f)^\infty \colon (u, f) \in U \times F\}$. Thus, our intertemporal framework is interpreted as one with one consumer (with time-stationary utility function $u \in U$) and one producer (with time-stationary production function $f \in F$).

Notice that given any $f \in F$, there is a golden-rule stock, $\hat{k}$, and a maximum sustainable stock, k. (This follows, as in Section 2a, from assumptions (A.1)–(A.5).) Furthermore, our definition of F ensures that $\hat{k} < \bar{k} \leq k < \xi$.

We specify the space of allocations by $A_t = \Re_+$ for all $t \geq 1$, and the state space by $S_t = (0, \bar{k})$ for all $t \geq 0$. Thus, a typical allocation in period t should be interpreted as the consumption in that period; a typical state in period t will be the input stock at the end of that period.

Proposition 6.1: There is a decentralized evolutionary mechanism, such that if the initial state is $\mathrm{x} \in (0, \bar{k})$, then the state sequence (x_t^*) generated by the mechanism is the rolling plan from x.

Proof: We specify the required mechanism, and simply check that it satisfies the property stated in the Proposition.

Define the message space in period t, $M_t = \Re^4_{++}$ for all $t \geq 1$; we write (suggestively) the typical element of M_t as

$$m_t = (x_t, c_t, d_{t+1}, r_t) \; .$$

The verification function in period t, G_t, is specified by defining A_t and B_t as follows for $t \geq 1$:

$$A_t(u, x_{t-1}, m_t) \;=\; \begin{bmatrix} [u'(c_t)/u'(d_{t+1})] - r_t \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B_t(f, x_{t-1}, m_t) \;=\; \begin{bmatrix} 0 \\ x_t + c_t - f(x_{t-1}) \\ x_{t-1} + d_{t+1} - f(x_t) \\ f'(x_t) - r_t \end{bmatrix}.$$

Now, $G_t(u, f, x_{t-1}, m_t)$ is defined as $A_t(u, x_{t-1}, m_t) + B_t(f, x_{t-1}, m_t)$ for $t \geq 1$. Note that, given (u, f) and $x_{t-1} \in (0, \bar{k})$, there is a unique maximal 2-plan from x_{t-1}, given by $(x_{t-1}, x_t^*, x_{t-1})$ with associated consumption sequence (c_t^*, d_{t+1}^*) satisfying $c_t^* = f(x_{t-1}) - x_t^*$ and $d_{t+1}^* = f(x_t^*) - x_{t-1}$. Furthermore, by the Ramsey-Euler equations, we have $[u'(c_t^*)/u'(d_{t+1}^*)] = f'(x_t^*)$. Thus, if we define $r_t^* = f'(x_t^*)$, and consider the message $m_t^* \equiv (x_t^*, c_t^*, d_{t+1}^*, r_t^*)$ we note that $A_t(u, x_{t-1}, m_t^*) = B_t(f, x_{t-1}, m_t^*) = 0$, so $G_t(u, f, x_{t-1}, m_t^*) = 0$; that is, m_t^* is an equilibrium message in period t.

Consider, next, any equilibrium message $\tilde{m} \equiv (\tilde{x}_t, \tilde{c}_t, \tilde{d}_{t+1}, \tilde{r}_t)$. Then $G_t(u, f, x_{t-1}, \tilde{m}_t) = 0$, and so $A_t(u, x_{t-1}, \tilde{m}_t) = 0 = B_t(f, x_{t-1}, \tilde{m}_t)$. This means that we have

$$\begin{aligned} u'(\tilde{c}_t)/u'(\tilde{d}_{t+1}) &= \tilde{r}_t \; , \\ \tilde{x}_t + \tilde{c}_t &= f(x_{t-1}) \; , \\ x_{t-1} + \tilde{d}_{t+1} &= f(\tilde{x}_t) \; , \\ f'(\tilde{x}_t) &= \tilde{r}_t \; . \end{aligned}$$

Thus $(x_{t-1}, \tilde{x}_t, x_{t-1})$ is a 2-plan from x_{t-1} with associated consumption sequence $(\tilde{c}_t, \tilde{d}_{t+1})$, which satisfies $\tilde{c}_t > 0$, $\tilde{d}_{t+1} > 0$ and

$$u'(\tilde{c}_t)/u'(\tilde{d}_{t+1}) = f'(\tilde{x}_t) \; .$$

Then by Proposition 2.1, $(x_{t-1}, \tilde{x}_t, x_{t-1})$ is a maximal 2-plan from x_{t-1}. Since $(x_{t-1}, x_t^*, x_{t-1})$ is the unique maximal 2-plan from x_{t-1}

we have $\tilde{x}_t = x_t^*$, $\tilde{c}_t = c_t^*$, $\tilde{d}_{t+1} = d_{t+1}^*$ and $\tilde{r}_t = r_t^*$. That is, m_t^* is the unique equilibrium message. We have now checked that G_t, as defined above, is a verification function.

Finally, define the outcome function, H_t, as the map

$$H_t(x_t, c_t, d_{t+1}, r_t) = (x_t, c_t) \; .$$

For the equilibrium message m_t^*, we have by Lemma 3.1, $x_t^* < \bar{k}$, so $x_t^* \in S_t$ as required; also $c_t^* > 0$, so $c_t^* \in A_t$ as required. We have now demonstrated that (M_t, G_t, H_t) as defined above is a decentralized, evolutionary mechanism.

Consider period 1, with $\mathrm{x} \in (0, \bar{k})$ the state in period 0. Our above demonstration has shown that if the unique maximal 2-plan from x is $(\mathrm{x}, x_1^*, \mathrm{x})$, then x_1^* is the equilibrium state (generated by the mechanism) in period 1. This means that $x_1^* = h(\mathrm{x})$, where h is defined in Section 2c. Repeating this step for $t = 2, 3, \ldots$ shows that the state sequence (x_t^*) generated by the mechanism (M_t, G_t, H_t) satisfies $x_0^* = \mathrm{x}$, and $x_{t+1}^* = h(x_t^*)$ for $t \geq 0$. That is, it is the rolling plan from x.

Q.E.D.

Define the optimality goal correspondence, Q, as follows. For each $(u, f, \mathrm{x}) \in U \times F \times (0, \bar{k})$, let

$$\begin{aligned} Q(u, f, \mathrm{x}) \;=\; & \{(c_t)\text{: there is an optimal plan } (x_t) \text{ from x,} \\ & \quad \text{whose associated consumption sequence is } (c_t)\} \; . \end{aligned}$$

That is, the goal is to attain a plan which is optimal as defined in Section 2.1. We can now state our "possibility result" as follows.

Theorem 6.1: There is a decentralized evolutionary mechanism which realizes the optimality goal correspondence.

Proof: Consider the decentralized evolutionary mechanism (M_t, G_t, H_t) constructed in the proof of Proposition 6.1. Then, given any $(u, f) \in U \times F$, and any $\mathrm{x} \in (0, \bar{k})$, the sequence of states (x_t^*) generated by the mechanism is the rolling plan from x. Thus, the sequence of allocations (c_t^*) generated by the mechanism is the consumption sequence associated with the rolling plan from x. By Theorem 4.3, the rolling plan from x is an optimal plan from x. Hence, the sequence of allocations (c_t^*) generated by the mechanism belongs to $Q(u, f, \mathrm{x})$. That is, (M_t, G_t, H_t) realizes the optimality goal correspondence.

Q.E.D.

Remarks:

(i) We make the following somewhat informal observation about the verification functions, G_t, which appear in our constructed mechanism.

Notice that in period t; given the message $m_t = (x_t, c_t, d_{t+1}, r_t)$ the consumer is asked to verify

$$u'(c_t)/u'(d_{t+1}) = r_t \ .$$

The consumer, knowing u, can surely do this. The condition to be verified is simply the equality of the marginal rate of intertemporal substitution on the consumption side with an appropriate "shadow" price ratio, r_t.

The producer is asked to verify

$$\begin{aligned} x_t + c_t &= f(x_{t-1}) \ , \\ x_{t-1} + d_{t+1} &= f(x_t) \ , \\ f'(x_t) &= r_t \ . \end{aligned}$$

The producer, knowing f and the previous period's input stock x_{t-1}, can do this. The first two conditions are to be interpreted as verification of feasibility. The third condition is simply the equality of the marginal rate of transformation on the production side with an appropriate "shadow" price ratio, r_t.

The above verifications imply that the following conditions hold:

$$\begin{aligned} x_t + c_t &= f(x_{t-1}) \ , \\ x_{t-1} + d_{t+1} &= f(x_t) \ , \\ u'(c_t)/u'(d_{t+1}) &= f'(x_t) \ . \end{aligned}$$

But these conditions constitute a complete characterization of the maximal 2-plan from x_{t-1}, according to Proposition 2.1. Thus, the equilibrium state of the mechanism in period t is precisely the same as $h(x_{t-1})$, the rolling plan input in period t.

(ii) If (m_t) is the sequence of equilibrium messages of our constructed mechanism (given u, f and $\mathbf{x}$), then

$$r_t = f'(x_t) \ ,$$

where (x_t) is the sequence of equilibrium states generated by the mechanism, and therefore also the rolling plan from $\mathbf{x}$. If we define

$$p_0 = 1 \text{ and } p_{t+1} = (p_t/r_t) \quad \text{for } t \geq 0 \ ,$$

then we have for $t \geq 0$

$$p_{t+1}f(x_t) - p_t x_t \geq p_{t+1}f(x) - p_t x \quad \text{for all } x \geq 0 .$$

That is, (p_t) is a sequence of intertemporal profit maximizing prices, the shadow prices that Malinvaud (1953) was concerned with.

The shadow prices that Gale and Sutherland (1968) are concerned with would, in addition, have to satisfy for $t \geq 1$

$$u(c_t) - p_t c_t \geq u(c) - p_t c \quad \text{for all } c \geq 0 .$$

This in our framework would require

$$[u'(c_t)/u'(c_{t+1})] = f'(x_t) \quad \text{for } t \geq 1 ,$$

a condition which is *not* satisfied by the rolling plan (x_t) unless $\mathrm{x} = \hat{k}$, the golden-rule input stock. That is, while for the maximal 2-plan formulated in period t, we have

$$[u'(c_t)/u'(d_{t+1})] = f'(x_t) ,$$

the "second-period consumption" of the maximal 2-plan formulated in period t, d_{t+1}, is not carried out in period $(t+1)$; instead the "first-period consumption" of the maximal 2-plan formulated in period $(t+1)$, c_{t+1}, is carried out in period $(t+1)$.

(iii) An "undiscounted" optimality notion which has figured prominently in the literature is the "overtaking" criterion of Atsumi (1965) and von Weizsäcker (1965); this was subsequently refined by Gale (1967) in terms of the "catching up" criterion. If a plan is "catching-up optimal," it is both efficient and good, and hence maximizes long-run average utility of consumption. That is, it is optimal in the sense of our definition in Section 2a. The converse is not true; the rolling plan from any $\mathrm{x} \in (0, k)$ with $\mathrm{x} \neq \hat{k}$ is optimal in our sense, but is not "catching-up optimal."

We conjecture, but do not attempt to prove, that an analogue of the Hurwicz-Weinberger result can be shown for the undiscounted case; that is, it is impossible for a decentralized evolutionary mechanism to realize the "catching-up optimality" criterion.

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