LECTURES ON MATHEMATICAL ANALYSIS FOR ECONOMISTS

Tapan Mitra
Cornell University

WORKED OUT SOLUTIONS TO PROBLEM SETS

Christopher Handy
Cornell University

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Contents

Part I

Linear Algebra

Chapter 1

Vectors

1.1 Vector Spaces

In defining vector spaces, we will consider the *field* to be given by the set of reals, denoted by $\mathbb{R}$. [One can define this set formally, but we will not do so here.] The elements of $\mathbb{R}$ are called *scalars* or *numbers*.

An *m-vector* x is an ordered set of m numbers $(x_1, \ldots, x_m)$. The number x_i is called the *ith coordinate* of x. We use the notation $x = (x_i)$, meaning x is the vector whose ith coordinate is x_i.

The set of all m-vectors is called *m-space* and is denoted by $\mathbb{R}^m$.

Some Special Vectors:

The *ith unit vector* is the vector whose ith coordinate is 1, and whose other coordinates are zero. We denote the ith unit vector by e^i. The *sum vector*, denoted by u, is the vector all of whose coordinates are 1. The *null vector*, denoted by 0, is the vector all of whose coordinates are 0.

Vector Operations:

Two m-vectors, x and y, are said to be *equal* (written $x = y$) if $x_i = y_i$ for $i = 1, \ldots, m$.

We now define two algebraic operations on the vectors of $\mathbb{R}^m$.

Addition:

If $x = (x_i)$ and $y = (y_i)$ are m-vectors, their *sum* $x + y$ is the vector $(x_i + y_i)$.

Scalar Multiplication:

If $x = (x_i)$ is an m-vector and λ is a number, the *product* λx is the vector (λx_i).

Given these two definitions, a number of properties, listed below, follow immediately. For addition, we have:

(A.1)	$(x+y)+z = x+(y+z)$	[Associative Law]
(A.2)	$x+y = y+x$	[Commutative Law]

(A.3) For every x and y, there is z such that $x + z = y$ [Law of Subtraction]

For multiplication, we have:

(M.1)	$\lambda(x+y) = \lambda x + \lambda y$	[Vector Distributive Law]
(M.2)	$(\lambda+\mu)x = \lambda x + \mu x$	[Scalar Distributive Law]
(M.3)	$\lambda(\mu x) = (\lambda\mu)x$	[Scalar Associative Law]
(M.4)	$1x = x$	[Identity Law]

The properties listed above may be taken as axioms for an abstract algebraic system. Such systems are called *vector spaces*.

The vector space that we will study consists of a field, $\mathbb{R}$; the m-space, $\mathbb{R}^m$; the operations of addition and scalar multiplication. We will generally refer to this vector space itself simply as $\mathbb{R}^m$, although this is clearly a shorthand.

1.2 Linear Dependence of Vectors

A set of vectors $x^1,\ldots,x^n$ in $\mathbb{R}^m$ is *linearly dependent* if there exist numbers $\lambda_1,\ldots,\lambda_n$, not all zero, such that

$$\sum_{i=1}^{n} \lambda_i x^i = 0$$

If the vectors are not linearly dependent, they are called *linearly independent*.

A vector y is a *linear combination* of the vectors $x^1,\ldots,x^n$ if

$$y = \sum_{i=1}^{n} \lambda_i x^i$$

for some numbers $\lambda_i \in \mathbb{R}$. A set of vectors $x^1,\ldots,x^n$ *spans* $\mathbb{R}^m$ if every $y \in \mathbb{R}^m$ can be expressed as a linear combination of $x^1,\ldots,x^n$.

Theorem 1. *(Fundamental Theorem on Vector Spaces): If each of the vectors $y^0, y^1, \ldots, y^m$ in the vector space $\mathbb{R}^n$ is a linear combination of the vectors $x^1,\ldots,x^m$, then the y^i are linearly dependent.*

Corollary 1. *Any set of $(m+1)$ vectors in $\mathbb{R}^m$ is linearly dependent.*

Proof. For any vector x in $\mathbb{R}^m$,

$$x = \sum_{i=1}^{m} x_i e^i$$

Consequently, for any set of $(m+1)$ vectors in $\mathbb{R}^m$, we can write each as a linear combination of the vectors $(e^1, \ldots, e^m)$. So the set of $(m+1)$ vectors must be linearly dependent by Theorem 1. ■

Corollary 2. *Any system of m homogeneous linear equations in $(m+1)$ unknowns has a non-zero solution.*

Proof. We can write the system of m homogeneous linear equations in $(m+1)$ unknowns in the following way:

$$\left.\begin{array}{c} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1m}x_m = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m0}x_0 + a_{m1}x_1 + a_{m2}x_2 + \ldots. + a_{mm}x_m = 0 \end{array}\right\} \tag{1.1}$$

Define $a^j = (a_{1j}, a_{2j}, \ldots., a_{mj})$ for $j = 0, \ldots, m$. Then the a^j are a set of $(m+1)$ vectors in m-space; hence by Corollary 1, they are linearly dependent. So there exist numbers $\lambda_0, \ldots., \lambda_m$, not all zero, such that

$$\sum_{j=0}^{m} \lambda_j a^j = 0$$

Then $(x_j) = (\lambda_j)$ gives the desired solution of (1.1). ■

1.3 Rank and Basis

Let S be a subset of the vector space $\mathbb{R}^m$. The *rank* of S is the maximum number of linearly independent vectors which can be chosen from S.

If r is the rank of S, a set of r linearly independent vectors of S is called a *basis* for S.

Corollary 3. *$\mathbb{R}^m$ has rank m.*

Proof. By Corollary 1, the rank of $\mathbb{R}^m$ is at most m. On the other hand, the unit vectors $e^1, \ldots, e^m$ are clearly linearly independent, so the rank of $\mathbb{R}^m$ is at least m. ■

Theorem 2. *(Basis Theorem) Suppose $x^1, \ldots, x^r$ are linearly independent vectors in S. Then $x^1, \ldots, x^r$ is a basis for S if and only if every vector y in S is a linear combination of the vectors $x^1, \ldots, x^r$.*

Proof. Suppose every $y \in S$ is a linear combination of the x^i. Then any set of more than r vectors in S is linearly dependent by Theorem 1, and so S has rank r, and the x^i are a basis.

Suppose the x^i form a basis of S. Then, S has rank r; so if $y \in S$, the vectors $x^1, \ldots, x^r, y$ are linearly dependent. Thus there exist $\lambda_0, \lambda_1, \ldots, \lambda_r$ in $\mathbb{R}$, not all zero, such that

$$\sum_{i=1}^{r} \lambda_i x^i + \lambda_0 y = 0$$

Clearly $\lambda_0 \neq 0$, otherwise the x^i are linearly dependent. Hence, defining $\mu_i = -\lambda_i/\lambda_0$ for $i = 1, \ldots, r$, we have

$$y = \sum_{i=1}^{r} \mu_i x^i$$

so y is a linear combination of $x^1, \ldots, x^r$. ∎

1.4 Inner Product and Norm

We now introduce a third operation in $\mathbb{R}^m$. If x and y are vectors in $\mathbb{R}^m$, their *inner product* is denoted by xy and is defined as the number:

$$xy = \sum_{i=1}^{m} x_i y_i$$

Let x be an m-vector. The inner product of x and the ith unit vector e^i is

$$xe^i = x_i \;\; for \; i = 1, \ldots, m$$

The inner product of x and the sum vector u is

$$xu = \sum_{i=1}^{m} x_i$$

Some properties of the inner product are listed below:

(I.1)	$xy = yx$	(Commutative Law)
(I.2)	$(\lambda x)y = \lambda(xy)$	(Mixed Associative Law)
(I.3)	$(x+y)z = xz + yz$	(Distributive Law)
(I.4)	$x^2 \equiv xx$ is 0 if and only if $x = 0$	

If $x \in \mathbb{R}^m$, then the (Euclidean) *norm* of x, denoted by $\|x\|$, is the (non-negative) number

$$\|x\| = \left[\sum_{i=1}^{m} x_i^2\right]^{1/2}$$

Thus the norm is the (non-negative) square root of the inner product of x with itself.

The following properties of the norm can be verified:

(N.1) $\|x\| = 0$ if and only if $x = 0$
(N.2) $\|\lambda x\| = |\lambda|\ \|x\|$
(N.3) $\|x+y\| \leq \|x\| + \|y\|$

Property (N.3) is usually referred to as the "triangle inequality" for norm.

Two vectors x and y are called *orthogonal* if their inner product is zero; that is, if $xy = 0$. For orthogonal vectors, we have the Pythagoras theorem:

(N.4) If $xy = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

If x and y are orthogonal m-vectors, they are called *orthonormal* if $\|x\| = \|y\| = 1$.

1.5 Worked Out Problems on Chapter 1

Problem 1 (Linear Dependence and Independence)**.**

Let $S = \{e^1, e^2, \ldots, e^n\}$ be the set of unit vectors in $\mathbb{R}^n$.
(a) Let $T = \{x^1, x^2, \ldots, x^n\}$ be a set of vectors in $\mathbb{R}^n$, defined by:

$$x^1 = e^1 + e^2, x^2 = e^2 + e^3, \cdots, x^{n-1} = e^{n-1} + e^n, x^n = e^n + e^1.$$

Is T a set of linearly independent vectors in $\mathbb{R}^n$? Explain. [Hint: consider two cases, n odd, n even].

(b) Let x be an arbitrary vector in $\mathbb{R}^n$, with $x_n \neq 0$. Let U be the set of vectors defined by $U = \{e^1, e^2, \cdots, e^{n-1}, x\}$. Is U a set of linearly independent vectors in $\mathbb{R}^n$? Explain.

Solution.

(a) We will show that the vectors $x^1, \ldots, x^n$ are linearly dependent when n is even and linearly independent when n is odd.

Consider a linear combination of the vectors of T:

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + \lambda_{n-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_n \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_n + \lambda_1 \\ \lambda_1 + \lambda_2 \\ \lambda_2 + \lambda_3 \\ \lambda_3 + \lambda_4 \\ \vdots \\ \lambda_{n-3} + \lambda_{n-2} \\ \lambda_{n-2} + \lambda_{n-1} \\ \lambda_{n-1} + \lambda_n \end{bmatrix}$$

If n is even, we can find $\lambda_1, \ldots, \lambda_n$ not all zero such that this linear combination gives the zero vector:

$$\lambda_i = \begin{cases} 1, & \text{if } i \text{ odd} \\ -1, & \text{if } i \text{ even} \end{cases} \qquad \text{for all } i = 1, \ldots, n$$

Thus for n even, the vectors of T are linearly dependent.

But if n is odd, these λ_i do not give the zero vector, because then $\lambda_n + \lambda_1 \neq 0$. In fact, we can show that there are no $\lambda_1, \ldots, \lambda_n$ not all zero such that the linear combination above is equal to the zero vector. Suppose there are $\lambda_1, \ldots, \lambda_n$ not all zero such that

$$\begin{bmatrix} \lambda_n + \lambda_1 \\ \lambda_1 + \lambda_2 \\ \vdots \\ \lambda_{n-1} + \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then, reading from the last equation to the first, since n is odd it must be that

$$\lambda_n = -\lambda_{n-1} = \lambda_{n-2} = \cdots = -\lambda_2 = \lambda_1.$$

Since the $\lambda_1, \ldots, \lambda_n$ are not all zero, it must be that each $\lambda_i \neq 0$, so that $\lambda_n + \lambda_1 = 2\lambda_1 \neq 0$. This is a contradiction, so we conclude that when n is odd, the vectors of T are linearly independent.

(b) We can prove the vectors of U are linearly independent by contradiction. Suppose they are linearly dependent, so that there are $\mu_1, \ldots, \mu_n$ not all zero such that

$$\mu_1 e^1 + \cdots + \mu_{n-1} e^{n-1} + \mu_n x = \begin{bmatrix} \mu_1 + \mu_n x_1 \\ \vdots \\ \mu_{n-1} + \mu_n x_{n-1} \\ \mu_n x_n \end{bmatrix} = 0.$$

Since $x_n \neq 0$, from the last row we must have $\mu_n = 0$. But then the equation above only holds if $\mu_i = 0$ for all $i = 1, \ldots, n$. This contradicts the assumption that the $\mu_1, \ldots, \mu_n$ are not all zero, so we conclude that the vectors of U are linearly independent.

Problem 2 (Rank)**.**

(a) Let S be a set of vectors in $\mathbb{R}^n$, defined by:

$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = 2\}$$

What is the rank of S? Explain.

(b) Let T be a set of vectors in $\mathbb{R}^n$, defined by:

$$T = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + 2x_2 + \cdots + nx_n = 0\}$$

What is the rank of T? Explain.

Solution.

(a) First, note that since $S \subseteq \mathbb{R}^n$, it must be that $\operatorname{rank}(S) \leq n$. If we can find n linearly independent vectors in S, we will know that $\operatorname{rank}(S) \geq n$ and this will allow us to conclude that $\operatorname{rank}(S) = n$.

Define vectors $d^i = 2e^i$ for $i = 1, \dots, n$, where the e^i are unit vectors in $\mathbb{R}^n$. Note that each $d^i \in S$. It is easy to check that these vectors are linearly independent: assume they are not, so that there are some $\lambda_1, \dots, \lambda_n$ not all zero such that

$$\lambda_1 d^1 + \cdots + \lambda_n d^n = \begin{bmatrix} 2\lambda_1 \\ \vdots \\ 2\lambda_n \end{bmatrix} = 0.$$

This implies that each $\lambda_i = 0$, which is a contradiction. Thus the n vectors $d^1, \dots, d^n$ are linearly independent, and we conclude that $\operatorname{rank}(S) = n$ as outlined above.

(b) Let's try to gain some intuition from the two-dimensional case: consider the set $T_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 = 0\}$. Now, $\operatorname{rank}(T_2) = 1$ because every vector in T_2 can be expressed as $x_1(1, -\frac{1}{2})$. This suggests that in the n-dimensional case, the restriction $x_1 + 2x_2 + \cdots + nx_n = 0$ reduces the rank of T to $n-1$. [Note this does not happen in part (a) because the restriction on the set S does not involve a zero on the right hand side. Think about this distinction geometrically in $\mathbb{R}^2$.]

To prove that T, which is a subset of $\mathbb{R}^n$, has rank $n-1$, we really need to show two things. First, we must construct $n-1$ linearly independent vectors from T. Second, we must show that any n vectors in T are necessarily linearly dependent. By the definition of $\operatorname{rank}(T)$ as the maximum number of linearly independent vectors that can be chosen from T, this will show that $\operatorname{rank}(T) = n-1$.

We first claim that the $n-1$ vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} -n \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

are elements of T and are linearly independent. These can be written more compactly as $x^i = e^i - ie^1$ for $i = 2, \dots, n$. It is clear that each x^i satisfies $x^i_1 + 2x^i_2 + \cdots + nx^i_n = 0$. Now, suppose the x^i are linearly dependent, so that there are $\mu_2, \dots, \mu_n$ not all zero such that

$$\mu_2 x^2 + \cdots + \mu_n x^n = \begin{bmatrix} -2\mu_2 - \cdots - n\mu_n \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_n \end{bmatrix} = 0.$$

This can hold only if all $\mu_2, \ldots, \mu_n$ are equal to zero, which is a contradiction. Therefore the x^i, as defined above, constitute $n-1$ linearly independent vectors from T, so $\text{rank}(T) \geq n-1$.

Now, it remains only to show that any n vectors from T are linearly dependent. So, consider n arbitrary vectors $y^1, \ldots, y^n$, each an element of T. These are linearly dependent if there exist $\theta_1, \ldots, \theta_n$ not all zero such that

$$\theta_1 \begin{bmatrix} -(2y_2^1 + \cdots + ny_n^1) \\ y_2^1 \\ \vdots \\ y_n^1 \end{bmatrix} + \cdots + \theta_n \begin{bmatrix} -(2y_2^n + \cdots + ny_n^n) \\ y_2^n \\ \vdots \\ y_n^n \end{bmatrix} = 0.$$

The last $n-1$ rows form a system of $n-1$ homogeneous linear equations in the n unknowns $\theta_1, \ldots, \theta_n$, and Corollary 2 tells us this system must have a non-zero solution in the thetas. Now, let's look at the first equation in the above system: regroup terms for

$$-2(\theta_1 y_2^1 + \cdots + \theta_n y_2^n) - \cdots - n(\theta_1 y_n^1 + \cdots + \theta_n y_n^n) = 0.$$

For the $\theta_1, \ldots, \theta_n$ that solve the last $n-1$ equations, this equation also holds, because the left hand side is zero term by term. Remembering that Corollary 2 ensures the $\theta_1, \ldots, \theta_n$ are not all zero, we conclude that any n vectors in T are linearly dependent. This last step shows that $\text{rank}(T) < n$, and together with our conclusion above that $\text{rank}(T) \geq n-1$, we finally have the result: $\text{rank}(T) = n-1$.

Note that it is also possible to prove that any vector $z \in T$ can be written as a linear combination of the vectors $x^2, \ldots, x^n$. By the Basis Theorem, the vectors $x^2, \ldots, x^n$ are then a basis for T, which is only possible if $\text{rank}(T) = n-1$. But the proof above is more fundamental in that it avoids the idea of basis altogether.

Problem 3 (Basis)**.**

Let $S = \{x^1, \ldots, x^m\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, with $m < n$, and let $T = \{y^1, \ldots, y^n\}$ be a set of basis vectors of $\mathbb{R}^n$.

Show that there are $(n-m)$ vectors in the set T, such that the m vectors in S, together with these $(n-m)$ vectors in T, constitute a basis of $\mathbb{R}^n$.

Solution.

The easiest way to prove this is by induction. The idea is that we can first find one of the y^j to replace by x^1 and the resulting set of n vectors will still be a basis for $\mathbb{R}^n$. We can then do the same for x^2, then x^3, and so on until we have replaced m of the y^j by all the $x^1, \ldots, x^m$ and the n vectors still form a basis for $\mathbb{R}^n$. This works because the x^i are linear combinations of the y^j, so whatever property of the y^j is making them a basis is somehow embodied in the x^i. We just have to be careful about which y^j we replace with one of the x^i.

The base case in the induction proof is to show that there is some y^j that can be replaced by x^1 so that the remaining vectors in T, together with x^1, constitute a basis for $\mathbb{R}^n$. Now, note that none of the x^i may be the zero vector; otherwise, S would not be linearly independent. Similarly, none of the y^j may be the zero vector. That means that when we use the fact that $y^1, \ldots, y^n$ is a basis for $\mathbb{R}^n$ to write

$$x^1 = \lambda_1 y^1 + \cdots + \lambda_n y^n, \tag{1.2}$$

we know that $\lambda_1, \ldots, \lambda_n$ are not all zero. Strictly speaking we do not know which λ_i are nonzero, but for notational simplicity we can assume without loss of generality that $\lambda_1 \neq 0$. (Alternatively, you can think of this as reordering the y^j to fit our notation.) This allows us to use (1) to write

$$y^1 = \frac{1}{\lambda_1} x^1 - \frac{\lambda_2}{\lambda_1} y^2 - \cdots - \frac{\lambda_n}{\lambda_1} y^n. \tag{1.3}$$

We want to conclude that $x^1, y^2, \ldots, y^n$ are a basis for $\mathbb{R}^n$. Since $\operatorname{rank}(\mathbb{R}^n) = n$, we only need to show that the n vectors $x^1, y^2, \ldots, y^n$ are linearly independent. Suppose they are not, so that there are $\mu_1, \ldots, \mu_n$ not all zero such that

$$\mu_1 x^1 + \mu_2 y^2 + \cdots + \mu_n y^n = 0. \tag{1.4}$$

Now, $\mu_1 \neq 0$ because if $\mu_1 = 0$ then the vectors $y^2, \ldots, y^n$ are linearly dependent, which contradicts $y^1, \ldots, y^n$ being a basis for $\mathbb{R}^n$. Then we can use (3) to write

$$x^1 = -\frac{\mu_2}{\mu_1} y^2 - \cdots - \frac{\mu_n}{\mu_1} y^n. \tag{1.5}$$

Substituting (4) into (2), we have that $y^1, \ldots, y^n$ are linearly dependent, which contradicts $y^1, \ldots, y^n$ being a basis for $\mathbb{R}^n$. At last we conclude that $x^1, y^2, \ldots, y^n$ are indeed linearly independent, so they form a basis for $\mathbb{R}^n$.

The inductive case in the proof is, fortunately, very similar to the base case. We want to show that if $x^1, \ldots, x^{k-1}, y^k, y^{k+1}, \ldots, y^n$ is a basis for $\mathbb{R}^n$, where $k - 1 < m < n$, then $x^1, \ldots, x^{k-1}, x^k, y^{k+1}, \ldots, y^n$ is also a basis for $\mathbb{R}^n$, again subject to reordering the $y^k, \ldots, y^n$

to suit our notation. Now, since by hypothesis $x^1,\dots,x^{k-1},y^k,y^{k+1},\dots,y^n$ form a basis for $\mathbb{R}^n$, and since $x^k \neq 0$, there are $\theta_1,\dots,\theta_n$ not all zero such that

$$x^k = \theta_1 x^1 + \cdots + \theta_{k-1}x^{k-1} + \theta_k y^k + \theta_{k+1}y^{k+1} + \cdots + \theta_n y^n. \tag{1.6}$$

It is not possible that all $\theta_k,\dots,\theta_n$ are zero, because then x^k would be a linear combination of $x^1,\dots,x^{k-1}$, which would be a contradiction of the linear independence of S. So, assume without loss of generality that $\theta_k \neq 0$. Then we can use (5) to write

$$y^k = -\frac{\theta_1}{\theta_k}x^1 - \cdots - \frac{\theta_{k-1}}{\theta_k}x^{k-1} + \frac{1}{\theta_k}x^k - \frac{\theta_{k+1}}{\theta_k}y^{k+1} - \cdots - \frac{\theta_n}{\theta_k}y^n. \tag{1.7}$$

We want to conclude that $x^1,\dots,x^{k-1},x^k,y^{k+1},\dots,y^n$ are a basis for $\mathbb{R}^n$. Since $\operatorname{rank}(\mathbb{R}^n) = n$, we only need to show that the n vectors $x^1,\dots,x^{k-1},x^k,y^{k+1},\dots,y^n$ are linearly independent. Suppose they are not, so that there are $\alpha_1,\dots,\alpha_n$ not all zero such that

$$\alpha_1 x^1 + \cdots + \alpha_{k-1}x^{k-1} + \alpha_k x^k + \alpha_{k+1}y^{k+1} + \cdots + \alpha_n y^n = 0. \tag{1.8}$$

Now, $\alpha_k \neq 0$ because if $\alpha_k = 0$ then the vectors $x^1,\dots,x^{k-1},y^{k+1},\dots,y^n$ are linearly dependent, which contradicts $x^1,\dots,x^{k-1},y^k,y^{k+1},\dots,y^n$ being a basis for $\mathbb{R}^n$. Then we can use (7) to write

$$x^k = -\frac{\alpha_1}{\alpha_k}x^1 - \cdots - \frac{\alpha_{k-1}}{\alpha_k}x^{k-1} - \frac{\alpha_{k+1}}{\alpha_k}y^{k+1} - \cdots - \frac{\alpha_n}{\alpha_k}y^n. \tag{1.9}$$

Substituting (8) into (6), we have that $x^1,\dots,x^{k-1},y^k,y^{k+1},\dots,y^n$ are linearly dependent, which contradicts $x^1,\dots,x^{k-1},y^k,y^{k+1},\dots,y^n$ being a basis for $\mathbb{R}^n$. At last we conclude that $x^1,\dots,x^{k-1},x^k,y^{k+1},\dots,y^n$ are indeed linearly independent, so they form a basis for $\mathbb{R}^n$.

By induction, then, we have that the n vectors $x^1,\dots,x^m,y^{m+1},\dots,y^n$ are a basis for $\mathbb{R}^n$, after suitable reordering of the original y^j.

Problem 4 (Inner Product)**.**

Suppose $S = \{x^1,\dots,x^n\}$ is a set of non-null vectors in $\mathbb{R}^n$, which are mutually orthogonal: that is, $x^i x^j = 0$ whenever $i \neq j$. Show that S is linearly independent.

Solution.

We will prove by contradiction that S is linearly independent. Suppose that S is linearly dependent, so that for some $\lambda_1,\dots,\lambda_n$ not all zero, we have $\lambda_1 x^1 + \cdots + \lambda_n x^n = 0$. Now, consider some $k \in \{1, \dots, n\}$ for which $\lambda_k \neq 0$. We have

$$\begin{aligned} 0 = x^k \cdot 0 = x^k \cdot (\lambda_1 x^1 + \cdots + \lambda_n x^n) &= \sum_{i=1}^{n} \lambda_i (x^k \cdot x^i) \\ &= \lambda_k (x^k \cdot x^k) && \text{(since } x^k \cdot x^i = 0 \text{ for } i \neq k\text{)} \\ &\neq 0 && \text{(since } \lambda_k \neq 0 \text{ and } x^k \neq 0\text{)} \end{aligned}$$

This contradiction establishes that S is linearly independent.

References

Much of this material is standard in texts on linear algebra, like *Linear Algebra* by *G. Hadley* (Chapter 2), or *Elementary Matrix Algebra* by *Franz Hohn* (Chapters 4,5). A good exposition can also be found in *The Theory of Linear Economic Models* by David Gale (Chapter 2). You will find a proof of the fundamental theorem on vector spaces (by using mathematical induction) in Gale's book.■

Chapter 2

Matrices

2.1 Matrix Algebra

An $m \times n$ *matrix* is a rectangular array of numbers $a_{ij}, i = 1, \ldots, m; j = 1, \ldots, n$. Thus, we write

$$A = (a_{ij}) = \begin{bmatrix} a_{11} \ldots\ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ a_{m1} \ldots\ldots\ldots a_{mn} \end{bmatrix}$$

The n-vector $A_i = (a_{i1}, \ldots, a_{in})$ is called the *ith row vector* of A; the m-vector $A^j = (a_{1j}, \ldots, a_{mj})$ is called the *jth column vector* of A. The matrix A has m row vectors $A_1, \ldots, A_m$ and it has n column vectors $A^1, \ldots, A^n$. Thus an $m \times n$ matrix may be interpreted as an ordered set of m row vectors, or as an ordered set of n column vectors.

In view of this above statement, the operations on matrices follow from the operations on vectors.

2.1.1 Matrix Operations

Two $m \times n$ matrices A and B are *equal* (written $A = B$) if $a_{ij} = b_{ij}$ for $i = 1, \ldots, m; j = 1, \ldots, n$.

If A and B are $m \times n$ matrices, their *sum* $A + B$ is an $m \times n$ matrix, $(a_{ij} + b_{ij})$.

If A is an $m \times n$ matrix, and λ is a scalar, their *product* λA is an $m \times n$ matrix (λa_{ij}).

Let A be an $m \times n$ matrix, and B an $n \times r$ matrix. Then B can be premultiplied by A or A can be post-multiplied by B. The *matrix product*, denoted by AB is an $m \times r$ matrix given by

$$(\sum_{k=1}^{n} a_{ik} b_{kj}) \qquad i = 1, \ldots, m; \ j = 1, \ldots, r$$

For both operations AB and BA to be defined, if A is $m \times n$, then B must be $n \times m$.

The following properties related to matrix addition and multiplication can be verified [assuming the relevant matrices can be added and/or multiplied].

(MA.1) $A + B = B + A$ [Commutative Law]
(MA.2) $(A+B)+C = A+(B+C)$ [Associative Law]
(MM.1) $(AB)C = A(BC) = ABC$ [Associative Law]
(MM.2) $A(B+C) = AB + AC$ [Distributive Law]
$(B+C)A = BA + CA$ [Distributive Law]

2.1.2 Some Words of Caution

Some results which are true for real numbers are not necessarily true for matrices. It is useful to be aware of some of these.

(i) AB is not necessarily equal to BA.

Example 1. *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}, \qquad BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}$$

(ii) $AB = 0$ is possible with neither A nor B being the null matrix.

Example 2. *Let*

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(iii) $CD = CE$ with C not null is possible without D and E being equal.

Example 3. *Let*

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

Then

$$CD = CE = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix}, \quad C \neq 0 \quad \textit{and} \quad D \neq E.$$

2.1.3 Transpose of a Matrix

If A is an $m \times n$ matrix, then the $n \times m$ matrix B defined by

$$b_{ij} = a_{ji} \qquad i = 1, \ldots, n; \; j = 1, \ldots, m$$

is called the *transpose* of A, and is denoted by A'.

The following properties of transposes can be easily verified:

(T.1) $(A')' = A$

(T.2) $(A + B)' = A' + B'$

(T.3) $(AB)' = B'A'$

2.1.4 Some Special Matrices

There are some special types of matrices, which are now discussed below. An $m \times n$ matrix is a *square matrix* if $m = n$. An $n \times n$ matrix is *symmetric* if

$$a_{ij} = a_{ji} \quad i \neq j$$

[That is, a square matrix, A, is symmetric if $A = A'$.] An $n \times n$ matrix is a *diagonal matrix* if

$$a_{ij} = 0 \qquad i \neq j$$

An $n \times n$ matrix is an *identity matrix* (denoted by I_n or I) if

$$\begin{aligned} a_{ii} &= 1 \qquad i = 1, \ldots, n \\ a_{ij} &= 0 \qquad i \neq j \end{aligned}$$

An $m \times n$ matrix is a *null matrix* (denoted by 0) if

$$a_{ij} = 0 \qquad i = 1, \ldots, m; \quad j = 1, \ldots, n.$$

Note that a null matrix need not be a square matrix.

The identity and null matrices are especially useful in matrix algebra. The following properties of identity and null matrices can be verified (assuming the relevant matrices can be added and/or multiplied)

(I.1) $AI = IA = A$

(N.1) $A + 0 = 0 + A = A$

(N.2) $A0 = 0; 0A = 0$

2.2 Rank of a Matrix

Let A be an $m \times n$ matrix. The rank of the set of row vectors (column vectors) of A is called the *row rank* (*column rank*) of A. We note that the row and column ranks of a matrix are, in fact, equal:

Theorem 3. *(Rank Theorem)*

For any $m \times n$ matrix A, the row rank and the column rank are equal.

In view of the rank theorem, we will, henceforth, simply refer to the *rank* of A [denoted $r(A)$].

If A is an $m \times n$ matrix, and B is an $n \times r$ matrix, then the following properties can be established:

(R.1) $r(A) \leq \min(m, n)$

(R.2) $r(AB) \leq \min(r(A), r(B))$.

If A is an $m \times m$ matrix, then A is called *non-singular* if the rank of A is m. A is called *singular* if the rank of A is less than m.

2.3 Inverse of a Matrix

Let A be an $m \times m$ matrix. If B is an $m \times m$ matrix satisfying

$$AB = BA = I \tag{2.1}$$

then A is called *invertible* and B is called the *inverse* of A (denoted by A^{-1}). The following properties can be established regarding inverses of matrices. [Here A and B are $m \times m$ invertible matrices].

(IN.1) $(A^{-1})^{-1} = A$

(IN.2) $(AB)^{-1} = B^{-1}A^{-1}$

(IN.3) $(A')^{-1} = (A^{-1})'$

2.4 Relationship Between Invertible and Non-Singular Matrices

Consider an $m \times m$ non-singular matrix, A. Then the m column vectors of A are linearly independent and is therefore a basis of $\mathbb{R}^m$. By the basis theorem, given any $b \in \mathbb{R}^m$, b can be expressed as a linear combination of the m column vectors. That is, for each $b \in \mathbb{R}^m$, there is $x \in \mathbb{R}^m$ such that

$$Ax = b \tag{2.2}$$

Applying this to each of the column unit vectors e^i $(i = 1, \dots, m)$ in turn, we will get vectors x^i $(i = 1, \dots, m)$ such that

$$Ax^i = e^i \tag{2.3}$$

Defining a matrix X with x^i representing the ith column, we get

$$AX = I \tag{2.4}$$

The m row vectors of A are linearly independent and is therefore a basis of $\mathbb{R}^m$. By the basis theorem, given any $c \in \mathbb{R}^m$, c can be expressed as a linear combination of the m row vectors. That is, for each $c \in \mathbb{R}^m$ there is $y \in \mathbb{R}^m$ such that

$$yA = c \tag{2.5}$$

Applying this to each of the row unit vectors e^i $(i = 1, \dots, m)$ in turn, we will get vectors y^i $(i = 1, \dots, m)$ such that

$$y^i A = e^i \tag{2.6}$$

Defining a matrix Y with y^i representing its ith row, we get

$$YA = I \tag{2.7}$$

Using (2.4) and (2.7), we get

$$Y = Y(AX) = (YA)X = X \tag{2.8}$$

Thus, we have a matrix X such that

$$AX = XA = I$$

which proves that A is invertible, and X is the inverse of A.

Conversely, consider an $m \times m$ *invertible* matrix, A. Then, there is an $m \times m$ matrix, B, such that

$$AB = BA = I \tag{2.9}$$

We claim that the column vectors of A are linearly independent. For if they are linearly dependent, there is some non-zero vector x such that

$$Ax = 0 \tag{2.10}$$

Multiplying (2.10) by B we get

$$0 = B(Ax) = (BA)x = Ix = x \tag{2.11}$$

which is a contradiction. Thus, the m column vectors of A are linearly independent, and the rank of A is m. Thus, A is a non-singular matrix.

2.5 Worked Out Problems on Chapter 2

Problem 5 (Matrix Multiplication)**.**

Suppose A is an $m \times n$ matrix, B is an $n \times r$ matrix, and C is an $r \times s$ matrix. Verify that:

$$A(BC) = (AB)C$$

Solution.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rs} \end{bmatrix}$$

$$\begin{aligned}
A(BC) &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} (b_{11}c_{11} + \cdots + b_{1r}c_{r1}) & \cdots & (b_{11}c_{1s} + \cdots + b_{1r}c_{rs}) \\ \vdots & \ddots & \vdots \\ (b_{n1}c_{11} + \cdots + b_{nr}c_{r1}) & \cdots & (b_{n1}c_{1s} + \cdots + b_{nr}c_{rs}) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^{n} a_{1j}(\sum_{k=1}^{r} b_{jk}c_{k1}) & \cdots & \sum_{j=1}^{n} a_{1j}(\sum_{k=1}^{r} b_{jk}c_{ks}) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{mj}(\sum_{k=1}^{r} b_{jk}c_{k1}) & \cdots & \sum_{j=1}^{n} a_{mj}(\sum_{k=1}^{r} b_{jk}c_{ks}) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^{r} c_{k1}(\sum_{j=1}^{n} a_{1j}b_{jk}) & \cdots & \sum_{k=1}^{r} c_{ks}(\sum_{j=1}^{n} a_{1j}b_{jk}) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{r} c_{k1}(\sum_{j=1}^{n} a_{mj}b_{jk}) & \cdots & \sum_{k=1}^{r} c_{ks}(\sum_{j=1}^{n} a_{mj}b_{jk}) \end{bmatrix} \\
&= \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{1n}b_{n1}) & \cdots & (a_{11}b_{1r} + \cdots + a_{1n}b_{nr}) \\ \vdots & \ddots & \vdots \\ (a_{m1}b_{11} + \cdots + a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1r} + \cdots + a_{mn}b_{nr}) \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{r1} & \cdots & c_{rs} \end{bmatrix} \\
&= (AB)C
\end{aligned}$$

Problem 6 (Transpose of a Matrix)**.**

Suppose A is an $m \times n$ matrix, B is an $n \times r$ matrix. Verify that:

$$(AB)' = B'A'$$

Solution.

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}, \quad B = \begin{bmatrix} B^1 & \cdots & B^r \end{bmatrix}$$

Recall that the inner product has some nice properties: for two vectors x and y of the same size, we have $xy = yx$ and $x'y = xy' = x'y' = xy$. Then we can write

$$\begin{aligned}
(AB)' &= \left(\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}\begin{bmatrix} B^1 & \cdots & B^r \end{bmatrix}\right)' \\
&= \begin{bmatrix} A_1B^1 & \cdots & A_1B^r \\ \vdots & \ddots & \vdots \\ A_mB^1 & \cdots & A_mB^r \end{bmatrix}' \\
&= \begin{bmatrix} A_1B^1 & \cdots & A_mB^1 \\ \vdots & \ddots & \vdots \\ A_1B^r & \cdots & A_mB^r \end{bmatrix} \\
&= \begin{bmatrix} B^1A_1 & \cdots & B^1A_m \\ \vdots & \ddots & \vdots \\ B^rA_1 & \cdots & B^rA_m \end{bmatrix} \\
&= \begin{bmatrix} (B^1)'(A_1)' & \cdots & (B^1)'(A_m)' \\ \vdots & \ddots & \vdots \\ (B^r)'(A_1)' & \cdots & (B^r)'(A_m)' \end{bmatrix} \\
&= \begin{bmatrix} (B^1)' \\ \vdots \\ (B^r)' \end{bmatrix}\begin{bmatrix} (A_1)' & \cdots & (A_m)' \end{bmatrix} \\
&= \begin{bmatrix} B^1 & \cdots & B^r \end{bmatrix}'\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}' \\
&= B'A'
\end{aligned}$$

Problem 7 (Rank of a Matrix)**.**

Let A be an $m \times n$ matrix, and suppose the n column vectors of A are linearly independent.

(a) Show that $m \geq n$.

(b) Show that the row rank of $A \geq n$.

[Do not use the Rank Theorem to answer any of the parts of the problem. You can, of course, use any result, which appears in the lecture notes before the statement of the Rank Theorem].

Solution.

(a) We are given that $A^1, \ldots, A^n$ are linearly independent vectors in $\mathbb{R}^m$. If $m < n$ we have a violation of Corollary 1 in Chapter 1. Therefore it must be that $m \geq n$.

(b) We will show row rank$(A) \geq n$ by contradiction. Suppose row rank$(A) = r < n \leq m$. Then the row vectors $A_1, \ldots, A_r$, each of size $1 \times n$, are a basis for $A_1, \ldots, A_m$. [If using $A_1, \ldots, A_r$ seems troublesome, remember that we can reorder the rows of A without changing the rank.]

Consider the system of r equations

$$\begin{aligned} \lambda_1 a_{11} + \cdots + \lambda_n a_{1n} &= 0 \\ &\vdots \\ \lambda_1 a_{r1} + \cdots + \lambda_n a_{rn} &= 0 \end{aligned}$$

where a_{ij} is the jth element of the ith row vector A_i. If we define $\lambda = [\lambda_1 \cdots \lambda_n]'$ then this can be written more compactly as

$$\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix} \lambda = 0.$$

This is a system of r homogeneous linear equations in the $n > r$ unknowns $\lambda_1, \ldots, \lambda_n$, so by Corollary 2 in Chapter 1, the system has a solution $\lambda_1, \ldots, \lambda_n$ not all zero.

Now, since $A_1, \ldots, A_r$ are a basis for the rows of A, any row A_k can be expressed as a linear combination of $A_1, \ldots, A_r$. That is, there are $\mu_1, \ldots, \mu_r$ such that

$$A_k = \mu_1 A_1 + \cdots + \mu_r A_r.$$

Taking the inner product of both sides with λ, we have

$$\begin{aligned} A_k \lambda &= \mu_1 (A_1 \lambda) + \cdots + \mu_r (A_r \lambda) \\ &= \mu_1 (0) + \cdots + \mu_r (0) \\ &= 0 \end{aligned}$$

This must hold for every row k, so the following equations hold:

$$\begin{aligned} A_1 \lambda = \ \lambda_1 a_{11} + \cdots + \lambda_n a_{1n} &= 0 \\ &\vdots \\ A_m \lambda = \lambda_1 a_{m1} + \cdots + \lambda_n a_{mn} &= 0 \end{aligned}$$

This is the same as writing

$$\lambda_1 A^1 + \cdots + \lambda_n A^n = 0$$

for $\lambda_1, \dots, \lambda_n$ not all zero. That means the columns of A are linearly dependent, which is a contradiction. So it must be that row $\operatorname{rank}(A) \geq n$.

Problem 8 (Singular and Non-Singular Matrices)**.**

(a) Let A be an $m \times 1$ matrix and let B be a $1 \times m$ matrix, where $m \geq 2$. Let C be the $m \times m$ matrix, defined by $C = AB$. Show that C must be a singular matrix.

(b) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Let C be the $m \times m$ matrix, defined by $C = AB$. If $n < m$, can C be non-singular ? Explain your answer carefully.

Solution.

(a) Let

$$A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \end{bmatrix}, \quad C = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1m} \end{bmatrix}$$

To show that C is singular, we want to show that $\operatorname{rank}(C) < m$. Suppose to the contrary that $\operatorname{rank}(C) = m$. Then the columns of C are linearly independent. Now, if $b_{1k} = 0$ for some $k \in \{1, \dots, m\}$ then $C^k = 0$ and we reach a contradiction immediately. So, suppose $b_{1i} \neq 0$ for all $i = 1, \dots, m$. Consider the system of equations

$$\lambda_1 C^1 + \lambda_2 C^2 + \cdots + \lambda_m C^m = \lambda_1 b_{11} \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \lambda_2 b_{12} \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + \lambda_m b_{1m} \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} = 0$$

Choosing $\lambda_1 = \frac{1}{b_{11}} \neq 0, \lambda_2 = -\frac{1}{b_{12}} \neq 0$, and $\lambda_j = 0$ for $j = 3, \dots, m$, the above equations hold. This means that the columns of C are linearly dependent, which is a contradiction. So we conclude that $\operatorname{rank}(C) < m$, which means C is singular.

(b) We will first prove result (R.2) from Chapter 2:

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

This result will make it very easy to show that C cannot be non-singular when $n < m$.

We get (R.2) by showing both $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ and $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$. Let's prove the first statement and hope the second one follows similarly. Consider any linear combination of the columns of AB: if y is such a linear combination, then there exists some $m \times 1$ vector x such that $y = (AB)x$. Since $(AB)x = A(Bx)$, we can say that y is also a linear combination of the columns of A, using the vector Bx as the weights.

[Note that we cannot say the reverse. If y' is a linear combination of the columns of A, then there is some z' such that $y' = Az'$. How do we fit B in on the right hand side? If $m = n$ and B is invertible, we can write $y' = (AB)(B^{-1}z') = Az'$, so that the vector $B^{-1}z'$ provides the weights. But for this problem we are assuming $n < m$.]

Seeking contradiction, suppose $\text{rank}(AB) = r > q = \text{rank}(A)$. Now, if $(AB)^k$ denotes the kth column of AB, then some linearly independent vectors $(AB)^1, \ldots, (AB)^r$ form a basis for the columns of AB. [If using $(AB)^1, \ldots, (AB)^r$ seems troublesome, remember that we can reorder the columns of AB without changing the rank.] Similarly, some linearly independent column vectors $A^1, \ldots, A^q$ form a basis for the columns of A.

Since every column of AB is trivially a linear combination of the columns of AB, it is also a linear combination of the columns of A. In particular, each $(AB)^1, \ldots, (AB)^r$ is a linear combination of $A^1, \ldots, A^n$. Moreover, since each $A^1, \ldots, A^n$ is a linear combination of $A^1, \ldots, A^q$, we have that each $(AB)^1, \ldots, (AB)^r$ is a linear combination of $A^1, \ldots, A^q$. Now, since $r > q$ we have by the Fundamental Theorem on Vector Spaces that the $(AB)^1, \ldots, (AB)^r$ are linearly dependent. This is a contradiction, so we conclude that $\text{rank}(AB) \leq \text{rank}(A)$.

A similar argument establishes that $\text{rank}(B'A') \leq \text{rank}(B')$. Now, note that for any matrix D,

$$\text{rank}(D) = \text{row rank}(D) = \text{col rank}(D') = \text{rank}(D').$$

Therefore we have that

$$\text{rank}(AB) = \text{rank}((AB)') \leq \text{rank}(B') = \text{rank}(B).$$

With our finding above that $\text{rank}(AB) \leq \text{rank}(A)$, we have now proved that

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\},$$

which is result (R.2).

By result (R.1) in Chapter 2, we have $\text{rank}(A) \leq \min\{m,n\} = n$ and $\text{rank}(B) \leq \min\{n,m\} = n$. Then we only have to apply result (R.2) to see that $\text{rank}(C) = \text{rank}(AB) \leq n < m$, which shows that C must be singular.

Problem 9 (Inverse of a Matrix)**.**

Let A be an $n \times n$ matrix, which satisfies:

$$a_{ij} = \begin{cases} 1 & \text{for all } i, j \in \{1, \ldots, n\} \text{ with } j \le i \\ 0 & \text{otherwise} \end{cases}$$

Show that A has an inverse.[Do not use your computer to obtain the inverse matrix].

Solution.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

We know that

$$\begin{aligned} A \text{ has an inverse} &\iff A \text{ is non-singular} \\ &\iff \operatorname{rank}(A) = n \\ &\iff \text{the columns of } A \text{ are linearly independent} \end{aligned}$$

Therefore, to show A has an inverse we only need to show that the columns of A are linearly independent. Suppose to the contrary that $A^1, \ldots, A^n$ are linearly dependent, so that there exist $\lambda_1, \ldots, \lambda_n$ not all zero satisfying

$$\lambda_1 A^1 + \cdots + \lambda_n A^n = \begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \\ \vdots \\ \lambda_1 + \cdots + \lambda_n \end{bmatrix} = 0$$

This implies that $\lambda_i = 0$ for all $i = 1, \ldots, n$, which is a contradiction. So we conclude that the columns of A are linearly independent, which shows that A has an inverse as outlined above.

References:

This material on matrix algebra can be found in standard texts like *F. Hohn: Elementary Matrix Algebra* (Chapters 1 and 6) and *G. Hadley: Linear Algebra* (Chapter 3). A good discussion on the rank of a matrix is in *D. Gale: Theory of Linear Economic Models* (Chapter 2), and a proof of the "rank theorem" can be found there. You will also find a good exposition of matrix algebra in *R. Dorfman, P.A. Samuelson and R.M. Solow: Linear Programming and Economic Analysis* (Appendix B).

Chapter 3

Simultaneous Linear Equations

3.1 System of Linear Equations

Consider a system of m linear equations in n unknowns, written as

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ - - - - - - - - - - & \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{*}$$

In matrix-vector notation, we can write this as

$$Ax = b$$

where A is an $m \times n$ matrix, b is a (column) vector in $\mathbb{R}^m$ (or an $m \times 1$ matrix) and x is a (column) vector in $\mathbb{R}^n$ (or an $n \times 1$ matrix).

In analyzing a system of linear equations like (*), the following questions naturally arise:

(i) (Existence) Does there exist a solution to (*)? [Are the equations "consistent"?]

(ii) (Uniqueness) If there exists a solution to (*), is it unique? [Are the equations "determinate"?]

(iii) (Computation) If there exists a solution to (*), how can we find such a solution?

3.2 Existence of Solutions

The system of equations (*) is called *homogeneous* if $b = 0$, and *non-homogeneous* if $b \neq 0$.

If the system is homogeneous, there is always a trivial solution, namely $x = 0$. If the system is non-homogeneous then, in general, there need not exist a solution to (*). For instance consider the system:

(i) $2x_1 + 4x_2 = 5$

(ii) $x_1 + 2x_2 = 1$

Notice that if we multiply equation (ii) by 2, we get

$$2x_1 + 4x_2 = 2$$

wich *contradicts* (is inconsistent with) the equation (i). So, there does not exist a solution to the system of equations given by (i) and (ii): the equations (i) and (ii) are inconsistent.

In general, if we look at a system of equations like (*), we would like to tell, given A and b, whether there is a solution to (*).

We start our discussion of the existence of solutions to linear equations by noting a simple consequence of the "rank theorem".

Proposition 1. *Suppose $a^1, \ldots, a^n$ are linearly independent vectors in $\mathbb{R}^m$, and b is in $\mathbb{R}^n$, then there is a vector y in $\mathbb{R}^m$ such that*

$$ya^i = b_i \qquad i = 1, \ldots, n$$

Proof. Let A be the $m \times n$ matrix with columns $a^1, \ldots, a^n$. By the rank theorem, the rows of A have rank n. Let $A_1, \ldots, A_n$ be a row basis; then we have n linearly independent vectors in $\mathbb{R}^n$, and hence by the basis theorem, b is a linear combination of $A_1, \ldots, A_n$. That is, there exist numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i A_i = b$$

Define y in $\mathbb{R}^m$ to be $(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$. Then

$$\sum_{i=1}^{m} y_i A_i = b, \qquad \text{or } yA = b$$

Consequently $ya^i = b_i$ for $i = 1, \ldots, n$ since $a^i = A^i$, the ith column of A. ∎

Using the above result, one can provide the following criterion for the solvability of linear equations.

Theorem 4. *Let A be an $m \times n$ matrix, and let c be in $\mathbb{R}^m$. Then exactly one of the following alternatives holds. Either the equation*

$$Ax = c \tag{3.1}$$

has a solution, or the equations

$$yA = 0, \qquad yc = 1 \tag{3.2}$$

have a solution.

Proof. Suppose (3.1) has no solution. Let $A^1, \ldots, A^r$ be a column basis of A. Then $A^1, \ldots, A^r, c$ is a linearly independent set of vectors. Otherwise, c can be expressed as a linear combination of $A^1, \ldots, A^r$ and therefore of $A^1, \ldots, A^n$. This contradicts our hypothesis that (3.1) has no solution.

Define b in $\mathbb{R}^{r+1}$ by $b = (0, \ldots, 0, 1)$. Then by Proposition 1, there is y in $\mathbb{R}^m$, such that

$$yA^i = 0 \qquad for\, i = 1, \ldots, r, \text{ and } yc = 1 \tag{3.3}$$

Since $A^1, \ldots, A^r$ is a column basis of A, given any $A^i (i = 1, \ldots, n), A^i$ can be expressed as a linear combination of $A^1, \ldots, A^r$. So (3.3) implies that $yA^i = 0$ for $i = 1, \ldots, n$. Thus, y is a solution to (3.2).

On the other hand, if (3.1) has a solution (say x), then (3.2) cannot have a solution. For if it did (say y), then multiplying (3.1) by y we get

$$1 = yc = y(Ax) = (yA)x = 0$$

a contradiction. ■

Finally, there is the criterion for solvability stated in terms of the rank of the relevant matrices.

Let A be an $m \times n$ matrix and c be a vector in $\mathbb{R}^m$. Then the $m \times (n+1)$ matrix given by $A_c \equiv (A^1, \ldots, A^n, c)$ is known as the *augmented matrix.*

Theorem 5. *Let A be an $m \times n$ matrix and c be a vector in $\mathbb{R}^m$. Then the system of equations*

$$Ax = c \tag{3.4}$$

has a solution if and only if

$$rank A = rank A_c \tag{3.5}$$

Proof. Let r be the rank of A. We clearly have $rank\ A_c \geq rank A$. Suppose $rank A_c > rank A$. Then we can find $(r+1)$ linearly independent vectors from the columns of A_c. Clearly this set must include the vector c, otherwise $rank A > r$. Thus there is a set of r column vectors of A, call them $A^1, \ldots, A^r$, which, together with c, form a set of linearly independent vectors. Thus, $A^1, \ldots, A^r$ are linearly independent and so is a column basis of A. If (3.4) has a solution, say x, then c is a linear combination of $A^1, \ldots, A^n$ and therefore of $A^1, \ldots, A^r$. But then the set $(A^1, \ldots, A^r, c)$ is linearly dependent, a contradiction. Thus if (3.5) does not hold, then (3.4) does not have a solution.

On the other hand, suppose (3.5) holds. Thus if $A^1, \ldots, A^r$ is a column basis of A, then $(A^1, \ldots, A^r, c)$ is a linearly dependent set. Since $(A^1, \ldots, A^r)$ is a linearly independent set, c can be expressed as a linear combination of the vectors $(A^1, \ldots, A^r)$ and therefore of the vectors $(A^1, \ldots, A^n)$. So (3.4) has a solution. ■

3.3 Uniqueness of Solutions

Theorem 6. *Let A be an $m \times n$ matrix and let c be a vector in $\mathbb{R}^m$. Then the system of equations*

$$Ax = c \tag{3.6}$$

has a unique solution if and only if

$$rank A = rank A_c = n \tag{3.7}$$

Proof. If (3.7) holds, then by Theorem 5, there is a solution to (3.6). Suppose, contrary to the assertion of the Theorem, there are x^1, x^2 in $\mathbb{R}^n$, $x^1 \neq x^2$ which both solve (3.6). Then

$$A(x_1 - x_2) = 0$$

Thus, the column vectors of A are linearly dependent [since $(x_1 - x_2) \neq 0$]. So the $rank A < n$, a contradiction.

Suppose, next, that (3.7) does not hold. If $rank A \neq rank A_c$, there is no solution to (3.6) by Theorem 5, and we are done. If $rank A = rank A_c$, and (3.7) is violated, then we have $rank A = rank A_c < n$. It follows that (a) there is a solution, x, to (3.6); and (b) the n column vectors of A, namely $A^1, \ldots, A^n$ are linearly dependent. Using (b), there is $y \in \mathbb{R}^n$, $y \neq 0$ such that

$$Ay = 0 \tag{3.8}$$

But, then, clearly $(x+y)$ also solves (3.6), and $(x+y) \neq x$ (since $y \neq 0$), so (3.6) does not have a unique solution. ■

3.4 Calculation of Solutions

The most important case to be considered in the actual calculation of solutions is the case of n linear equations in n unknowns. Let A be an $n \times n$ matrix, and c be a vector in $\mathbb{R}^n$. We have the system of equations given by

$$Ax = c \tag{3.9}$$

If (3.9) has a unique solution, then we have noted that $rank A = n$. Conversely if $rank A = n$, then $A^1, \ldots, A^n$ is a basis of $\mathbb{R}^n$, and so c can be expressed as a linear combination of $A^1, \ldots, A^n$, yielding a solution to (3.9). Furthermore, such a solution must be unique since $rank A = rank A_c = n$.

We consider, therefore, in what follows how to calculate *the* solution to (3.9) when $rank A = n$. Since $rank A = n$, A is a non-singular matrix. It follows that it has an $n \times n$ inverse matrix (denoted by A^{-1}) such that

$$A^{-1}A = AA^{-1} = I \tag{3.10}$$

Pre-multiplying (3.9) by A^{-1} and using (3.10), we obtain

$$x = A^{-1}c \tag{3.11}$$

as *the* solution to (3.9).

In terms of calculating this solution, then, it remains to learn how to compute the inverse of a non-singular matrix. This leads us naturally into the study of "determinants".

3.5 Determinants

Let A be an $n \times n$ matrix. We can associate with A a number, denoted by $|A|$, called the determinant of A.

The determinant of the $n \times n$ matrix, is defined recursively as follows.

(1) For a 1×1 matrix, which is, of course, a number, we define the determinant to be the number itself.

(2) For any $m \times m (m \geq 2)$ matrix, the *cofactor* A_{ij} of the element a_{ij} is $(-1)^{i+j}$ times the determinant of the submatrix obtained from A by deleting row i and column j. The *determinant* of the $m \times m$ matrix is then given by

$$|A| = \sum_{j=1}^{n} a_{1j} A_{1j}$$

Thus using (2), and knowing (1), the determinant of a 2×2 matrix is:

$$a_{11}a_{22} - a_{12}a_{21}$$

This information can then be used in (2) again to obtain the determinant of a 3×3 matrix:

$$a_{11}[a_{22}a_{33} - a_{32}a_{23}] - a_{12}[a_{21}a_{33} - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}a_{22}]$$

This procedure can be continued to obtain the determinant of any $n \times n$ matrix.

It is implicit in the definition of $|A|$ that the "expansion" is done by the first row. However, it can be shown that for every $i \in [1, \ldots, n]$,

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij}$$

so that expansion by any row will given the same result. Indeed, expansion by any column will also give the same result. That is, for every $j \in [1, \ldots, n]$,

$$|A| = \sum_{i=1}^{n} a_{ij}A_{ij}$$

The following properties of determinants can be established:

(D.1) $|A| = |A'|$

(D.2) The interchange of any two rows will alter the sign, but not the numerical value, of the determinant.

(D.3) The multiplication of any one row by a scalar k will change the determinant k-fold.

(D.4) The addition of a multiple of any row to another row will leave the determinant unaltered.

(D.5) If one row is a multiple of another row, the determinant is zero.

(D.6) The expansion of a determinant by "alien" co-factors yields a value of zero. That is,

$$\sum_{j=1}^{n} a_{ij}A_{kj} = 0 \qquad if\ i \neq k$$

[Here, the expansion is by the ith row, using co-factors of kth row].

(D.7) $|AB| = |A|\,|B|$

The above properties (D.2) - (D.5) hold if the word "row" is replaced uniformly by "column" in each statement.

3.6 Matrix Inversion

Now, we get back to the problem of computing the inverse of a non-singular matrix. We first note the following result.

Theorem 7. *Let A be an $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0$. Furthermore, in case A is invertible, $|A^{-1}| = |A|^{-1}$.*

Proof. Suppose A is invertible. Then

$$A \quad A^{-1} = I$$

so $1 = |I| = |AA^{-1}| = |A| \quad |A^{-1}|$ using property (D.7) of determinants, noted above. Consequently $|A| \neq 0$, and $|A^{-1}| = |A|^{-1}$.

Suppose, next, that A is not invertible. Then, A is singular and so one of its columns (say, A^1) can be expressed as a linear combination of its other columns $A^2, \ldots, A^n$. That is,

$$A^1 = \sum_{i=2}^{n} \lambda_i A^i$$

Consider the matrix, B, whose first column is $\left[A^1 - \sum_{i=2}^{n} \lambda_i A^i\right]$ and whose other columns are the same as those of A. Then, the first column of B is zero, and so $|B| = 0$. By property (D.4) of determinants (noted above), $|B| = |A|$, and so $|A| = 0$. ■

For an $n \times n$ matrix, A, we define the *co-factor matrix of* A to be the $n \times n$ matrix given by

$$C = \begin{bmatrix} A_{11} & A_{12} & \ldots\ldots & A_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \ldots\ldots & A_{nn} \end{bmatrix}$$

The transpose of C is called the *adjoint of* A, and denoted by $adj\,A$.

Now, by the rules of matrix multiplication,

$$AC' = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}A_{1j} & \sum_{j=1}^{n} a_{1j}A_{2j} & \ldots\ldots & \sum_{j=1}^{n} a_{1j}A_{nj} \\ \vdots & & & \vdots \\ \sum_{j=1}^{n} a_{nj}A_{1j} & \sum_{j=1}^{n} a_{nj}A_{2j} & \ldots\ldots & \sum_{j=1}^{n} a_{nj}A_{nj} \end{bmatrix} = \begin{bmatrix} |A| & 0 & \ldots\ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots\ldots & |A| \end{bmatrix}$$

This yields the equation

$$AC' = |A|I \tag{3.12}$$

If A is non-singular (that is invertible) then there is A^{-1} such that

$$AA^{-1} = A^{-1}A = I \tag{3.13}$$

Pre-multiplying (3.12) by A^{-1} and using (3.13),

$$C' = |A|A^{-1}$$

Since A is non-singular, we have $|A| \neq 0$, and

$$A^{-1} = \frac{C'}{|A|} = \frac{adjA}{|A|} \tag{3.14}$$

Thus (3.14) gives us a formula for computing the inverse of a non-singular matrix in terms of the determinant and cofactors of A.

3.7 Cramer's Rule

Recall that we wanted to calculate *the* (unique) solution of a system of n equations in n unknowns given by

$$Ax = c \tag{3.15}$$

where A is an $n \times n$ matrix, and c is a vector in $\mathbb{R}^n$.

To obtain a unique solution, we saw that we must have A non-singular, which now translaltes to the condition "$|A| \neq 0$". The unique solution to (3.15) is then

$$x = A^{-1}c = \frac{adjA}{|A|}c \tag{3.16}$$

Let us evaluate x_1, using (3.16). This can be done by finding the inner product of x with the first unit vector, $e^1 = (1, 0, \ldots, 0)$. Thus,

$$x_1 = e^1 x = \frac{e^1 adjA}{|A|}c$$

$$= \frac{[A_{11}A_{21} \qquad A_{n1}]c}{|A|}$$

$$= [c_1 A_{11} + c_2 A_{21} + \ldots.. + c_n A_{n1}] / |A|$$

$$= \begin{vmatrix} c_1 & a_{12} & .. & a_{1n} \\ \vdots & & & \\ c_n & a_{n2} & .. & a_{nn} \end{vmatrix} \quad |A|^{-1}$$

This gives us an easy way to compute the solution of x_1. In general, in order to calculate x_i, replace the ith column of A by the vector c and find the determinant of this matrix. Dividing this number by the determinant of A yields the solution x_i. This rule is known as *Cramer's Rule*.

3.8 Appendix: System of Homogeneous Linear Equations

Let A be an $n \times n$ matrix. Consider the following system of homogeneous linear equations:

$$Ax = 0 \tag{1}$$

Suppose A is singular, with $rank(A) = r$, where $1 \leq r < n$. Let $S = \{x \in \mathbb{R}^n : Ax = 0\}$.

(a) Show that S contains some non-zero vector.

(b) Let q be the rank of S. Show that $1 \leq q < n$.

(c) Let $\{x^1, \ldots, x^q\}$ be a set of vectors in S, which is linearly independent [the existence of such a set of vectors is guaranteed by (b)]. Show that there are vectors $y^{q+1}, \ldots, y^n$ in $\mathbb{R}^n$, such that the set $\{x^1, \ldots, x^q, y^{q+1}, \ldots, y^n\}$ is a basis of $\mathbb{R}^n$. [Use the result of problem 3 in problem set 2].

(d) Let $T = \{v \in \mathbb{R}^n : v = Az \text{ for some } z \in \mathbb{R}^n\}$. Show that any $v \in T$ can be expressed as a linear combination of $\{z^{q+1}, \ldots, z^n\}$, where $z^j = Ay^j$ for $j = q+1, \ldots, n$.

(e) Show that the set $\{z^{q+1}, \ldots, z^n\}$ is linearly independent, and therefore the rank of T is equal to $(n-q)$.

(f) Show that rank of T is also equal to r.

(g) Conclude that:

$$rank(S) = n - rank(A) \tag{*}$$

(a) Since the rank of A is less than n, the n column vectors of A are linearly dependent. Consequently, there is some $c \in \mathbb{R}^n$, with $c \neq 0$, such that $Ac = 0$. This $c \in S$.

(b) Since $c \in S$, with $c \neq 0$, the rank of S is ≥ 1. Since S is a subset of $\mathbb{R}^n$, and the rank of $\mathbb{R}^n$ is n, the rank of S is $\leq n$. To show that $q < n$, suppose on the contrary that $q = n$. Then, we can find n vectors $x^1, .., x^n$ in S, such that $\{x^1, .., x^n\}$ is linearly independent, and is therefore a basis of $\mathbb{R}^n$.

Since $r \geq 1$, we can find at least one column vector among the column vectors of A, which is non-zero; let A^k be such a column vector of A. The k-th unit vector $e^k \in \mathbb{R}^n$ can be written as a linear combination of the vectors in $\{x^1, .., x^n\}$:

$$e^k = \lambda_1 x^1 + \cdots + \lambda_n x^n \tag{2}$$

Pre-multiplying (2) by A, we obtain:

$$A^k = Ae^k = \lambda_1 Ax^1 + \cdots + \lambda_n Ax^n = 0$$

since $x^1, .., x^n$ are in S. But, this contradicts the fact that A^k is non-zero. Thus, we must have $q < n$.

(c) Since the rank of S is q, we can find a set of vectors $x^1,..,x^q$ in S, such that $\{x^1,..,x^q\}$ is linearly independent. Since the rank of $\mathbb{R}^n$ is n, we can find vectors $y^1,\ldots,y^n$ in $\mathbb{R}^n$, such that $E = \{y^1,\ldots,y^n\}$, is a basis of $\mathbb{R}^n$. Using the result of problem 3 of problem set 2, we can then find $(n-q)$ vectors from E, such that these vectors, together with the q vectors $x^1,..,x^q$ constitute a basis of $\mathbb{R}^n$. Denoting the $(n-q)$ vectors (without loss of generality) from E so chosen by $y^{q+1},\ldots,y^n$, the set $\{x^1,\ldots,x^q,y^{q+1},\ldots,y^n\}$ is a basis of $\mathbb{R}^n$.

(d) Since $\{x^1,\ldots,x^q,y^{q+1},\ldots,y^n\}$ is a basis of $\mathbb{R}^n$, any $z \in \mathbb{R}^n$ can be expressed as a linear combination of the set of vectors $\{x^1,\ldots,x^q,y^{q+1},\ldots,y^n\}$:

$$z = \lambda_1 x^1 + \cdots + \lambda_q x^q + \lambda_{q+1} y^{q+1} + \cdots + \lambda_n y^n \tag{3}$$

Pre-multiplying (3) by A, we obtain:

$$\begin{aligned} Az &= \lambda_1 Ax^1 + \cdots + \lambda_q Ax^q + \lambda_{q+1} Ay^{q+1} + \cdots + \lambda_n Ay^n \\ &= \lambda_{q+1} Ay^{q+1} + \cdots + \lambda_n Ay^n \\ &= \lambda_{q+1} z^{q+1} + \cdots + \lambda_n z^n \end{aligned} \tag{4}$$

the second line of (4) following from the fact that the vectors $x^1,..,x^q$ belong to S, and the third line of (4) following from the definition of z^j for $j = q+1,\ldots,n$. Since any $v \in T$ is equal to Az for some $z \in \mathbb{R}^n$, this shows that any $v \in T$ can be expressed as a linear combination of $\{z^{q+1},\ldots,z^n\}$.

(e) Suppose that $\{z^{q+1},\ldots,z^n\}$ is linearly dependent. Then, there are numbers $\alpha_{q+1},..,\alpha_n$, not all equal to zero such that:

$$\alpha_{q+1} z^{q+1} + \cdots + \alpha_n z^n = 0 \tag{5}$$

Using the definition of z^j for $j = q+1,\ldots,n$, we have:

$$A(\alpha_{q+1} y^{q+1} + \cdots + \alpha_n y^n) = 0 \tag{6}$$

Denoting $(\alpha_{q+1} y^{q+1} + \cdots + \alpha_n y^n)$ by w, we observe that (6) implies that $w \in S$. Since the rank of S is q, and $\{x^1,..,x^q\}$ is linearly independent, we know that $\{x^1,..,x^q\}$ is a basis of S. Consequently, w can be expressed as a linear combination of the set of vectors $\{x^1,..,x^q\}$:

$$w = \beta_1 x^1 + \cdots + \beta_q x^q \tag{7}$$

Using the definition of w, and (7), we get:

$$\alpha_{q+1} y^{q+1} + \cdots + \alpha_n y^n - \beta_1 x^1 - \cdots - \beta_q x^q = 0.$$

But this means that $\{x^1,\ldots,x^q,y^{q+1},\ldots,y^n\}$ is linearly dependent, a contradiction. Thus, $\{z^{q+1},\ldots,z^n\}$ must be linearly independent.

The vectors in $\{z^{q+1},\ldots,z^n\}$ belong (by definition) to T. Using (d) and the linear independence of $\{z^{q+1},\ldots,z^n\}$, we infer that $\{z^{q+1},\ldots,z^n\}$ is a basis of T. Consequently, the rank of T is $(n-q)$.

(f) Since the rank of A is r, we can find a set of r linearly independent vectors among the n column vectors of A. Without loss of generality, we let this set be $\{A^1,\ldots,A^r\}$. Note that $A^i = Ae^i$ for $i = 1,\ldots,r$, so the vectors in $\{A^1,\ldots,A^r\}$ belong to T. Since any $v \in T$ can be expressed as Az for some $z \in \mathbb{R}^n$, any $v \in T$ can be expressed as a linear combination of the vectors in $U = \{A^1,\ldots,A^n\}$.

The rank of U is r, and so $\{A^1,\ldots,A^r\}$ is a basis of U. Thus, any $A^j \in U$ can be expressed as a linear combination of the set of vectors in $\{A^1,\ldots,A^r\}$.

It follows that any $v \in T$ can be expressed as a linear combination of the set of vectors in $\{A^1,\ldots,A^r\}$. Since $\{A^1,\ldots,A^r\}$ is linearly independent, $\{A^1,\ldots,A^r\}$ is a basis of T. Thus, the rank of T is r.

(g) Using (e) and (f), we have $r = n-q$. This establishes (*).

3.9 Worked Out Problems on Chapter 3

Problem 10 (System of Linear Equations: Existence and Uniqueness of Solutions)**.**

Consider the following system of linear equations:

$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ 3x_1 + 3x_2 &= 9 \\ 2x_1 + 3x_2 &= 7 \end{aligned} \tag{1}$$

(a) Show, using the existence criterion discussed in class, that the system of equations (1) has a solution.
(b) Does the system of equations (1) have a unique solution ? Explain.

Solution.

We can write the system (1) as $Ax = b$, where

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \\ 2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 9 \\ 7 \end{bmatrix}, \quad A_b = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 3 & 9 \\ 2 & 3 & 7 \end{bmatrix}$$

(a) To show that $Ax = b$ has a solution, we must show that $\text{rank}(A) = \text{rank}(A_b)$. First, $\text{rank}(A) = 2$: $\text{rank}(A) \leq 2$ because A has only two columns, and $\text{rank}(A) \geq 2$ because A^1, A^2 are linearly independent. Second, $\text{rank}(A_b) = 2$: $\text{rank}(A_b) \geq 2$ since $A_b^1 = A^1$ and $A_b^2 = A^2$ are linearly independent, and $\text{rank}(A_b) < 3$ since $A_b^3 = 2A_b^1 + A_b^2$, so the three columns of A_b are linearly dependent. Because $\text{rank}(A_b) = \text{rank}(A) = 2$, the system $Ax = b$ has a solution by the Existence Theorem from Chapter 3.

(b) We can apply the Uniqueness Theorem from Chapter 3, since we have $m = 3, n = 2$. From part (a), $\text{rank}(A) = \text{rank}(A_b) = 2 = n$, so the system $Ax = b$ has a unique solution.

Problem 11 (System of Linear Equations: Existence of Solutions)**.**

Consider the following system of linear equations:

$$\begin{aligned} 3x_1 + x_2 + x_3 &= t \\ x_1 - x_2 + 2x_3 &= 1 - t \\ x_1 + 3x_2 - 3x_3 &= 1 + t \end{aligned} \tag{2}$$

where t is a real number. For what values of t will the system of equations (2) have a solution ? Explain.

Solution.

[Note: You are free to just solve the system, but the method below may help you solve certain other existence or uniqueness problems.] We can write the system (2) as $Ax = b$, where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 3 & -3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} t \\ 1-t \\ 1+t \end{bmatrix}, \quad A_b = \begin{bmatrix} 3 & 1 & 1 & t \\ 1 & -1 & 2 & 1-t \\ 1 & 3 & -3 & 1+t \end{bmatrix}$$

By the Existence Theorem from Chapter 3, the system $Ax = b$ will have a solution when $\text{rank}(A) = \text{rank}(A_b)$. Now, $\text{rank}(A) = 2$ because A^1, A^2 are linearly independent, but $A_1 = 2A_2 + A_3$. So we want to find the values of t for which $\text{rank}(A_b) = 2$. Since A^1, A^2 are linearly independent, if $\text{rank}(A_b) = 2$ then A^1, A^2 will form a basis for the columns of A_b. In particular, we can write

$$\begin{bmatrix} t \\ 1-t \\ 1+t \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

This is a system of three equations. By adding the first and second equations, we have $4\lambda_1 = 1$, or $\lambda_1 = \frac{1}{4}$. By subtracting the second equation from the first equation, then subtracting twice the third equation, we have $-4\lambda_2 = -3$, or $\lambda_2 = \frac{3}{4}$. With these values of λ_1, λ_2, the second equation gives $t = \frac{3}{2}$. This is the only value of t for which the system $Ax = b$ has a solution.

Problem 12 (System of Linear Equations: Uniqueness of Solution)**.**

Let A be an $m \times n$ matrix and let b be a vector in $\mathbb{R}^m$. Consider the following system of linear equations:

$$Ax = b \tag{3}$$

Suppose (3) has a unique solution for *every* $b \in \mathbb{R}^m$. Can m be different from n ? Explain.

Solution.

Using the Uniqueness Theorem from Chapter 3, we are given that for all $b \in \mathbb{R}^m$, $\text{rank}(A) = \text{rank}(A_b) = n$. That means the $A^1, \ldots, A^n \in \mathbb{R}^m$ are linearly independent and any $b \in \mathbb{R}^m$ is a linear combination of $A^1, \ldots, A^n$. By the Basis Theorem, then, the $A^1, \ldots, A^n$ are a basis for $\mathbb{R}^m$. Since $\text{rank}(\mathbb{R}^m) = m$, it must be that $m = n$.

Problem 13 (System of Homogeneous Linear Equations: Existence and Uniqueness of Solutions)**.**

Let A be an $n \times n$ matrix. Consider the following system of homogeneous linear equations:

$$Ax = 0 \tag{4}$$

(a) Suppose A is non-singular. Show that there is a unique solution to the system of equations (4).

(b) Suppose A is singular. Show that there are an infinite number of distinct solutions to the system of equations (4).

Solution.

(a) In class, the following result was presented for an $n \times n$ matrix A: the system $Ax = b$ has a unique solution if and only if $\text{rank}(A) = n$. Since we are given that A is non-singular, we know $\text{rank}(A) = n$ and so the system $Ax = 0$ has a unique solution.

(b) Let $\text{rank}(A) = k < n$. Then $A_b = [A\ 0]$ also has rank k. By the Existence Theorem in Chapter 3, then, the system $Ax = 0$ has a solution, since $\text{rank}(A) = \text{rank}(A_b) = k$.

Since $Ax = 0$ has a unique solution if and only if A is non-singular, it must be that $Ax = 0$ has more than one solution. Suppose vectors x and x' solve $Ax = 0$, with $x \neq x'$. Consider $x'' = \lambda x + (1-\lambda)x'$ for some $\lambda \in (0,1)$. Note that $x'' \neq x$ and $x'' \neq x'$. Now $Ax'' = \lambda Ax + (1-\lambda)Ax' = 0$, so x'' solves $Ax = 0$. Letting λ vary over (0, 1), we have an infinite number of distinct solutions to $Ax = 0$.

Problem 14 (Determinant of Upper Triangular Matrix)**.**

Let A be an $n \times n$ matrix, with $a_{ij} = 0$ whenever $i > j$.

(a) Show that:

$$\det A = \prod_{i=1}^{n} a_{ii}$$

(b) Use (a) to verify that A is non-singular if and only if $a_{ii} \neq 0$ for each $i \in \{1, \ldots, n\}$.

Solution.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

(a) Let's prove this by induction.

Base case: Let $n = 1$. If B is a 1×1 matrix, then $\det B = b_{11} = \prod_{i=1}^{1} b_{ii}$ by the definition of a determinant.

Inductive case: Let $n > 1$. Assume that for any $(n-1) \times (n-1)$ matrix C with $c_{ij} = 0$ for all $i > j$, we have $\det C = \prod_{i=1}^{n-1} c_{ii}$. Now consider any $n \times n$ matrix A with $a_{ij} = 0$ for all $i > j$. Expanding by the last row, we have

$$\begin{aligned}
\det A &= a_{n1}A_{n1} + \cdots + a_{nn}A_{nn} \\
&= a_{nn}(-1)^{n+n} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ 0 & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \end{vmatrix} \\
&= a_{nn} \prod_{i=1}^{n-1} a_{ii} \\
&= \prod_{i=1}^{n} a_{ii}
\end{aligned}$$

where the third equality follows from the inductive hypothesis.

(b) As an "if and only if" statement, this requires proofs in both directions.

[Note: You are free to cite Theorem 7, Chapter 3, the proof of which is contained in the answer below, in order to write a shorter proof for this problem.]

Claim: If the upper triangular matrix A is non-singular, then $a_{ii} \neq 0$ for all $i = 1, \ldots, n$.

Proof: Let A be non-singular. Then A has an inverse, A^{-1}. Since

$$1 = \det I = \det A^{-1}A = (\det A^{-1})(\det A),$$

we know that $\det A \neq 0$. If $a_{ii} = 0$ for any $i \in 1, \ldots, n$, then by (a) we would have $\det A = 0$, a contradiction. So it must be that $a_{ii} \neq 0$ for all $i = 1, \ldots, n$.

Claim: If A is upper triangular and $a_{ii} \neq 0$ for all $i = 1, \ldots, n$, then A is non-singular.

Proof: Let $a_{ii} \neq 0$ for all $i = 1, \ldots, n$. Then by (a), $\det A \neq 0$. Seeking contradiction, suppose A is singular. Without loss of generality, we can write $A^1 = \sum_{i=2}^n \lambda_i A^i$. Let

$$B = \begin{bmatrix} A^1 - \sum_{i=2}^n \lambda_i A^i & A^2 & \cdots & A^n \end{bmatrix}$$
$$= \begin{bmatrix} 0 & A^2 & \cdots & A^n \end{bmatrix}$$

We know, by property (D.4) in Chapter 3, that $\det B = \det A$. But, expanding B by the first column, we have $\det B = 0$. This gives $\det A = 0$, a contradiction. So we have that A is non-singular.

Problem 15 (Test of Linear Dependence of Vectors)**.**

Let $S = \{x^1, x^2, \ldots, x^m\}$ be a set of vectors in $\mathbb{R}^n$, and let G be the $m \times m$ matrix defined by:

$$G = \begin{bmatrix} x^1x^1 & \cdots & x^1x^m \\ \vdots & \cdots & \vdots \\ x^mx^1 & \cdots & x^mx^m \end{bmatrix}$$

where x^ix^j is the inner product of x^i and x^j, for $i = 1, \ldots, m$ and $j = 1, \ldots, m$.

Show that S is linearly dependent if and only if:

$$\det G = 0.$$

Solution.

As an "if and only if" statement, this requires proofs in both directions.

Claim: If S is linearly dependent, then det $G = 0$.

Proof: Suppose S is linearly dependent. Then, without loss of generality, we can write $x^1 = \lambda_2 x^2 + \cdots + \lambda_m x^m$. Substituting this into G in a clever way, we have

$$G = \begin{bmatrix} (\lambda_2 x^2 + \cdots + \lambda_m x^m)x^1 & \cdots & (\lambda_2 x^2 + \cdots + \lambda_m x^m)x^m \\ x^2x^1 & \cdots & x^2x^m \\ \vdots & \ddots & \vdots \\ x^mx^1 & \cdots & x^mx^m \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_2 x^2x^1 + \cdots + \lambda_m x^mx^1 & \cdots & \lambda_2 x^2x^m + \cdots + \lambda_m x^mx^m \\ x^2x^1 & \cdots & x^2x^m \\ \vdots & \ddots & \vdots \\ x^mx^1 & \cdots & x^mx^m \end{bmatrix}$$

That is, the rows of G are linearly dependent: $G_1 = \lambda_2 G_2 + \cdots + \lambda_m G_m$. So G is singular, which is the same as saying G is not invertible, which is the same as det $G = 0$.

Claim: If det $G = 0$, then S is linearly dependent.

Proof: Suppose det $G = 0$. This is equivalent to saying G is not invertible, which is equivalent to saying G is singular. That is, the columns of G are linearly dependent: there exist $\lambda_1, \ldots, \lambda_m$ not all zero such that

$$\lambda_1 \begin{bmatrix} x^1 x^1 \\ \vdots \\ x^m x^1 \end{bmatrix} + \cdots + \lambda_m \begin{bmatrix} x^1 x^m \\ \vdots \\ x^m x^m \end{bmatrix} = 0$$

We can write this as the system of equations

$$\begin{aligned} x^1(\lambda_1 x^1 + \cdots + \lambda_m x^m) &= 0 \\ &\vdots \\ x^m(\lambda_1 x^1 + \cdots + \lambda_m x^m) &= 0 \end{aligned}$$

Now, multiply the ith equation by λ_i and sum all the equations for

$$(\lambda_1 x^1 + \cdots + \lambda_m x^m)(\lambda_1 x^1 + \cdots + \lambda_m x^m) = 0$$

Let $y = \lambda_1 x^1 + \cdots + \lambda_m x^m$. By (I.4) in Chapter 1, $yy = 0$ if and only if $y = 0$. This means $\lambda_1 x^1 + \cdots + \lambda_m x^m = 0$ for $\lambda_1, \ldots, \lambda_m$ not all zero. Thus S is linearly dependent.

References:

This material is based on *Linear Algebra* by *G. Hadley*(Chapters 3, 5), *Elementary Matrix Algebra* by *F. Hohn* (Chapters 2, 6, 7) and *The Theory of Linear Economic Models* by D. Gale (Chapter 2). Some of this material is also covered in *Linear Programming and Economic Analysis* by *R. Dorfman, P.A. Samuelson and R.M. Solow* (Appendix B).

Chapter 4

Characteristic Values and Vectors

4.1 The Characteristic Value Problem

Let $\mathbb{C}$ denote the set of complex numbers. Given an $n \times n$ real matrix, for what *non-zero* vectors $x \in \mathbb{C}^n$, and for what complex numbers λ is it true that

$$Ax = \lambda x \tag{4.1}$$

This is known as the *characteristic value problem* or the *eigenvalue problem*.

If $x \neq 0$ and λ satisfy equation (4.1), then λ is called a *characteristic value* or *eigenvalue* of A, and x is called a *characteristic vector* or *eigenvector* of A.

Clearly (4.1) holds if and only if

$$(A - \lambda I)x = 0 \tag{4.2}$$

But (4.2) is a homogeneous system of n equations in n unknowns. It has a non-zero solution for x if and only if $(A - \lambda I)$ is singular; that is, if and only if

$$|A - \lambda I| = 0 \tag{4.3}$$

This equation is called the *characteristic equation* of A. If we look at the expression

$$f(\lambda) \equiv |A - \lambda I| \tag{4.4}$$

we note that f is a *polynomial* in λ; it is called the *characteristic polynomial* of A.

Example: Consider the 2×2 matrix A given by

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Then equation (4.3) becomes

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \tag{4.5}$$

So, $(4-2\lambda+\lambda^2)-1=0$, which yields

$$(1-\lambda)(3-\lambda)=0$$

Thus, the characteristic roots are $\lambda=1$ and $\lambda=3$.

Putting $\lambda=1$ in (4.2), we get

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which yields

$$x_1+x_2=0$$

Thus the general solution of the characteristic vector corresponding to the characteristic root $\lambda=1$ is given by

$$(x_1,x_2)=\theta(1,-1) \qquad for\,\theta\neq 0$$

Similarly, corresponding to the characteristic root $\lambda=3$, we have the characteristic vector given by

$$(x_1,x_2)=\theta(1,1) \qquad for\,\theta\neq 0.$$

In developing the basic results on the characteristic-value problem, we note that, in general, the characteristic equation will have n roots in the complex plane (by the "Fundamental Theorem of Algebra"), since it is a polynomial equation (in λ) of degree n. [Of course some of these roots might be repeated]. In general, the corresponding eigenvectors will also have their components in the complex plane.

4.2 Characteristic Values, Trace and Determinant of a Matrix

If A is an $n\times n$ matrix, the *trace* of A, denoted by $tr(A)$, is the number defined by

$$tr(A)=\sum_{i=1}^{n}a_{ii}$$

The following properties of the trace can be verified easily [Here A, B and C are $n\times n$ matrices, and $\lambda\in\mathbb{R}$].

(TR.1) $tr(A+B) = tr(A) + tr(B)$
(TR.2) $tr(\lambda A) = \lambda \quad tr(A)$
(TR.3) $tr(AB) = tr(BA)$
(TR.4) $tr(ABC) = tr(BCA) = tr(CAB)$

Let A be an $n \times n$ matrix. The characteristic polynomial of A, defined in (4.4) above can generally be written as

$$|A - \lambda I| = (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + \ldots. + b_1(-\lambda) + b_0 \tag{4.6}$$

where $b_0, \ldots, b_{n-1}$ are the coefficients of the polynomial which are determined by the coefficients of the A-matrix.

On the other hand, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then the characteristic equation (4.3) can be written as

$$0 = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots.(\lambda_n - \lambda) \tag{4.7}$$

Using (4.3), (4.6), and (4.7) and "comparing coefficients" we can conclude that

$$b_{n-1} = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

and

$$b_0 = \lambda_1 \lambda_2 \ldots \lambda_n$$

Also, by looking at the terms in the characteristic polynomial of A which would involve $(-\lambda)^{n-1}$, we can conclude that

$$b_{n-1} = a_{11} + a_{22} + \ldots + a_{nn}$$

Finally, putting $\lambda = 0$ in (4.6), we get

$$b_0 = |A|$$

Thus we might note two interesting relationships between the characteristic values, the trace and the determinant of A:

$$trA = \sum_{i=1}^{n} \lambda_i$$

$$|A| = \prod_{i=1}^{n} \lambda_i$$

4.3 Characteristic Values and Vectors of Symmetric Matrices

There is considerable simplification in the theory of characteristic values if A is a *symmetric* matrix. In this case, it can be shown that all the roots of (4.3) are *real*.

Theorem 8. *Let A be a symmetric $n \times n$ matrix. Then all the characteristic values of A are real.*

Proof. Suppose λ is a complex characteristic value, with associated complex characteristic vector, x. Then we have

$$Ax = \lambda x \tag{4.8}$$

Define x^* to be the complex conjugate of x, and λ^* to be the complex conjugate of λ. Then

$$Ax^* = \lambda^* x^* \tag{4.9}$$

Pre-multiply (4.8) by $(x^*)'$ and (4.9) by x' to get

$$(x^*)'Ax = \lambda(x^*)'x \tag{4.10}$$

$$x'Ax^* = \lambda^* x'x^* \tag{4.11}$$

Subtracting (4.11) from (4.10)

$$(x^*)'Ax - x'Ax^* = (\lambda - \lambda^*)x'x^* \tag{4.12}$$

since $(x^*)'x = x'x^*$. Also,

$$x'Ax^* = (x'Ax^*)' = (x^*)'A'x = (x^*)'Ax$$

since $A' = A$ (by symmetry). Thus (4.12) yields

$$(\lambda - \lambda^*)x'x^* = 0 \tag{4.13}$$

Since $x \neq 0$, we know that $x'x^*$ is real and positive. Hence (4.13) implies that $\lambda = \lambda^*$, so λ is real. ■

We will develop the theory of eigenvalues and eigenvectors *only for symmetric matrices*.

Notice that once the eigenvalues are real, the system of equations

$$(A - \lambda I)x = 0$$

will yield a non-zero solution x in $\mathbb{R}^n$ if and only if

$$|A - \lambda I| = 0$$

So the eigenvectors corresponding to the eigenvalues of A will also be real vectors.

If x is an eigenvector corresponding to an eigenvalue λ, then so is tx, where t is *any non-zero scalar*. A *normalized eigenvector* is an eigenvector with (Euclidean) norm equal to 1.

4.4 Spectral Decomposition of Symmetric Matrices

An $n \times n$ matrix C is called an *orthogonal matrix* if it is invertible, and its inverse equals its transpose; that is $C' = C^{-1}$.

Theorem 9. *Suppose A is an $n \times n$ symmetric matrix with n distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$. If $x^1, \ldots, x^n$ are (normalized) eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then the matrix B, such that B^i, the ith column of B, is the vector x^i, is an orthogonal matrix.*

Proof. Pick any two distinct indices i and j (so $i \neq j$). Then we have

$$Ax^i = \lambda_i x^i \tag{4.14}$$

and

$$Ax^j = \lambda_j x^j \tag{4.15}$$

Multiplying (4.14) by $(x^j)'$ we get

$$(x^j)' Ax^i = \lambda_i (x^j)' x^i \tag{4.16}$$

Multiplying (4.15) by $(x^i)'$ we get

$$(x^i)' Ax^j = \lambda_j (x^i)' x^j \tag{4.17}$$

Now, $(x^i)' Ax^j$ is a number, so its transpose is the same number. Thus

$$(x^i)' Ax^j = ((x^i)' Ax^j)' = (x^j)' A' x^i$$

But $A' = A$ by symmetry of A. So

$$(x^i)' Ax^j = (x^j)' Ax^i$$

Using this in (4.16) and (4.17), we get

$$(\lambda_i - \lambda_j)x^i x^j = 0$$

Since the eigenvalues of A are distinct, $\lambda_i \neq \lambda_j$ for $i \neq j$. Thus $x^i x^j = 0$, so x^i is orthogonal to x^j.

It follows from this that $x^1, \ldots, x^n$ are linearly independent. For if they were dependent, there would exist $\mu_1, \ldots, \mu_n$, not all zero, such that

$$\mu_1 x^1 + \ldots + \mu_n x^n = 0 \tag{4.18}$$

Without loss of generality consider $\mu_1 \neq 0$. Then premultiplying (4.18) by $(x^1)'$, we get

$$\mu_1 \left\| x^1 \right\|^2 = 0$$

since $(x^1)' x^j = 0$ for all $j \neq 1$. Since $\|x^1\|^2 = 1$, we get $\mu_1 = 0$, a contradiction. Thus $x^1, \ldots, x^n$ are linearly independent.

If B is the matrix such that B^i, the ith column of B, is the vector x^i, then B is invertible, since we have shown that B is non-singular. Also,

$$BB' = I \tag{4.19}$$

since $x^i x^j = 0$ for $i \neq j$, and $x^i x^j = 1$ for $i = j$. Thus premultiplying (4.19) by B^{-1} we get

$$B' = B^{-1}$$

Hence B is an orthogonal matrix. ■

Theorem 10. *(Spectral Decomposition)*

Suppose A is an $n \times n$ symmetric matrix with n distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$. If B is the $n \times n$ matrix with B^i, the ith column of B, being a (normalized) eigenvector of A corresponding to the eigenvalue λ_i of $A(i = 1, \ldots, n)$, then

$$A = BLB' \tag{4.20}$$

where L is the diagonal matrix with the eigenvalues of A on its diagonal.

Proof. For each $i = 1, \ldots, n$, we have

$$AB^i = \lambda_i B^i$$

This can be written in compact form as

$$AB = BL \tag{4.21}$$

where the $n \times n$ matrix L is defined by

$$L = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

We have noted in Theorem 9 that B is an orthogonal matrix. So, post-multiplying (4.21) by B^{-1}, we get

$$A = BLB^{-1}$$

But since B is orthogonal, we have $B^{-1} = B'$, and thus we arrive at (4.20). ■

Remark: The formula (4.20) "decomposes" the matrix A into a matrix L consisting of its eigenvalues, and the matrices B, B' which consist of its eigenvectors.

4.5 Quadratic Forms

Definition 1. *Let A be a symmetric $n \times n$ matrix. Then*

(a) A is *negative semi-definite* if $hAh \leq 0$ for all h in $\mathbb{R}^n$.

(b) A is *negative definite* if $hAh < 0$ for all h in $\mathbb{R}^n$, $h \neq 0$.

(c) A is *positive semi-definite* if $hAh \geq 0$ for all h in $\mathbb{R}^n$.

(d) A is *positive definite* if $hAh > 0$ for all h in $\mathbb{R}^n$, $h \neq 0$.

Let us concentrate on definition (d). Notice that the relevant inequality must hold for *every* vector $h \neq 0$ in $\mathbb{R}^n$. This means that if we already know that a symmetric $n \times n$ matrix A is positive-definite, then we should be able to infer some useful properties of A quite easily. On the other hand, it also means that if we do not know that a symmetric $n \times n$ matrix A is positive definite, definition (d) by itself will not be very easy to check to determine whether A is positive definite or not.

We illustrate the first observation by noting that if a symmetric $n \times n$ matrix, A, is positive definite then we can infer that all its diagonal elements must be positive. To see this, note that the i^{th} diagonal element, a_{ii}, can be expressed as

$$(e^i)'Ae^i = a_{ii}$$

where e^i is the i^{th} unit vector of $\mathbb{R}^n$. Since A is positive definite and $e^i \neq 0$ is in $\mathbb{R}^n$, definition (d) tells us that the left-hand side of the above equation is positive. Thus the right-hand side is also positive.

The second observation leads one to explore convenient characterizations of quadratic forms from which it should be easy to check whether a given $n \times n$ symmetric matrix A is positive definite or not. We provide two such characterizations in the next two sections.

4.6 Characterization of Quadratic Forms

Let A be an $n \times n$ symmetric matrix, with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. We have seen that we can define a matrix B, such that B^i, the i^{th} column of B, is a normalized eigenvector of A corresponding to the eigenvalue $\lambda_i (i = 1, \ldots, n)$. This matrix is orthogonal, and furthermore

$$B'AB = L \tag{4.22}$$

where L is a diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on its diagonal.

Let x be an arbitrary non-zero vector in $\mathbb{R}^n$. Since B is an orthogonal matrix we can define the vector $y \in \mathbb{R}^n$ by $y = B^{-1}x = B'x$, so that $By = x$, and $y'B' = x'$. Then premultiplying (4.22) by y' and post-multiplying it by y, we get

$$y'B'ABy = y'Ly$$

which yields, by definition of y,

$$x'Ax = y'Ly \tag{4.23}$$

The right hand side of (4.23) is

$$y'Ly = \sum_{i=1}^{n} \lambda_i y_i^2 \tag{4.24}$$

where $y = (y_1, \ldots, y_n)$. Suppose all the eigenvalues of A are positive. Then since $x \neq 0$, by definition of y, we have $y \neq 0$, and (4.24) is positive. So, by (4.23), $x'Ax > 0$ for each $x \neq 0$, and A is positive definite.

Conversely, if y is an arbitrary vector in $\mathbb{R}^n$, we define $x = By$ and so $y'B' = x'$, and note that if $y \neq 0$, then $x \neq 0$ since B is orthogonal. Thus repeating the above calculations, we get (4.23) and (4.24). Now, suppose A is positive definite. Then choosing in turn $y = e^j$, the j^{th} unit vector, it follows from (4.24) that $y'Ly = \lambda_j$ and from (4.23) that $\lambda_j = x'Ax > 0$. Thus, all the eigenvalues of A are positive.

In the same way, characterization of the other quadratic forms can be obtained in terms of the signs of the eigenvalues of the matrix. We summarize these results as follows:

(a) *A is positive (negative) definite if and only if every eigenvalue of A* is positive (negative).

(b) *A is positive (negative) semi-definite if and only if every eigenvalue of A is non-negative (non-positive).*

Examples:
Consider the following matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}; C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of A are -1 and 0. So A is negative semi-definite. The eigenvalues of B are $(-2+\sqrt{2})$ and $(-2-\sqrt{2})$, which are both negative. So B is negative definite. The eigenvalues of C are 0 and 1. So C is positive semidefinite.

4.7 Alternative Characterization of Quadratic Forms

There is an alternative way to characterize quadratic forms in terms of the signs of the "principal minors" of the corresponding matrix.

If A is an $n \times n$ matrix, a *principal minor of order r* is the determinant of the $r \times r$ submatrix that remains when $(n-r)$ rows and $(n-r)$ columns *with the same indices* are deleted from A.

Examples: If A is a 3×3 matrix, then the principal minors of order 2 are

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}; \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

And, the principal minors of order 1 are

$$a_{11} \quad ; \quad a_{22} \quad ; \quad a_{33}$$

The principal minor of order 3 is the determinant of the matrix.

If A is an $n \times n$ matrix, the *leading principal minor of order r* is defined as

$$D_r = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}$$

Thus $D_1 = a_{11}$; $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$;etc.

We note below the characterization result for positive and negative definite matrices. Let A be an $n \times n$ symmetric matrix.

(1') (i) *A is a positive definite if and only if all the leading principal minors of A are positive.*

(ii) *A is a negative definite if and only if the leading principal minors of A alternate in sign, starting with negative* [That is, the r^{th} leading principal minor, D_r, (where $r = 1, \ldots, n$) has the same sign as $(-1)^r$].

Checking that a matrix is positive or negative semi-definite is somewhat more involved. The relevant results are stated below:

(2') (i) *A is positive semi-definite if and only if every principal minor of A of every order is non-negative.*

(ii) *A is negative semi-definite if and only if every principal minor of A of odd order is non-positive and every principal minor of even order is non-negative.*

Examples:

We re-examine the matrices which we studied above:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}; C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

All the principal minors of A of odd order are non-positive $[a_{11} < 0$ and $a_{22} = 0]$, and the (only) principal minor of A of even order is non-negative $[a_{11}a_{22} - a_{21}a_{12} = 0]$. So A is negative semidefinite.

For the matrix B, the leading principal minor of order 1 is negative $[b_{11} = -1 < 0]$, and the leading principal minor of order 2 is positive $[b_{11}b_{22} - b_{12}b_{21} = 2 > 0]$. So B is negative definite.

The principal minors of C of odd order are non-negative $[c_{11} = 0$ and $c_{22} > 0]$. The principal minor of C of even order is non-negative $[c_{11}c_{22} - c_{12}c_{21} = 0]$. So C is positive semi-definite.

4.8 Appendix: Spectral Decomposition of Non-Symmetric Matrices

The characteristic roots of non-symmetric matrices need not be real numbers. However, when they are real numbers, one can develop a theory of spectral decomposition of non-symmetric matrices, using the methods already developed in this course.

The following problem, split up into five parts, provides the steps involved in such a theory. You might wish to work through the steps; it is entirely optional.

Let A be an $n \times n$ matrix (not necessarily symmetric), whose characteristic roots are real and distinct; denote the roots by $\lambda_1, \ldots, \lambda_n$.

(a) Show that for each $i \in \{1, \ldots, n\}$, there exists a vector $x^i \in \mathbb{R}^n$, such that $x^i \neq 0$, and:

$$(A - \lambda_i I)x^i = 0$$

(b) The principal difference from the case in which A is symmetric is that the characteristic vectors $x^1, \ldots, x^n$ need not be orthogonal. To see this, consider the following example:

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

(i) Show that A has two real characteristic roots, which are distinct; call them λ_1 and λ_2.

(ii) Obtain characteristic vectors $x^1 \in \mathbb{R}^2$ and $x^2 \in \mathbb{R}^2$, corresponding to λ_1 and λ_2 respectively.

(iii) Show that x^1 is *not* orthogonal to x^2.

(c) Continuing now with the result obtained in (a) above, show that the set $\{x^1, \ldots, x^n\}$ is linearly independent. Denote by X the matrix whose i-th column is given by x^i for $i \in \{1, .., n\}$. Note that X has an inverse.

(d) Show that:

$$AX = X\Lambda$$

where Λ is a diagonal matrix with the characteristic roots $\lambda_1, \ldots, \lambda_n$ on its diagonal (in that order). Note then that:

$$A = X\Lambda X^{-1}$$

(e) Show that for $t = 1, 2, 3, \ldots$, we have:

$$A^t = XL(t)X^{-1}$$

where $L(t)$ is a diagonal matrix, with $\lambda_1^t, \ldots, \lambda_n^t$ on its diagonal (in that order).

Remark:

The above theory also goes through when the characteristic roots are complex, but distinct. However, the methods developed in this course will not suffice to cover this case, because concepts such as linear independence and matrix inverse were developed in this course starting with vectors in $\mathbb{R}^n$ (not $\mathbb{C}^n$).

4.9 Worked Out Problems on Chapter 4

Problem 16 (Non-Symmetric Matrices and Real Eigenvalues)**.**

Let A be the 2×2 matrix, given by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{12} \neq a_{21}$. Let p be the trace of A, and let q be the determinant of A. Assume that:

$$p > 0, q > 0,\; p^2 > 4q \text{ and } (p - q) < 1$$

Denote the characteristic roots of A by λ_1 and λ_2.

(i) Show that λ_1 and λ_2 are real and positive.

(ii) Show that exactly one of the following alternatives must occur: (A) $\lambda_1 < 1$ and $\lambda_2 < 1$; (B) $\lambda_1 > 1$ and $\lambda_2 > 1$.

Solution.

(a) Note that $p = \text{tr}(A) = a_{11} + a_{22}$ and $q = \det A = a_{11}a_{22} - a_{12}a_{21}$. Then we want to solve

$$\begin{aligned} 0 = f(\lambda) = \det(A - \lambda I) &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &= \lambda^2 - p\lambda + q \end{aligned}$$

By the quadratic formula, the roots are

$$\lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

We are given that $p^2 > 4q$, so there are two distinct real roots. Without loss of generality, let's call the larger root λ_1. Then we have

$$\lambda_1 = \frac{p + \sqrt{p^2 - 4q}}{2}, \quad \lambda_2 = \frac{p - \sqrt{p^2 - 4q}}{2}, \quad \lambda_1 > \lambda_2$$

Since $p > 0$, we have $\lambda_1 > 0$. The smaller root λ_2 is positive if and only if

$$\begin{aligned} & p - \sqrt{p^2 - 4q} > 0 \\ \iff \quad & p > \sqrt{p^2 - 4q} \\ \iff \quad & p^2 > p^2 - 4q \\ \iff \quad & q > 0 \end{aligned}$$

where $p > \sqrt{p^2 - 4q} \iff p^2 > p^2 - 4q$ holds since $p > 0$. Because we are given $q > 0$, we have $\lambda_2 > 0$.

(b) If $\lambda_1 = 1$ or $\lambda_2 = 1$, then $p - q = 1$, which is a contradiction of the given information that $p - q < 1$. So, in the proof below we consider only strict inequalities.

(A): $\lambda_1 < 1$ and $\lambda_2 < 1$. (not A): $\lambda_1 > 1$ or $\lambda_2 > 1$.

(B): $\lambda_1 > 1$ and $\lambda_2 > 1$. (not B): $\lambda_1 < 1$ or $\lambda_2 < 1$.

We want to show that (A) implies (not B) and that (not A) implies (B). This will establish that exactly one of (A) or (B) always occurs.

Claim: (A) implies (not B). That is, if $\lambda_1 < 1$ and $\lambda_2 < 1$, then $\lambda_1 < 1$ or $\lambda_2 < 1$.

Proof: By hypothesis, $\lambda_1 < 1$. So the claim holds.

Claim: (not A) implies (B). That is, if $\lambda_1 > 1$ or $\lambda_2 > 1$, then $\lambda_1 > 1$ and $\lambda_2 > 1$.

Proof: There are two cases to consider: one in which $\lambda_1 > 1$ and one in which $\lambda_2 > 1$. We must show that in both cases, $\lambda_1 > 1$ and $\lambda_2 > 1$.

Case 1: Suppose $\lambda_1 > 1$. Then it follows that $p - 2 > -\sqrt{p^2 - 4q}$. Seeking contradiction, assume $\lambda_2 < 1$. This implies $p - 2 < \sqrt{p^2 - 4q}$. So we have

$$\begin{aligned} & |p-2| < \sqrt{p^2 - 4q} \\ \iff \quad & p^2 - 4p + 4 < p^2 - 4q \\ \iff \quad & p - q > 1 \end{aligned}$$

This is a contradiction of the given information that $p - q < 1$, so it must be that $\lambda_2 > 1$.

Case 2: Suppose $\lambda_2 > 1$. Then $\lambda_1 > \lambda_2 > 1$.

Since $\lambda_1 > 1$ and $\lambda_2 > 1$ in both cases, the claim holds.

Problem 17 (Eigenvalues and Eigenvectors of Symmetric Matrices)**.**

Let $A = (a_{ij})$ be a symmetric 2×2 matrix. We know that it has only real eigenvalues; denote these by λ_1 and λ_2.

(a) Show that there is $b = (b_1, b_2) \in \mathbb{R}^2$, with $b \neq 0$, such that $(A - \lambda_1 I)b = 0$. This shows that there is a real eigenvector corresponding to the eigenvalue λ_1.

(b) Define $y = (y_1, y_2)$ as follows: $y_1 = b_1 + ib_1, y_2 = b_2 + ib_2$. Is y also an eigenvector of A (corresponding to the eigenvalue λ_1)? Explain.

Solution.

(a) Since λ_1 solves det $(A - \lambda_1 I) = 0$, we know that $(A - \lambda_1 I)$ is singular. Then there are b_1 and b_2 not both zero such that $b_1(A - \lambda_1 I)^1 + b_2(A - \lambda_1 I)^2 = 0$. Letting $b = (b_1, b_2)'$, we have that $(A - \lambda_1 I)b = 0$, so b is an eigenvector of A, corresponding to the eigenvalue λ_1.

(b) We can write $y = b + ib$. Then

$$(A - \lambda_1 I)y = (A - \lambda_1 I)b + (A - \lambda_1 I)ib = 0 + 0i = 0.$$

since b is an eigenvector of A, corresponding to the eigenvalue λ_1. So y is an eigenvector of A, corresponding to the eigenvalue λ_1.

Problem 18 (Application of Spectral Decomposition)**.**

Let A be the 2×2 matrix defined as follows:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Obtain the characteristic values and corresponding normalized characteristic vectors of this matrix.

(b) Use the Spectral Decomposition Theorem to show that for any positive integer n, the matrix A^n can be written as:

$$A^n = \begin{bmatrix} (3^n/2) + (1/2) & (3^n/2) - (1/2) \\ (3^n/2) - (1/2) & (3^n/2) + (1/2) \end{bmatrix}$$

Solution.

(a) The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned} f(\lambda) = \det(A - \lambda I) &= (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3) \end{aligned}$$

The roots are $\lambda_1 = 1, \lambda_2 = 3$.

A normalized eigenvector $b^1 = \left(b_1^1, b_2^1\right)'$ corresponding to $\lambda_1 = 1$ solves

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system implies $b_1^1 + b_2^1 = 0$, so that $b^1 = \left(b_1^1, -b_1^1\right)'$. The normalization constraint is

$$1 = \left\|b^1\right\| = \sqrt{\left(b_1^1\right)^2 + \left(-b_1^1\right)^2} = \sqrt{2\left(b_1^1\right)^2}$$

This is solved by $b_1^1 = \frac{1}{\sqrt{2}}$, so a normalized eigenvector corresponding to $\lambda_1 = 1$ is $b^1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)'$.

A normalized eigenvector $b^2 = \left(b_1^2, b_2^2\right)'$ corresponding to $\lambda_2 = 3$ solves

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1^2 \\ b_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system implies $b_1^2 = b_2^2$, so that $b^2 = \left(b_1^2, b_1^2\right)'$. The normalization constraint is

$$1 = \left\|b^2\right\| = \sqrt{\left(b_1^2\right)^2 + \left(b_1^2\right)^2} = \sqrt{2\left(b_1^2\right)^2}$$

This is solved by $b_1^2 = \frac{1}{\sqrt{2}}$, so a normalized eigenvector corresponding to $\lambda_2 = 3$ is $b^2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)'$.

(b) Let

$$B = \begin{bmatrix} b^1 & b^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

By the Spectral Decomposition Theorem, $A = B\Lambda B'$. Because $B' = B^{-1}$, we have

$$\begin{aligned} A^n &= B\Lambda^n B' \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 3^n \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 3^n \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3^n}{2} + \frac{1}{2} & \frac{3^n}{2} - \frac{1}{2} \\ \frac{3^n}{2} - \frac{1}{2} & \frac{3^n}{2} + \frac{1}{2} \end{bmatrix} \end{aligned}$$

Problem 19 (Symmetric Matrices with Repeated Characteristic Roots)**.**

Let A be the 3×3 matrix, defined by:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) Show that the characteristic values of A are 3 and 1, but 3 is a repeated root. That is, show that the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$ can be written as a product of three factors as follows:

$$f(\lambda) = (1-\lambda)(3-\lambda)(3-\lambda)$$

(b) Show that $(b^1)' = [0, 1/\sqrt{2}, -1/\sqrt{2}]$ is a normalized characteristic vector corresponding to the characteristic root $\lambda_1 = 1$.

(c) Show that $(b^2)' = [0, 1/\sqrt{2}, 1/\sqrt{2}]$ is a normalized characteristic vector corresponding to the characteristic root $\lambda_2 = 3$.

(d) Show that there is another normalized characteristic vector, b^3, corresponding to the characteristic root $\lambda_2 = 3$, which is orthogonal to both b^1 and b^2.

(e) Define B as the 3×3 matrix which has b^1, b^2, b^3 as its first, second and third columns. Define Λ to be the diagonal matrix:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Show that $A = B\Lambda B'$.

Solution.

(a) The characteristic polynomial is

$$\begin{aligned} f(\lambda) &= \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} \\ &= (3-\lambda)[(2-\lambda)^2 - 1] \\ &= (3-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= (3-\lambda)(3-\lambda)(1-\lambda) \end{aligned}$$

(b) We have that $\|b^1\| = 1$ and

$$(A - \lambda_1 I)b^1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

So b^1 is a normalized eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

(c) We have that $\|b^2\| = 1$ and

$$(A - \lambda_2 I)b^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

So b^2 is a normalized eigenvector corresponding to the eigenvalue $\lambda_2 = 3$.

(d) A normalized eigenvector $b^3 = \left(b_1^3, b_2^3, b_3^3\right)'$ corresponding to $\lambda_2 = 3$ solves

$$(A - \lambda_2 I)b^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} b_1^3 \\ b_2^3 \\ b_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system implies $b_2^3 = b_3^3$.

Any eigenvector b^3 satisfying $b_2^3 = b_3^3$ is orthogonal to b^1, so this condition gives us no information about the components of b^3.

We need b^3 to be orthogonal to b^2, so we require

$$0 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} b_1^3 \\ b_2^3 \\ b_2^3 \end{bmatrix} = \frac{1}{\sqrt{2}} b_2^3 + \frac{1}{\sqrt{2}} b_2^3 = \frac{2}{\sqrt{2}} b_2^3$$

This implies $b_2^3 = 0$, so any eigenvector b^3 satisfying $b_2^3 = b_3^3 = 0$ is orthogonal to b^2.

We can choose $b^3 = (1, 0, 0)'$. This is a normalized eigenvector corresponding to $\lambda_2 = 3$ that is orthogonal to both b^1 and b^2.

(e) Collecting the normalized eigenvectors above, we have

$$B = \begin{bmatrix} b^1 & b^2 & b^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Then

$$\begin{aligned} B\Lambda B' &= \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 3 \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ &= A \end{aligned}$$

Problem 20 (Positive Definite Matrices)**.**

Let A be an $n \times n$ symmetric matrix, which is positive definite. For each $k \in \{1, \ldots, n\}$, define the submatrix $A(k)$ by:

$$A(k) = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$$

(a) Show, using only the definition of a positive definite matrix (that is, without using any characterization result of positive definite matrices) that $a_{ii} > 0$ for each $i \in \{1, \ldots, n\}$.

(b) Show, using only the definition of a positive definite matrix (that is, without using any characterization result of positive definite matrices) that $A(k)$ is a positive definite matrix for each $k \in \{1, \ldots, n\}$.

(c) Show that the determinant of $A(k)$ is non-zero for each $k \in \{1, ..., n\}$.

(d) Show that the determinant of $A(k)$ is positive for each $k \in \{1, ..., n\}$.

Solution.

(a) Since A is positive definite, $a_{ii} = (e^i)'Ae^i > 0$ for all $i = 1, \ldots, n$.

(b) For any $z \in \mathbb{R}^k$, we can write

$$
\begin{aligned}
z'A(k)z &= \begin{bmatrix} z_1 & \cdots & z_k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \\
&= \begin{bmatrix} z_1 & \cdots & z_k \end{bmatrix} \begin{bmatrix} a_{11}z_1 + \cdots + a_{1k}z_k \\ \vdots \\ a_{k1}z_1 + \cdots + a_{kk}z_k \end{bmatrix} \\
&= z_1(a_{11}z_1 + \cdots + a_{1k}z_k) + \cdots + z_k(a_{k1}z_1 + \cdots + a_{kk}z_k) \\
&= \sum_{i=1}^{k} z_i(z_1 a_{i1} + \cdots + z_k a_{ik}) \\
&= \sum_{i=1}^{k} z_i \left(\sum_{j=1}^{k} z_j a_{ij} \right) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} z_i z_j a_{ij}
\end{aligned}
$$

Let $z \in \mathbb{R}^k$ be such that $z \neq 0$. Define the vector $z^n \in \mathbb{R}^n$ by $z_i^n = z_i$ if $i \in \{1, \ldots, k\}$ and $z_i^n = 0$ if $i \in \{k+1, \ldots, n\}$. That is,

$$
\begin{aligned}
z &= (z_1, \ldots, z_k)' \\
z^n &= (z_1, \ldots, z_k, 0, \ldots, 0)'
\end{aligned}
$$

Then we have

$$
\begin{aligned}
z'A(k)z &= \sum_{i=1}^{k} \sum_{j=1}^{k} z_i z_j a_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j a_{ij} && (\text{since } z_{k+1}^n = \cdots = z_n^n = 0) \\
&= (z^n)'Az^n \\
&> 0 && (\text{since } A \text{ is positive definite})
\end{aligned}
$$

Since our non-null $z \in \mathbb{R}^k$ is arbitrary, this shows that $A(k)$ is positive definite for each $k = 1, \ldots, n$.

(c) [Note: It is fine to first show the stronger result requested in (d) and then state that the result in (c) follows. The proof below establishes the result using only the definition of positive definiteness, together with material discussed before Chapter 4.]

Suppose $\det A(k) = 0$ for some $k \in \{1, \ldots, n\}$. Then $A(k)$ is singular, so the columns of $A(k)$ are linearly dependent. This means that there is some non-null $z = (z_1, \ldots, z_k)' \in \mathbb{R}^k$ such that $A(k)z = 0$. Premultiplying by z', we have that $z'A(k)z = 0$ for some $z \neq 0$. But this is a contradiction of $A(k)$ being positive definite, which was established in (b). So $\det A(k) \neq 0$ for all $k = 1, \ldots, n$.

(d) It was shown in (b) that $A(k)$ is positive definite. From Section 4.6, we have the result that $A(k)$ is positive definite if and only if every eigenvalue $\lambda_1, \ldots, \lambda_k$ of $A(k)$ is positive. Then using the relation between characteristic values (eigenvalues) and the determinant of $A(k)$ from Section 4.2, we have that $\det A(k) = \prod_{i=1}^{k} \lambda_i > 0$.

References

You will find most of this material covered in *Elementary Matrix Algebra* by F. Hohn (Chapters 9, 10) and *Linear Algebra* by *G. Hadley* (Chapter 7). A more advanced treatment of the subject can be found in *The Theory of Matrices*, vol. 1 by *F.R. Gantmacher* (Chapters 3, 10).

Part II

Real Analysis

Chapter 5

Basic Concepts of Real Analysis

5.1 Norm and Distance

We recall that $\mathbb{R}^n$ is the set of all n-vectors $x = (x_1,\ldots,x_n)$, where each x_i is a real number for $i = 1,\ldots,n$. The (Euclidean) *norm* of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$ and defined by

$$\|x\| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}$$

We have already noted some properties of the norm, where x, $y \in \mathbb{R}^n$and $\lambda \in \mathbb{R}$:

(N.1) $\|x\| \geq 0$; and $\|x\| = 0$ iff $x = 0$.

(N.2) $\|\lambda x\| = |\lambda| \ \ \|x\|$

(N.3) $\|x+y\| \leq \|x\| + \|y\|$

While (N.1) and (N.2) are easy to check, (N.3) is not. Start by establishing the following property:

(N.4) $|xy| \leq \|x\| \ \ \|y\|$

The inequality (N.4) is known as the *Cauchy-Schwarz inequality*. You can then use (N.4) to prove (N.3).

Using the norm, one can define the (Euclidean) *distance function* or *metric*. For x, y, $\in \mathbb{R}^n$, the *distance* between x and y, denoted by $d(x,y)$, is

$$d(x,y) = \|x-y\|$$

The following properties of d can be verified (when x, y, $z \in \mathbb{R}^n$):

(D.1) $d(x,y) \geq 0; d(x,y) = 0$ if and only if $x = y$

(D.2) $d(x,y) = d(y,x)$

(D.3) $d(x,z) \leq d(x,y) + (d(y,z)$

The property (D.3) is known as the "triangle inequality"; it can be established using (N.3).

5.2 Open and Closed Sets

Open Ball:

If $\bar{x} \in \mathbb{R}^n$, and r is a positive real number, an *open ball* (with *center* $\bar{x}$ and *radius* r) in $\mathbb{R}^n$ is

$$B(\bar{x}, r) = \{x \text{ in } \mathbb{R}^n : d(x, \bar{x}) < r\}$$

Open Set:

A set $S \subset \mathbb{R}^n$ is *open* (in $\mathbb{R}^n$) if for every $x \in S$, there is an open ball (with center x and radius $r > 0$) in $\mathbb{R}^n$ which belongs to S.

It follows that an open ball is an open set. You can check that the set $S = \{(x_1, x_2)$ in $\mathbb{R}^2 : x_1 > 0, x_2 > 0$ and $x_1^2 + x_2^2 < 1\}$ is an open set in $\mathbb{R}^2$.

In discussing the concept of an open set, it is important to specify the *space* in which we are considering the set. For instance, the set $\{x \in \mathbb{R}; 0 < x < 1\}$ is open in $\mathbb{R}$; but the set $\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, x_2 = 0\}$ is *not* open in $\mathbb{R}^2$, although, graphically, the two sets "look the same".

Complement of a Set:

If $S \subset \mathbb{R}^n$, the *complement* of S (in $\mathbb{R}^n$) is denoted by $\sim S$, and defined by $\sim S = \{x$ in $\mathbb{R}^n : x$ is not in $S\}$

Closed Set:

A set $S \subset \mathbb{R}^n$ is *closed* in $\mathbb{R}^n$ if the complement of S in $\mathbb{R}^n$ is open in $\mathbb{R}^n$.

You can check that the set $S = \{(x_1, x_2)$ in $\mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0$, and $x_1^2 + x_2^2 \leq 1\}$ is a closed set in $\mathbb{R}^2$.

There are many sets which are *neither* open *nor* closed in $\mathbb{R}^n$. For example, the set $S = \{(x_1, x_2)$ in $\mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0$, and $x_1^2 + x_2^2 < 1\}$ is neither open nor closed in $\mathbb{R}^2$.

If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then one of three possibilities must hold:

(1) There is an open ball $B(x, r)$ such that $B(x, r) \subset A$.

(2) There is an open ball $B(x, r)$ such that $B(x, r) \subset \sim A$.

(3) If $B(x, r)$ is any open ball, then $B(x, r)$ contains points of both A and $\sim A$.

Those points $x \in \mathbb{R}^n$ which satisfy (1) constitute the *interior* of A; those satisfying (2) the *exterior* of A; those satisfying (3) the *boundary* of A. Points in these sets are called, respectively, interior, exterior and boundary points (with respect to the set A).

Neighborhood:

If $x \in \mathbb{R}^n$, any set which contains an open set containing x is called a *neighborhood* of x, and is denoted by $N(x)$.

Thus, an open ball in $\mathbb{R}^n$ with center x and radius $r > 0$ is a neighborhood of x.

Empty Set:

A set $\phi \subset \mathbb{R}^n$ which contains no elements of $\mathbb{R}^n$ is called the *empty set*.

You can show that $\mathbb{R}^n$ and ϕ are the only two sets which are both open and closed in $\mathbb{R}^n$.

5.3 Convergent Sequences

Let $x^1, x^2, x^3, ...$ be a sequence of vectors in $\mathbb{R}^n$. A vector x in $\mathbb{R}^n$ is called a *limit* of the sequence $x^1, x^2, x^3, ...$if given any real number $\varepsilon > 0$, there is a positive integer N such that $d(x^s, x) < \varepsilon$ whenever $s > N$. If the sequence $x^1, x^2, x^3, ...$ has a limit, we call the sequence *convergent*. If x is a limit of the sequence we say that *the sequence converges* to x.

For example, the sequence of numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ is convergent, with limit equal to zero.

You can check that if a sequence $x^1, x^2, x^3, ...$ is convergent, it has a *unique* limit; so it makes sense to speak of *the* limit of a convergent sequence.

An important result on convergent sequences is that "weak inequalities are preserved in the limit".

Proposition 2. *Suppose $\{x^s\}_1^\infty$ is a convergent sequence of points in $\mathbb{R}^n$ with limit $x \in \mathbb{R}^n$, and let $a \in \mathbb{R}^n$. If $x^s \leq a$ for all s, then $x \leq a$.*

Remark: You should be able to prove, by using Proposition 1, that if $b \in \mathbb{R}^n$, and $\{y^s\}_1^\infty$ is convergent of points in $\mathbb{R}^n$ with limit $y \in \mathbb{R}^n$, and $y^s \geq b$ for all s, then $y \geq b$.

Using the notion of convergent sequence, we can obtain a more "usable" characterization of closed sets.

Theorem 11. *Let $S \subset \mathbb{R}^n$. Then S is closed in $\mathbb{R}^n$ iff whenever $x^1, x^2, ...$ is a sequence of points of S that is convergent in $\mathbb{R}^n$, we have*

$$\lim_{n\to\infty} x^n \in S.$$

Proposition 2 combined with Theorem 11 gives us a convenient way of checking that the typical constraint sets which arise in optimization problems are closed. For example, you can check that the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 2\}$ is a closed set by using Proposition 2 and Theorem 11.

5.4 Compact Sets

Bounded Set:

A set $S \subset \mathbb{R}^n$ is *bounded* if it is contained in some open ball in $\mathbb{R}^n$.

For example, the set $S = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\}$ is a bounded set, because it is contained in the open ball in $\mathbb{R}^2$ with center 0 and radius 2.

A set $S \subset \mathbb{R}^n$ is *compact* if it is both closed and bounded.

You can use the above definition to check that

(i) the set $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 2\}$ is a compact set;

(ii) the set $S_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 < 1 \text{ and } 0 \leq x_2 < 1\}$ is not a compact set, because it is not a closed set;

(iii) the set $S_3 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2)$ is not a compact set, because it is not a bounded set.

5.5 Continuous Functions

Functions:

Let $A \subset \mathbb{R}^n$. A *function*, f, from A to $\mathbb{R}^m$ (written $f : A \to \mathbb{R}^m$) is a rule which associates with each point in A a unique point in $\mathbb{R}^m$. In this case A is called the *domain* of f. We define $f(A) = \{y \in \mathbb{R}^m : y = f(x) \text{ for some } x \in A\}$. For example, if $f(x) = x^2$ for all $x \in \mathbb{R}$, then $A \equiv \mathbb{R}$, and $f(A) = \mathbb{R}_+$. In the special case where $m = 1$, f is called a *real valued* function.

If $f : A \to \mathbb{R}^m$ is a function, we can define $f^1(x)$ as the first component of the vector $f(x)$ for each $x \in A$. Then f^1 is a function from A to $\mathbb{R}$. Similarly, $f^2, \ldots, f^m$ can be defined. These functions $f^1, \ldots., f^m$ are called the *component functions* of f.

Conversely, if $g^1, \ldots, g^m$ are m real valued functions on A, we can define $g(x) = (g^1(x), \ldots, g^m(x))$ for each $x \in A$. Then g is a function from A to $\mathbb{R}^m$.

Limit of a Function:

If $f : A \to \mathbb{R}$, and $a \in A$, then

$$\lim_{x \to a} f(x) = b$$

means that given any $\varepsilon > 0$, there is a number $\delta > 0$, such that if $x \in A$, and $0 < d(x,a) < \delta$, then $|f(x) - b| < \varepsilon$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 3$, then $\lim_{x \to 2} f(x) = 1$.

Continuity of a Function:

A function $f : A \to \mathbb{R}$ is called *continuous at* $a \in A$, if $\lim_{x \to a} f(x) = f(a)$. The function f is *continuous* (on A) if it is continuous at each $x \in A$.

For example, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $f(x) = x + 1$, then f is continuous at 0. If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $g(x) = x + 1$ for $x \neq 0$, and $g(0) = 0$, then g is not continuous at 0.

5.6 Existence of Solutions to Constrained Optimization Problems

The most important result to decide whether or not a constrained optimization problem has a solution is known as Weierstrass theorem, and can be stated as follows:

Theorem 12. *(Weierstrass) Suppose A is a non-empty closed and bounded subset of* $\mathbb{R}^n$. *If* $f : A \to \mathbb{R}$ *is continuous on A, then there exist* x^1, x^2 *in A such that* $f(x) \leq f(x^1)$ *for all* $x \in A$ *and* $f(x) \geq f(x^2)$ *for all* $x \in A$.

5.7 Appendix I: Closed Sets

Result:

Let $f : \mathbb{R}^m_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^m_+$. Then the set:

$$C = \{x \in \mathbb{R}^m_+ : f(x) \geq 0\} \tag{1}$$

is closed in $\mathbb{R}^m$.

Proof:

Take an arbitrary convergent sequence $\{x^n\}$ of points in C, with limit $z \in \mathbb{R}^m$. That is, $x^1, x^2, x^3, \ldots.$ belong to C, and:

$$\lim_{n\to\infty} x^n = z \tag{2}$$

We have to show that $z \in C$.

Since $x^n \in C$ for each $n = 1,2,3,\ldots$, we have $x^n \geq 0$ for each $n = 1,2,3,\cdots$. Since (2) holds, and weak inequalities are preserved in the limit, we have $z \geq 0$; that is, $z \in \mathbb{R}^m_+$.

Since f is continuous on $\mathbb{R}^m_+$, given any $\varepsilon > 0$, there is $\delta > 0$, such that whenever $d(x,z) < \delta$, we have:

$$|f(x) - f(z)| < \varepsilon \tag{3}$$

Using this $\delta > 0$, we can find a positive integer N, such that whenever $n > N$, we have $d(x^n, z) < \delta$, since (2) holds. Thus, using (3), for all $n > N$, we must have $|f(x^n) - f(z)| < \varepsilon$. This implies that the sequence $\{f(x^n)\}$ is convergent with limit $f(z)$.

Since $x^n \in C$ for each $n = 1,2,3,\ldots$, we have $f(x^n) \geq 0$ for each $n = 1,2,3,\cdots$. Since $\{f(x^n)\}$ is convergent with limit $f(z)$, and weak inequalities are preserved in the limit, we have $f(z) \geq 0$.

We have now shown that $z \in \mathbb{R}^m_+$, and $f(z) \geq 0$, so $z \in C$.

Exercise 1:

Let $p \in \mathbb{R}^m_{++}$, $w > 0$ and consider the budget set:

$$C = \{x \in \mathbb{R}^m_+ : px \leq w\}$$

Show that C is closed in $\mathbb{R}^m$, by defining $f(x) = w - px$ for all $x \in \mathbb{R}^m_+$, checking the continuity of f on $\mathbb{R}^m_+$, and then using the above result.

Exercise 2:

Let $u : \mathbb{R}^m_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^m_+$, and let v be a real number in the range of the function u. Consider the upper contour set:

$$C = \{x \in \mathbb{R}^m_+ : u(x) \geq v\}$$

Show that C is a closed set in $\mathbb{R}^m$, by defining $f(x) = u(x) - v$, checking the continuity of f on $\mathbb{R}^m_+$, and then using the above result.

Exercise 3:
Let g_j be a continuous function from $\mathbb{R}^m_+$ to $\mathbb{R}$, for each $j \in \{1, \ldots, k\}$. Consider the set:

$$D = \{x \in \mathbb{R}^m_+ : g_j(x) \geq 0 \text{ for each } j \in \{1, \ldots, k\}\}$$

Show, by generalizing the argument used in establishing the above result, that D is a closed set in $\mathbb{R}^m$.

Sets like D represent many of the constraint sets encountered in modern optimization theory.

5.8 Appendix II: Continuity of Functions of Several Variables

The sketch of continuity of the function:

$$f(x_1, x_2) = x_1 x_2 \text{ for all } (x_1, x_2) \in \mathbb{R}^2$$

contains a more general idea, which is quite useful in checking continuity of some commonly encountered functions of several variables. This general idea is developed below.

(a) Let A be a subset of $\mathbb{R}^n$, and let f and g be functions from A to $\mathbb{R}$. Define $h : A \to \mathbb{R}$ by:

$$h(x) = f(x)g(x) \text{ for all } x \in A$$

The general result is the following: If f and g are continuous functions on A, then h is a continuous function on A.

To prove this result, let z be an arbitrary point in A, and let $\varepsilon > 0$ be given. We have to show that whenever $x \in A$ and $d(x, z) < \delta$, we have:

$$|h(x) - h(z)| < \varepsilon$$

Define:

$$\varepsilon' = \min\left\{1, \frac{\varepsilon}{1 + |f(z)| + |g(z)|}\right\} \tag{1}$$

Then $0 < \varepsilon' \leq 1$. Since f and g are continuous at z, given the $\varepsilon' > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever $x \in A$ and $d(x, z) < \delta_1$, we have:

$$|f(x) - f(z)| < \varepsilon' \tag{2}$$

and whenever $x \in A$ and $d(x, z) < \delta_2$, we have:

$$|g(x) - g(z)| < \varepsilon' \tag{3}$$

Define:

$$\delta = \min\{\delta_1, \delta_2\} \tag{4}$$

Then, clearly $\delta > 0$.

We can write for $x \in A$,

$$\begin{aligned} |h(x) - h(z)| &= |f(x)g(x) - f(z)g(z)| \\ &= |f(x)g(x) - f(x)g(z) + f(x)g(z) - f(z)g(z)| \\ &\leq |f(x)(g(x) - g(z))| + |(f(x) - f(z))g(z)| \\ &= |f(x)||(g(x) - g(z))| + |(f(x) - f(z))||g(z)| \end{aligned} \tag{5}$$

Since $x \in A$ and $d(x,z) < \delta = \min\{\delta_1, \delta_2\}$, we know that both (2) and (3) must hold. Using this in (5), we obtain:

$$\begin{aligned} |h(x) - h(z)| &\leq |f(x)||(g(x) - g(z))| + |(f(x) - f(z))||g(z)| \\ &< [1 + |f(z)|]\varepsilon' + \varepsilon'|g(z)| \leq \varepsilon \end{aligned} \tag{6}$$

the inequalities on the last line of (6) following from (1). This establishes that h is continuous at z.

Remark:

Note that, given z and ε, the appropriate definitions of ε' and of δ are suggested by (5). So, (5) is really the first step in the proof, although it appears later in the formal proof, compared to the appearance of ε' and δ.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x_1, x_2) = x_1 \tag{7}$$

Then f is continuous on $\mathbb{R}^2$, since given any $z \in \mathbb{R}^2$, and $\varepsilon > 0$, we can choose $\delta = \varepsilon$, and note that whenever $x \in \mathbb{R}^2$ and $d(x,z) < \delta = \varepsilon$, we must have $|x_1 - z_1| < \varepsilon$, and so $|f(x_1,x_2) - f(z_1,z_2)| < \varepsilon$.

(c) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$g(x_1, x_2) = x_2 \tag{8}$$

Then g is continuous on $\mathbb{R}^2$, since given any $z \in \mathbb{R}^2$, and $\varepsilon > 0$, we can choose $\delta = \varepsilon$, and note that whenever $x \in \mathbb{R}^2$ and $d(x,z) < \delta = \varepsilon$, we must have $|x_2 - z_2| < \varepsilon$, and so $|g(x_1,x_2) - g(z_1,z_2)| < \varepsilon$.

(d) Define $h : \mathbb{R}^2 \to \mathbb{R}$ by:

$$h(x_1, x_2) = x_1 x_2 \text{ for all } (x_1, x_2) \in \mathbb{R}^2$$

Then, we have:

$$h(x_1, x_2) = f(x_1, x_2) g(x_1, x_2) \text{ for all } (x_1, x_2) \in \mathbb{R}^2$$

where f and g are defined by (7) and (8) respectively. Then, using the results in (a),(b) and (c), h is continuous on $\mathbb{R}^2$.

5.9 Appendix III: On a Variation of Weierstrass Theorem

Theorem:

Let A be a non-empty subset of $\mathbb{R}^n$, and let f be a continuous function from A to $\mathbb{R}$. Suppose B is a non-empty, closed and bounded set in $\mathbb{R}^n$, and z is an element of B, such that B is a subset of A, and:

$$f(x) \leq f(z) \quad for\, all\, x \in A \sim B \tag{1}$$

Then there is $\bar{x} \in A$ such that:

$$f(x) \leq f(\bar{x}) \quad for\, all\, x \in A \tag{2}$$

Proof:

Since B is a subset of A and f is a continuous function from A to $\mathbb{R}$, f is continuous on B. Since B is a non-empty, closed and bounded set in $\mathbb{R}^n$, we can apply Weierstrass theorem to infer that there is $\bar{x} \in B$ such that:

$$f(x) \leq f(\bar{x}) \quad for\, all\, x \in B \tag{3}$$

Let x be an arbitrary element of A. There are two cases to consider: (i) $x \in B$, (ii) $x \notin B$. In case (i), we have:

$$f(x) \leq f(\bar{x}) \tag{4}$$

by (3).

In case (ii), we have $x \in A \sim B$, and so:

$$f(x) \leq f(z) \tag{5}$$

by (1). Also, since $z \in B$, we have:

$$f(z) \leq f(\bar{x}) \tag{6}$$

by (3). Combining (5) and (6), we obtain:

$$f(x) \leq f(\bar{x}) \tag{7}$$

Thus, we have shown that in either case, $f(x) \leq f(\bar{x})$ must hold. Since $\bar{x} \in B \subset A$, we know that $\bar{x} \in A$, and this establishes (2).//

Remark:

Note that in the statement of the theorem, the set A is assumed to be neither closed nor bounded. However, the theorem ensures that there is a solution to the constrained maximization problem:

$$\left.\begin{array}{ll} Maximize & f(x) \\ subject\ to & x \in A \end{array}\right\} (P)$$

This variation of Weierstrass theorem is often useful in applications.

It is important to realize that the constrained maximization problem you will be given will be in the form of (P); that is, you will be given A and f. You are **not** given B and $z \in B$. Thus, in order to apply the theorem to ensure that there exists a solution to (P), **you** have to define an appropriate set B and an element $z \in B$ with the properties stated in the theorem.

Example:

Let $A = \mathbb{R}_+$ and let f be a continuous function on $\mathbb{R}_+$, satisfying $f(0) = 0$ and $f(x) \leq 0$ for all $x > 1$. Show that there is a solution to problem (P) by applying the above theorem. How would you choose B and $z \in B$ to apply the above theorem?

Converse of "Extension of Weierstrass Theorem "

Following example establishes the necessity of the Extension of Weierstrass Theorem.

Example:

Let A be a non-empty subset of $\mathbb{R}^n$, and let f be a continuous function from A to $\mathbb{R}$. Suppose there exists $\overline{x} \in A$ such that:

$$f(\overline{x}) \geq f(x) \quad \text{for all} \quad x \in A. \tag{8}$$

Then there exists a non-empty, closed and bounded set, B, in $\mathbb{R}^n$, with $B \subset A$, and there is $z \in B$, such that

$$f(z) \geq f(x) \quad \text{for all} \quad x \in A \quad \text{satisfying} \quad x \notin B \tag{9}$$

Solution.

Note, intuitively, even though a closed ball may seem to be a candiadte for set B, we do not know much about the set A, except that it is non-empty and has a global maximum. Thus it could be possible that the set A may not contain a closed ball at all. In the extreme case, A could be a singleton containing $\overline{x}$. Then our only choice would be to let $B \equiv \{\overline{x}\}, z = \overline{x}$. We will verify now that this choice is suitable in the general case also for any given A.

Let $B \equiv \{\overline{x}\}, z = \overline{x}$. Set B is bounded since any open ball with center $\overline{x}$ and a positive radius contains it. B is also closed since if we pick a convergent sequence from B, all the points in this sequence must be $\overline{x}$, and terefore convergent to $\overline{x}$. Set $B \subset A$ and $f(z) \geq f(x)$ for all $x \in A$, $x \notin B$ since $\overline{x}$ is a global maximum of f on A.

Thus we have proved existence of B and z by construction.

5.10 Worked Out Problems on Chapter 5

Problem 21 (Open Sets)**.**

(a) Let S and T be open sets in $\mathbb{R}^n$. Show that $S \cap T$ is also an open set in $\mathbb{R}^n$.
(b) Let A be the set defined by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, x_1 x_2 > 1\}$$

Express A as the intersection of two sets, and use (a) to show that A is open in $\mathbb{R}^2$.
(c) Let B be the set defined by:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 x_2 > 1\}$$

Is B open in $\mathbb{R}^2$? Explain.

Solution.

(a) Consider an arbitrary $\bar{x} \in S \cap T$. Now, since $\bar{x} \in S$ and S is open in $\mathbb{R}^n$, there exists $r_S > 0$ such that $B(\bar{x}, r_S) \subset S$. And since $\bar{x} \in T$ and T is open in $\mathbb{R}^n$, there exists $r_T > 0$ such that $B(\bar{x}, r_T) \subset T$. Take $r = \min\{r_S, r_T\} > 0$. Then $B(\bar{x}, r) \subset B(\bar{x}, r_S) \subset S$ and $B(\bar{x}, r) \subset B(\bar{x}, r_T) \subset T$. Therefore $B(\bar{x}, r) \subset S \cap T$, so $S \cap T$ is open in $\mathbb{R}^n$.

(b) We can write $A = A_1 \cap A_2$ where

$$A_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}$$
$$A_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 > 1\}$$

Note that A_2 contains some points where $x_1 < 0$ and $x_2 < 0$. We want to show that both A_1 and A_2 are open in $\mathbb{R}^2$, which will show by part (a) that A is open in $\mathbb{R}^2$.

Claim: A_1 is open in $\mathbb{R}^2$.

Proof: Let $\bar{x} \in A_1$ and take $r = \min\{\bar{x}_1, \bar{x}_2\} > 0$. We want to show $B(\bar{x}, r) \subset A_1$. This is the same as showing that for any $x \notin A_1$, we have $x \notin B(\bar{x}, r)$. So, consider some $x \notin A_1$ and assume without loss of generality that $x_1 \leq 0$. Then we have that $x_1 \leq 0 < r \leq \bar{x}_1$, so $\bar{x}_1 - x_1 \geq r$. That means that $d(x, \bar{x}) = \sqrt{(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2} \geq r$, so we have $x \notin B(\bar{x}, r)$, which is what we wanted to show. Therefore A_1 is open.

Claim: A_2 is open in $\mathbb{R}^2$.

To show that A_2 is open in $\mathbb{R}^2$, it is easiest to show that $\sim A_2$ is closed in $\mathbb{R}^2$. First, define the function $f(x) = 1 - x_1 x_2$ for all $x \in \mathbb{R}^2$ and note that f is continuous on $\mathbb{R}^2$. Then we can write $\sim A_2$ as

$$\sim A_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 \leq 1\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x) \geq 0\}$$

By Prof. Mitra's result on closed sets, then, $\sim A_2$ is closed in $\mathbb{R}^2$. Therefore A_2 is open in $\mathbb{R}^2$.

(c) We can write

$$\begin{aligned} B &= A \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 > 0, x_1 x_2 > 1\} \\ &\quad \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 = 0, x_1 x_2 > 1\} \\ &\quad \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 = 0, x_1 x_2 > 1\} \\ &= A \cup \varnothing \cup \varnothing \cup \varnothing \\ &= A \end{aligned}$$

Since A is open in $\mathbb{R}^2$ and $B = A$, B is open in $\mathbb{R}^2$.

Problem 22 (Closed Sets)**.**

(a) Let S and T be open sets in $\mathbb{R}^n$. Show that $S \cup T$ is also an open set in $\mathbb{R}^n$.
(b) Let A be the set defined by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$$

and let B be the complement of A in $\mathbb{R}^2$. Express B as the union of three sets and show that B is open in $\mathbb{R}^2$.
(c) Show that A is closed in $\mathbb{R}^2$.

Solution.

(a) Consider an arbitrary $\bar{x} \in S \cup T$. Without loss of generality, assume $\bar{x} \in S$. Now, since $\bar{x} \in S$ and S is open in $\mathbb{R}^n$, there exists $r > 0$ such that $B(\bar{x}, r) \subset S$. But $S \subset S \cup T$, so $B(\bar{x}, r) \subset S \cup T$. Therefore $S \cup T$ is open in $\mathbb{R}^n$.

(b) We can write $\sim A = B = B_1 \cup B_2 \cup B_3$ where

$$\begin{aligned} B_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0\} \\ B_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 < 0\} \\ B_3 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 > 1\} \end{aligned}$$

We want to show that each of B_1, B_2, and B_3 are open in $\mathbb{R}^2$, which will show by part (a) that B is open in $\mathbb{R}^2$.

Claim: B_1 is open in $\mathbb{R}^2$.

Proof: Let $\bar{x} \in B_1$ and take $r = -\bar{x}_1 > 0$. We want to show that $B(\bar{x}, r) \subset B_1$. This is the same as showing that for any $x \notin B_1$, we have $x \notin B(\bar{x}, r)$. So, consider some $x \notin B_1$. Then we have that $x_1 \geq 0$, so $x_1 - \bar{x}_1 = x_1 + r \geq r$. That means that $d(x, \bar{x}) = \sqrt{(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2} \geq r$, so we have $x \notin B(\bar{x}, r)$, which is what we wanted to show. Therefore B_1 is open.

Claim: B_2 is open in $\mathbb{R}^2$.

Proof: Follow the steps from the proof above for the set B_2 instead of B_1.

Claim: B_3 is open in $\mathbb{R}^2$.

To show that B_3 is open in $\mathbb{R}^2$, it is easiest to show that $\sim B_3$ is closed in $\mathbb{R}^2$. First, define the function $f(x) = 1 - x_1 - x_2$ for all $x \in \mathbb{R}^2$ and note that f is continuous on $\mathbb{R}^2$. Then we can write $\sim B_3$ as

$$\sim B_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x) \geq 0\}$$

By Prof. Mitra's result on closed sets, then, $\sim B_3$ is closed in $\mathbb{R}^2$. Therefore B_3 is open in $\mathbb{R}^2$.

(c) We know that $B = \sim A$. Since B is open, by the definition of closed sets A is closed in $\mathbb{R}^2$.

Problem 23 (Continuity of Functions)**.**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n$, and let $\bar{x}$ be a vector in $\mathbb{R}^n$, satisfying $f(\bar{x}) > 0$. Show that there is a positive real number r such that:

$$f(x) > 0 \text{ for all } x \in B(\bar{x}, r)$$

Solution.

Let $\varepsilon = f(\bar{x}) > 0$. Now, since f is continuous on $\mathbb{R}^n$, we know that given this $\varepsilon = f(\bar{x}) > 0$ and any $x \in \mathbb{R}^n$, there is some $\delta > 0$ such that whenever $d(x, \bar{x}) < \delta$, we have $|f(x) - f(\bar{x})| < f(\bar{x})$. This implies that $f(x) - f(\bar{x}) < f(\bar{x})$ and $f(x) - f(\bar{x}) > -f(\bar{x})$, which we can rearrange for $0 < f(x) < 2f(\bar{x})$. Now, take $r = \delta$ and note that whenever $x \in \mathbb{R}^n$ and $x \in B(\bar{x}, r)$, we have $d(x, \bar{x}) < \delta$ and thus $f(x) > 0$, which is what we wanted to show.

Problem 24 (Extension of Weierstrass theorem)**.**

Let a and b be positive real numbers, and let f be a function from $\mathbb{R}_+$ to $\mathbb{R}_+$, defined by:

$$f(x) = ax + b[x/(1+x)] \text{ for all } x \geq 0$$

Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \textit{Maximize} & f(x) - x \\ \textit{subject to} & x \geq 0 \end{array}\right\} (P)$$

(a) If $a < 1$, show that there exists a solution to problem (P).
(b) If $a \geq 1$, show that there is no solution to problem (P).

Solution.

Define $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(x) = f(x) - x = (a-1)x + \frac{bx}{1+x} \quad \text{for all } x \geq 0$$

Then problem (P) is to maximize $g(x)$ subject to $x \geq 0$. The function g is continuous on the nonempty and closed constraint set $C = [0, \infty)$, but we cannot use the Weierstrass Theorem because the constraint set is not bounded.

(a) Note that g is continuously differentiable on $x \geq 0$. The first order condition is

$$a - 1 + \frac{b}{(1+x)^2} = 0 \quad \Longrightarrow \quad (1+x)^2 = \frac{b}{1-a}$$

This suggests using the Extension of Weierstrass Theorem on the bounded set $B = \left[0, \frac{b}{1-a}\right]$. We need to show that whenever $c > 1$, we have $g\left(\frac{b}{1-a}\right) \geq g\left(c\frac{b}{1-a}\right)$.

$$g\left(\frac{b}{1-a}\right) = -b + \frac{\frac{b^2}{1-a}}{1 + \frac{b}{1-a}} = -b + \frac{b^2}{1-a+b} = \frac{ab-b}{1-a+b}$$

$$\begin{aligned} g\left(c\frac{b}{1-a}\right) &= -bc + \frac{\frac{b^2c}{1-a}}{1 + \frac{bc}{1-a}} = -bc + \frac{b^2c}{1-a+bc} = \frac{abc - bc - b^2c^2 + b^2c}{1-a+bc} \\ &= \frac{(ab-b)(c+1-1) - b^2c(c-1)}{1-a+bc} = \frac{(ab-b) + (c-1)(ab-b-b^2c)}{1-a+bc} \\ &= \frac{ab-b-(c-1)b(bc+1-a)}{1-a+bc} \end{aligned}$$

Whenever $c > 1$, the numerator in $g\left(c\frac{b}{1-a}\right)$ is smaller and its denominator is larger, so we have $g\left(\frac{b}{1-a}\right) \geq g\left(c\frac{b}{1-a}\right)$. Therefore we can apply the Extension of Weierstrass Theorem on the nonempty, closed, and bounded set $B = \left[0, \frac{b}{1-a}\right]$ to conclude that problem (P) has a solution.

(b) For all $x \geq 0$, we have that

$$g'(x) = a - 1 + \frac{b}{(1+x)^2} > 0$$

since $a \geq 1$ and $b > 0$. Therefore g is strictly increasing on $x \geq 0$, so problem (P) cannot have a solution. Formally, we can suppose for the sake of contradiction that some $\bar{x} \geq 0$ is a solution to (P). Then define $x' = \bar{x} + 1$ and note that because g is strictly increasing on $x \geq 0$, we have $g(x') > g(\bar{x})$. But this contradicts $\bar{x}$ solving (P), so it must be that problem (P) has no solution.

Problem 25 (Extension of Weierstrass Theorem)**.**

Let p and q be arbitrary positive numbers, and let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^2_+$. Suppose there is $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_+$ which satisfies $f(\bar{x}_1, \bar{x}_2) = 1$. Consider the constrained minimization problem:

$$\left.\begin{array}{ll} Minimize & px_1 + qx_2 \\ subject \quad to & f(x_1,x_2) \geq 1 \\ and & (x_1,x_2) \in \mathbb{R}^2_+ \end{array}\right\} (Q)$$

Show that there is a solution to problem (Q).

Solution.

First, define $g : \mathbb{R}^2_+ \to \mathbb{R}$ by $g(x_1, x_2) = f(x_1, x_2) - 1$ for all $x \in \mathbb{R}^2_+$. Since f is continuous on $\mathbb{R}^2_+$, g is also continuous on $\mathbb{R}^2_+$. Then by Prof. Mitra's result on closed sets, the constraint set

$$C = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, g(x_1,x_2) \geq 0\}$$

is closed in $\mathbb{R}^2$. The objective function is continuous on C and C is nonempty because $(\bar{x}_1, \bar{x}_2) \in C$. But, depending on f, C is not necessarily bounded. To use the Extension of Weierstrass Theorem, define the set

$$B = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, f(x_1,x_2) \geq 1, px_1 + qx_2 \leq p\bar{x}_1 + q\bar{x}_2\}$$

We could show that B is closed by showing that B is the intersection of two closed sets. Also, B is nonempty since $(\bar{x}_1, \bar{x}_2) \in B$ and B is bounded because it is contained in the open ball $B(0,r)$ where $r = \max\left\{\frac{p\bar{x}_1 + q\bar{x}_2}{p}, \frac{p\bar{x}_1 + q\bar{x}_2}{q}\right\} + 1$. Finally, for any $x' \in C, x' \notin B$, we have $px'_1 + qx'_2 > p\bar{x}_1 + q\bar{x}_2$. Since (Q) is a minimization problem, it cannot be solved by any $x' \in C, x' \notin B$. Using B, then, the Extension of Weierstrass Theorem guarantees a solution to (Q).

References:

This material is standard in many texts on Real Analysis. You might consult *Principles of Mathematical Analysis* by *W. Rudin*(Chapters 2, 3, 4) or *Introduction to Analysis* by *M. Rosenlicht* (Chapters 3, 4). Some of the material is also covered in *Calculus on Manifolds* by *M. Spivak* (Chapter 1).

Chapter 6

Differential Calculus

6.1 Partial Derivatives

Let A be an open set in $\mathbb{R}^n$, and let $x \in A$. If $f : A \to \mathbb{R}$, the limit

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

if it exists, is called the i^{th} (first-order) *partial derivative* of f at x, and is denoted by $D_i f(x)$, where $i = 1, \ldots, n$. This means, of course, that we can compute partial derivatives just like ordinary derivatives of a function of *one* variable. That is, if $f(x_1, \ldots, x_n)$ is given by some formula involving $(x_1, \ldots, x_n)$, then we find $D_i f(x)$ by differentiating the function whose value at x_i is given by the formula when all x_j (for $j \neq i$) are thought of as constants.

For example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x_1, x_2) = x_1^3 + 3x_2^3 + 2x_1x_2$, then $D_1 f(x) = 3x_1^2 + 2x_2$, $D_2 f(x) = 9x_2^2 + 2x_1$.

The i^{th} partial derivative of f at x is also, alternatively, denoted by $f_i(x)$. When the i^{th} partial derivative of f at x exists for all $i = 1, \ldots, n$, we can define the n-vector of these partial derivatives

$$\nabla f(x) \equiv [D_1 f(x), \ldots, D_n f(x)]$$

This vector is called the *gradient vector* of f at x. In the example described above, the gradient vector is

$$\nabla f(x) \equiv [3x_1^2 + 2x_2,\ 9x_2^2 + 2x_1] \qquad \text{for all } x \in \mathbb{R}^2$$

When $f : A \to \mathbb{R}$ has (first-order) partial derivatives at each $x \in A$, we say that f has (first-order) *partial derivatives on A*.

Second-Order Partial Derivatives and the Hessian Matrix

When $f : A \to \mathbb{R}$ has (first-order) partial derivatives on A, these first-order partial derivatives are themselves functions from A to $\mathbb{R}$. If these (first-order) partial derivatives are continuous on A, then we say that f is *continuously differentiable* on A. If these functions have (first-order) partial derivatives on A, *these* partial derivatives are called the *second-order partial derivatives* of f on A.

To elaborate, if $D_i f(x)$ exists for all $x \in A$, we can define the function $D_i f : A \to \mathbb{R}$. If this function has (first-order) partial derivatives on A, then the j^{th} (first-order) partial derivative of $D_i f$ at x [that is, $D_j(D_i f(x))$] is a second-order partial derivative of f at x, and is denoted by $D_{ij} f(x)$. [Here $i = 1,\ldots,n$ and $j = 1,\ldots,n$].

In the example described above, $D_{11} f(x) = 6x_1$; $D_{12} f(x) = 2 = D_{21} f(x)$; $D_{22} f(x) = 18x_2$. We note in this example that the "cross partials" $D_{12} f(x)$ and $D_{21} f(x)$ are equal. This is not a coincidence; it is a more general phenomenon as noted in the following result, known as "Young's theorem".

Theorem 13. *(Young) Suppose A is an open set in $\mathbb{R}^n$, and f has first and second-order partial derivatives on A. If $D_{ij} f$ and $D_{ji} f$ are continuous on A, then $D_{ij} f(x) = D_{ji} f(x)$ for all $x \in A$*

When all of the hypotheses of Theorem 1 hold, we will say that f is *twice continuously differentiable* on A; this will be the typical situation in many applications.

When the first and second-order partial derivatives of $f : A \to \mathbb{R}$ exist on A, the $n \times n$ matrix of second-order partial derivatives of f described below:

$$H_f(x) = \begin{bmatrix} D_{11} f(x) & D_{12} f(x) \ldots D_{1n} f(x) \\ \text{------------------} & \\ D_{n1} f(x) & D_{n2} f(x) \ldots D_{nn} f(x) \end{bmatrix}$$

is called the *Hessian matrix* of f at $x \in A$, and is denoted by $H_f(x)$. When f is twice continuously differentible on A, the Hessian matrix of f is *symmetric* at all $x \in A$.

In the example described above

$$H_f(x) = \begin{bmatrix} 6x_1 & 2 \\ 2 & 18x_2 \end{bmatrix}$$

is the Hessian matrix of f for all $(x_1, x_2) \in \mathbb{R}^2$.

6.2 Composite Functions and the Chain Rule

Let $g : A \to \mathbb{R}^m$ be a function with component functions $g^i : A \to \mathbb{R}(i = 1,\ldots,m)$ which are defined on an open set $A \subset \mathbb{R}^n$. Let $f : B \to \mathbb{R}$ be a function defined on an open set

$B \subset \mathbb{R}^m$ which contains the set $g(A)$. Then, we can define $F : A \to \mathbb{R}$ by $F(x) \equiv f[g(x)] \equiv f[g^1(x), \ldots, g^m(x)]$ for each $x \in A$. This function is known as a *composite function* [of f and g].

The "Chain Rule" of differentiation provides us with a formula for finding the partial derivatives of a composite function, F, in terms of the partial derivatives of the individual functions, f and g.

Theorem 14. *(Chain Rule) Let $g : A \to \mathbb{R}^m$ be a function with component functions $g^i : A \to \mathbb{R}(i = 1, \ldots, m)$ which are continuously differentiable on an open set $A \subset \mathbb{R}^n$. Let $f : B \to \mathbb{R}$ be a continuously differentiable function on an open set $B \subset \mathbb{R}^m$ which contains the set $g(A)$. If $F : A \to \mathbb{R}$ is defined by $F(x) = f[g(x)]$ on A, and $a \in A$, then F is differentiable at a and we have for $i = 1, \ldots, n$,*

$$D_i F(a) = \sum_{j=1}^{m} D_j f(g^1(a), \ldots, g^m(a)) D_i g^j(a)$$

Examples:
(i) Here $m = 2$, $n = 1$. Let $g^1(x) = x^2$ on $\mathbb{R}$, and $g^2(x) = 1 + x$ on $\mathbb{R}$; let $f(y_1, y_2) = y_1 + y_2^2$ on $\mathbb{R}^2$. Then $F(x) = f[g(x)] = f[g^1(x), g^2(x)] = g^1(x) + [g^2(x)]^2 = x^2 + (1+x)^2$ is a composite function on $\mathbb{R}$. If $a \in \mathbb{R}$,

$$\begin{aligned} F'(a) &= D_1F(a) = D_1 f(g^1(a), g^2(a)) \cdot D_1 g^1(a) + D_2 f(g^1(a), g^2(a)) \cdot D_1 g^2(a) \\ &= 2a + 2g^2(a) = 2a + 2(1+a) \end{aligned}$$

(ii) Here $m = 1$, $n = 2$. Let $g^1(x) = g^1(x_1, x_2) = x_1^2 + x_2$ on $\mathbb{R}^2$; $f(y) = 4y$ on $\mathbb{R}$. Then $F(x) = F(x_1, x_2) = f[g^1(x_1, x_2)] = 4[x_1^2 + x_2]$. Then if $a \in \mathbb{R}^2$,

$$D_1 F(a) = D_1 f[g^1(a_1, a_2)] D_1 g^1(a_1, a_2)$$

$$D_2 F(a) = D_1 f[g^1(a_1, a_2)] D_2 g^1(a_1, a_2)$$

Thus, $D_1 F(a) = 4(2a_1)$; $D_2 F(a) = 4(1)$.

6.3 Homogeneous Functions and Euler's Theorem

A function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is *homogeneous of degree r* on $\mathbb{R}^n_+$ if for all x in $\mathbb{R}^n_+$, and all $t > 0$,

$$f(tx) = t^r f(x)$$

Consider $f:\mathbb{R}^2_+\to\mathbb{R}$ given by $f(x_1,x_2)=x_1^a x_2^b$ where $a>0$ and $b>0$. Then if $t>0$, we have $f(tx_1,tx_2)=(tx_1)^a(tx_2)^b=t^{a+b}x_1^a x_2^b=t^{a+b}f(x_1,x_2)$. So, f is homogeneous of degree $(a+b)$.

We can calculate the partial derivatives of f on $\mathbb{R}^2_{++}$. Thus, $D_1f(x_1,x_2)=ax_1^{a-1}x_2^b$; $D_2f(x_1,x_2)=bx_1^a,x_2^{b-1}$. Now, if $t>0$, then $D_1f(tx_1,tx_2)=a(tx_1)^{a-1}(tx_2)^b=t^{a+b-1}ax_1^{a-1}x_2^b=t^{a+b-1}D_1f(x_1,x_2)$. So D_1f is homogeneous of degree $(a+b-1)$. Similarly, one can check that D_2f is homogeneous of degree $(a+b-1)$. More generally, whenever a function, f, is homogeneous of degree r, its partial derivatives are homogeneous of degree $(r-1)$ (under suitable differentiability assumptions), and this is demonstrated in Theorem 15 below.

We can verify that $x_1D_1f(x_1,x_2)+x_2D_2(x_1,x_2)=ax_1^a x_2^b+bx_1^a x_2^b=(a+b)x_1^a x_2^b=(a+b)f(x_1,x_2)$. More generally, when a function, f, is homogeneous of degree r, then (under suitable differentiability assumptions) $x\nabla f(x)=rf(x)$, a result known as Euler's theorem, which we prove below in Theorem 16.

Theorem 15. *Suppose $f:\mathbb{R}^n_+\to\mathbb{R}$ is homogeneous of degree r on $\mathbb{R}^n_+$, and continuously differentiable on $\mathbb{R}^n_{++}$. Then for each $i=1,\ldots,n$, D_if is homogeneous of degree $(r-1)$ on $\mathbb{R}^n_{++}$.*

To prove this let $t>0$ be given and define the function $g^1,\ldots,g^n$ from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ by

$$g^i(x)=tx_i \qquad i=1,\ldots,n$$

and the composite function F from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ by

$$F(x)=f[g^1(x),\ldots,g^n(x)]=f(tx_1,\ldots.,tx_n)$$

Then applying the Chain Rule, we have for each $i=1,\ldots,n$

$$D_iF(x)=D_if(tx_1,\ldots,tx_n)\cdot t \tag{6.1}$$

But since f is homogeneous of degree r, we have

$$F(x)=t^rf(x_1,\ldots,x_n)$$

So,

$$D_iF(x)=t^rD_if(x_1,\ldots,x_n) \tag{6.2}$$

Using (6.1) and (6.2), we get

$$D_if(tx_1,\ldots,tx_n)=t^{r-1}D_if(x_1,\ldots,x_n) \tag{6.3}$$

since $t>0$.

Theorem 16. *(Euler's Theorem) Suppose $f : \mathbb{R}^n_+ \to \mathbb{R}$ is homogeneous of degree r on $\mathbb{R}^n_+$ and continuously differentiable on $\mathbb{R}^n_{++}$. Then*

$$x \nabla f(x) = r f(x) \quad for\, all\, x \in \mathbb{R}^n_{++}$$

To prove this, let $(x_1, \ldots, x_n)$ be given in $\mathbb{R}^n_{++}$, and define the functions $g^1, \ldots, g^n$ from $\mathbb{R}_{++}$ to $\mathbb{R}_+$ by

$$g^i(t) = tx_i, \qquad i = 1, \ldots, n$$

and the composite function F from $\mathbb{R}_{++}$ to $\mathbb{R}$ by

$$F(t) = f[g^1(t), \ldots, g^n(t)] = f(tx_1, \ldots, tx_n)$$

Then, applying the Chain Rule, we have

$$F'(t) \equiv D_1 F(t) = \sum_{i=1}^{n} D_i f[g^1(t), \ldots, g^n(t)] x_i \tag{6.4}$$

But since f is homogeneous of degree r, we have

$$F(t) = t^r f(x_1, \ldots, x_n)$$

and,

$$F'(t) \equiv D_1 F(t) = r t^{r-1} f(x_1, \ldots, x_n) \tag{6.5}$$

Also, for $i = 1, \ldots., n$ we have

$$D_i f[g^1(t), \ldots, g^n(t)] = D_i f[tx_1, \ldots, tx_n] = t^{r-1} D_i f(x_1, \ldots, x_n) \tag{6.6}$$

by using Theorem 15. Thus, combining (6.4), (6.5) and (6.6),

$$r t^{r-1} f(x_1, \ldots, x_n) = \sum_{i=1}^{n} t^{r-1} D_i f(x_1, \ldots, x_n) x_i \tag{6.7}$$

Cancelling the common term $t^{r-1} > 0$ on both sides of (6.7), we get

$$r f(x_1, \ldots, x_n) = \sum_{i=1}^{n} D_i f(x_1, \ldots, x_n) x_i$$

which is the desired result.

6.4 The Inverse and Implicit Function Theorems

Jacobians:

Suppose A is an open set in $\mathbb{R}^n$, and f is a function from A to $\mathbb{R}^n$, with component functions $f^1, \ldots, f^n$. If $a \in A$, and the partial derivatives of $f^1, \ldots., f^n$ exist at a, then the $n \times n$ matrix

$$Df(a) \equiv (D_j f^i(a))$$

is defined as the *Jacobian matrix* of f at a. The determinant of this matrix, denoted by $J_f(a)$, is defined as the *Jacobian* of f at a.

For example, if $f^1(x_1, x_2) = x_1^2 + 2x_1x_2$ on $\mathbb{R}^2$, and $f^2(x_1, x_2) = x_1 + 3x_2^3$ on $\mathbb{R}^2$, and $a = (a_1, a_2)$ is in $\mathbb{R}^2$, then

$$Df(a) = \begin{bmatrix} 2a_1 + 2a_2 & 2a_1 \\ 1 & 9a_2^2 \end{bmatrix}$$

is the Jacobian matrix at a, and the Jacobian at a is $J_f(a) \equiv [18a_2^3 + 18a_1a_2^2 - 2a_1]$. Note that, typically, the Jacobian matrix is *not* a symmetric matrix (unlike a Hessian matrix).

Inverse Functions:

Let A be a set in $\mathbb{R}^m$, and let f be a function from A to $\mathbb{R}^n$. Then f is *one-to-one* on A if whenever $x^1, x^2 \in A$ and $x^1 \neq x^2$, we have $f(x^1) \neq f(x^2)$. If there is a function g, from $f(A)$ to A, such that $g[f(x)] = x$ for each $x \in A$, then g is called the *inverse function* of f on $f(A)$.

If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2x$, then we note that f is one-to-one on $\mathbb{R}$; also we can define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(y) = (1/2)y$, and note that it has the property $g[f(x)] = x$; g is then the *inverse function* of f on $\mathbb{R}$. Furthermore $g'[f(x)] = 1/f'(x)$ for all $x \in \mathbb{R}$.

More generally, let A be an open set in $\mathbb{R}$, and $f : A \to \mathbb{R}$ be continuously differentiable on A. Let $a \in A$, and suppose that $f'(a) \neq 0$. If $f'(a) > 0$, then there is an open ball $B(a,r)$ such that $f'(x) > 0$ for all x in $B(a,r)$, and f is increasing on $B(a,r)$. Thus, for every $y \in f[B(a,r)]$, there is a *unique* x in $B(a,r)$ such that $f(x) = y$. That is, there is a unique function $g : f[B(a,r)] \to B(a,r)$ such that $g[f(x)] = x$ for all $x \in B(a,r)$. Thus, g is an inverse function of f *on* $f[B(a,r)\}$; we say that g is the inverse of f "locally" around the point $f(a)$. [Notice that there is no guarantee that the inverse function is defined on the entire set $f(A)$]. Similarly, if $f'(a) < 0$, an inverse function could be defined "locally" around $f(a)$. The important restriction to carry out the kind of analysis noted above is that $f'(a) \neq 0$.

To illustrate this, consider $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = x^2$. Consider the point $a = 0$. Clearly f is continuously differentiable on $\mathbb{R}$, but $f'(a) = f'(0) = 0$. Now, we cannot define a unique inverse function of f even "locally" around $f(a)$. That is, choose any open ball $B(0,r)$, and consider any point $y \neq 0$ in the set $f[B(0,r)]$. There will be *two* values x_1, x_2 in $B(0,r), x_1 \neq x_2$, such that $f(x_1) = y$ *and* $f(x_2) = y$.

Of course, $f'(a) \neq 0$ is not a necessary condition to get a unique inverse function of f. For example if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then we have f to be continuously differentible on $\mathbb{R}$, with $f'(0) = 0$. However f is an increasing function, and clearly has a unique inverse function $g(y) = y^{1/3}$ on $\mathbb{R}$, and hence locally around $f(0)$.

Theorem 17. *(Inverse Function Theorem) Let A be an open set of $\mathbb{R}^n$, and $f : A \to \mathbb{R}^n$ be continuously differentiable on A. Suppose $a \in A$ and the Jacobian of f at a is non-zero. Then there is an open set $X \subset A$ containing a, and an open set $Y \subset \mathbb{R}^n$ containing $f(a)$, and a unique function $g : Y \to X$, such that:*

(i) $f(X) = Y$;

(ii) f is one-to-one on X;

(iii) $g(Y) = X$, and $g[f(x)] = x$ for all $x \in X$.

Further, g is continuously differentiable on Y.

Note that (i) and (iii) of Theorem 17 imply that g is an inverse function of f on $f(X)$. There are two preliminary implications of the theorem that are worth noting.

First, g is one-to-one on Y. To see this, let $y^1, y^2 \in Y$, and suppose that $g(y^1) = g(y^2)$. Denote the common vector by x; by (iii) of Theorem 17, $x \in X$. Using (i) of Theorem 17, we can find $x^1, x^2 \in X$ such that $f(x^1) = y^1$ and $f(x^2) = y^2$. By (iii) of Theorem 17,

$$\begin{aligned} x &= g(y^1) = g(f(x^1)) = x^1 \\ x &= g(y^2) = g(f(x^2)) = x^2 \end{aligned}$$

Thus, we must have $x^1 = x^2 = x$, and consequently $y^1 = y^2$. This establishes that g is one-to-one on Y.

Second, we must have:

$$f(g(y)) = y \ \ for\ all\ y \in Y \tag{1}$$

To see this, let y be an arbitrary vector in Y. By (iii) of Theorem 17, $g(y) \in X$, and by (i) of Theorem 17, $f(g(y)) \in Y$. Denote $g(y)$ by x and $f(g(y))$ by z. Then, by (iii) of Theorem 17, we have $g(f(x)) = x$. But, $g(f(x)) = g(f(g(y))) = g(z)$, by definition of x and z. Thus, $g(z) = x$. But, by definition of x, we have $g(y) = x$. Thus, using the fact that g is one-to-one on Y, we must have $y = z$. This establishes (1).

The advantage of noting both the identity in (iii) of Theorem 17 and the identity in display (1) is that, depending on the application, one might use either form to obtain the partial derivatives of the inverse function, g, at $f(a)$.

We illustrate how these partial derivatives can be obtained by using the identity in (iii) of Theorem 17. We can define for $x \in X, F^1(x) = g^1[f(x)]$ as a composite function of f

and g^1. Using the Chain Rule we get:

$$D_i F^1(x) = \sum_{j=1}^{n} D_j g^1[f(x)] D_i f^j(x) \qquad for\ i = 1, \ldots, n$$

But since $F^1(x) = x_1$, we have $D_i F^1(x) = 1$ for $i = 1$, while $D_i F^1(x) = 0$ for $i \neq 1$. We can repeat these calculations with $F^2(x) = g^2[f(x)]$, and get $D_i F^2(x) = 1$ for $i = 2$, while $D_i F^2(x) = 0$ for $i \neq 2$. The results for $F^3(x), \ldots, F^n(x)$ should now be obvious. This information can then be written in familiar matrix multiplication form:

$$I = \begin{bmatrix} D_1 g^1[f(x)] \ldots . D_n g^1[f(x)] \\ ------------ \\ D_1 g^n[f(x)] \ldots . D_n g^n[f(x)] \end{bmatrix} \begin{bmatrix} D_1 f^1(x) \ldots . D_n f^1(x) \\ D_1 f^2(x) \ldots . D_n f^2(x) \\ ------------ \\ D_1 f^n(x) \ldots . D_n f^n(x) \end{bmatrix}$$

That is, $I = Dg[f(x)]\, Df(x)$. Thus, $Df(a)$ is invertible, and we have $Dg[f(a)] = [Df(a)]^{-1}$. This yields, in turn, $J_g[f(a)] = 1/J_f(a)$, since $J_f(a) \neq 0$.

Example:

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (y, x + y^2)$ for $(x,y) \in \mathbb{R}^2$. Let us consider the point $(a,b) = (1,1)$. The Jacobian matrix of f at $(1,1)$ is

$$Df(1,1) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

The Jacobian of f at $(1,1)$ is

$$J_f(1,1) = -1$$

You can check that f is continuously differentiable on $\mathbb{R}^2$. Thus, we can invoke the inverse function theorem and get an open set A containing $(1,1)$, an open set B containing $f(A)$, and a unique function $g : B \to A$, such that g is continuously differentiable on B, and $g[f(x,y)] = (x,y)$ for all (x,y) in A. So $g^1(y, x + y^2) = x$, and $g^2(y, x + y^2) = y$ for all (x,y) in A.

Let $(Z_1, Z_2) \in B$. Then we can define $y = Z_1$, $x = Z_2 - Z_1^2$. Then $g^1(Z_1, Z_2) = g^1(y, x + y^2) = x = Z_2 - Z_1^2$. Thus $g^1(Z_1, Z_2) = Z_2 - Z_1^2$ for (Z_1, Z_2) in B. Similarly we have $g^2(Z_1, Z_2) = g^2(y, x + y^2) = y = Z_1$ for (x_1, x_2) in B. Thus, in this case we can actually *compute* the inverse function. We can calculate the Jacobian matrix of g at $f(1,1) = (1,2)$:

$$Dg[f(1,1)] = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

It can be checked easily that

$$Dg[f(1,1)]\,Df(1,1) = I$$

by simply multiplying the two matrices.

Implicit Functions:

Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + y^2 - 1$. If we choose (a,b) with $f(a,b) = 0$, and $a \neq 1$, $a \neq -1$, there are open intervals $X \subset \mathbb{R}$ containing a, and $Y \subset \mathbb{R}$ containing b, such that if $x \in X$, there is a *unique* $y \in Y$ with $f(x,y) = 0$. Thus, we can define a unique function $g: X \to Y$ such that $f(x,g(x)) = 0$ for all $x \in X$. [If $a > 0$ and $b > 0$, then $g(x) = (1-x^2)^{1/2}$ on X.] Such a function is said to be defined *implicitly* by the equation $f(x,y) = 0$. Note that if $a = 1$, and $b = 0$, so that $f(a,b) = 0$, we *cannot* find such a unique function, g.

The above example can be generalized considerably to obtain a very important result, which is known as the Implicit Function Theorem.

Theorem 18. *(Implicit Function Theorem) Let A be an open set of $\mathbb{R}^{n+m}$, and let $f^1,\ldots,f^m$ be continuously differentiable functions from A to $\mathbb{R}$. Let $(a;b) \in A$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, such that $f^i(a;b) = 0$ for $i = 1,\ldots,m$. Suppose the $m \times m$ matrix $D_{n+j} f^i(a;b)$ $[i = 1,\ldots,m$ and $j = 1,\ldots,m]$ has a non-zero determinant. Then there exists an open set X containing a and an open set Y containing b, and a unique function $g: X \to Y$, such that*

(i) $f(x,g(x)) = 0$ *for all* $x \in X$

(ii) $g(a) = b$

Further, g is continuously differentiable on X.

Example:

Consider the function, $f:\mathbb{R}^2_{++} \to \mathbb{R}$, defined by $f(x,y) = x^\alpha y^\beta - 1$ for $x > 0$, $y > 0$, where α, β are positive constants. Note that at $(a,b) = (1,1)$, we have $f(a,b) = f(1,1) = 0$. We can calculate the (one by one) matrix $D_2 f(a,b) = D_2 f(1,1) = \beta$, and this has a determinant equal to $\beta \neq 0$. Since f is continuously differentiable on $\mathbb{R}^2_{++}$, so we can invoke the implicit function theorem, to obtain an open set A containing a, and an open set B containing b, and a unique function $g: A \to B$, such that $g(a) = b$, and $f(x,g(x)) = 0$ for all $x \in A$. Further, g is continuously differentiable on A. Thus, defining $F: A \to R$ by $F(x) = f(x,g(x))$, we have by the chain-rule

$$F'(x) = D_1 f(x,g(x)) + D_2 f(x,g(x))g'(x)$$

But since $F(x) = 0$ for all $x \in A$, we get

$$0 = D_1 f(x,g(x)) + D_2 f(x,g(x))g'(x)$$

Transposing terms, and evaluating the derivatives at $x = a$,

$$-g'(a) = \frac{D_1 f(a,b)}{D_2 f(a,b)} \text{ since } D_2 f(a,b) \neq 0$$

Thus,

$$-g'(1) = [D_1 f(1,1)/D_2 f(1,1)] = (\alpha/\beta).$$

6.5 Worked Out Problems on Chapter 6

Problem 26 (Converse of Euler's Theorem)**.**

(a) Let $f:\mathbb{R}^n_{++}\to\mathbb{R}_{++}$ be a continuously differentiable function on its domain, which satisfies

$$x\nabla f(x)=f(x)$$

for all $x\in\mathbb{R}^n_{++}$. Show that f is homogeneous of degree one on its domain.

(b) Generalize the result in (a) to provide an appropriate converse of Euler's theorem.

Solution.

The proofs of parts (a) and (b) are very similar, so let's construct and prove the more general result requested in part (b). Then part (a) will follow for the special case where the degree of homogeneity is one.

Claim: Let $f:\mathbb{R}^n_{++}\to\mathbb{R}_{++}$ be a continuously differentiable function on $\mathbb{R}^n_{++}$, and suppose that $x\nabla f(x)=rf(x)$ for all $x\in\mathbb{R}^n_{++}$. Then f is homogeneous of degree r on $\mathbb{R}^n_{++}$.

Proof: Let $\bar{x}$ be an arbitrary vector in $\mathbb{R}^n_{++}$. We want to show that for all $t>0$, we have $f(t\bar{x})=t^r f(\bar{x})$. Given this $\bar{x}\in\mathbb{R}^n_{++}$, define the function $g:\mathbb{R}_{++}\to\mathbb{R}_{++}$ by

$$g(t)=f(t\bar{x}_1,\ldots,t\bar{x}_n)\quad\text{for all } t>0$$

Using the Chain Rule and the hypothesis $x\nabla f(x)=rf(x)$, we have that for all $t>0$,

$$g'(t)=\sum_{i=1}^n D_i f(t\bar{x}_1,\ldots,t\bar{x}_n)\bar{x}_i=\bar{x}\nabla f(t\bar{x})=\frac{r}{t}f(t\bar{x})=\frac{r}{t}g(t)$$

Rearranging, we have that

$$tg'(t)=rg(t)\quad\text{for all } t>0$$

Now, consider $\frac{g(t)}{t^r}$. Differentiating with respect to t, we have that for all $t>0$,

$$\frac{\partial}{\partial t}\left(\frac{g(t)}{t^r}\right)=\frac{g'(t)t^r-g(t)rt^{r-1}}{t^{2r}}=\frac{t^{r-1}}{t^{2r}}\left(tg'(t)-rg(t)\right)=0$$

This implies that for all $t>0$, $\frac{g(t)}{t^r}=c$ for some $c\in\mathbb{R}$. Evaluating at $t=1$, we have that $c=g(1)=f(\bar{x})$, so $\frac{g(t)}{t^r}=f(\bar{x})$. Since $g(t)=f(t\bar{x})$, we have that $f(t\bar{x})=t^r f(\bar{x})$ for all $t>0$, which is what we wanted to show.

Problem 27 (Homothetic Functions)**.**

A function $F:\mathbb{R}^n_+ \to \mathbb{R}_+$ is called a *homothetic function* on $\mathbb{R}^n_+$ if there exists a function $f:\mathbb{R}^n_+ \to \mathbb{R}_+$ which is homogeneous of degree one on $\mathbb{R}^n_+$, and there exists a function $g:\mathbb{R}_+ \to \mathbb{R}_+$ which is an increasing function on $\mathbb{R}_+$, such that $F(x) = g(f(x))$ for all $x \in \mathbb{R}^n_+$.

(a) Let $F:\mathbb{R}^n_+ \to \mathbb{R}_+$ be a function which is homogeneous of degree $r > 0$ on $\mathbb{R}^n_+$. Show that F is a homothetic function on $\mathbb{R}^n_+$.

(b) Let $F:\mathbb{R}^2_+ \to \mathbb{R}_+$ be defined by:

$$F(x_1,x_2) = \frac{x_1^\alpha x_2^{1-\alpha}}{1+x_1^\alpha x_2^{1-\alpha}} \quad \text{for all } (x_1,x_2) \in \mathbb{R}^2_+$$

where $\alpha \in (0,1)$ is a parameter. Verify that F is a homothetic function on $\mathbb{R}^2_+$.

(c) Suppose $f:\mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is a continuously differentiable function on $\mathbb{R}^2_{++}$, which is homogeneous of degree one, and which satisfies $D_1 f(x) > 0$ and $D_2 f(x) > 0$ for all x in $\mathbb{R}^2_{++}$. Suppose $g:\mathbb{R}_{++} \to \mathbb{R}_{++}$ is a continuously differentiable function satisfying $g'(y) > 0$ for all y in $\mathbb{R}_{++}$. Let $F:\mathbb{R}^2_{++} \to \mathbb{R}_{++}$ be defined by $F(x) = g(f(x))$. If t is an arbitrary positive real number, show that

$$[D_1F(tx)/D_2F(tx)] = [D_1F(x)/D_2F(x)]$$

for all x in $\mathbb{R}^2_{++}$.

Solution.

Note that part (a) shows that all homogeneous functions are also homothetic functions, while part (b) shows that a homothetic function is not necessarily homogeneous.

(a) Homogeneity of degree r of F requires $F(tx) = t^r F(x)$, which is equivalent to $g(tf(x)) = t^r g(f(x))$ since f must be homogeneous of degree one. This suggests choosing $g(x) = x^r$. That choice implies $F(x) = (f(x))^r$, which suggests choosing $f(x) = (F(x))^{\frac{1}{r}}$. Now we must formally use these choices to show that F is homothetic.

Define $f:\mathbb{R}^n_+ \to \mathbb{R}_+$ and $g:\mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(x) = (F(x))^{\frac{1}{r}} \quad \text{for all } x \in \mathbb{R}^n_+, \quad g(x) = x^r \quad \text{for all } x \in \mathbb{R}_+$$

Note that g is increasing on $\mathbb{R}_+$ because we are given that $r > 0$. Also, f is homogeneous of degree one on $\mathbb{R}^n_+$ because for any $x \in \mathbb{R}^n_+$ and any $t > 0$, we have

$$f(tx) = (F(tx))^{\frac{1}{r}} = (t^r F(x))^{\frac{1}{r}} = t(F(x))^{\frac{1}{r}} = tf(x)$$

Finally, verify that for all $x \in \mathbb{R}^n_+$,

$$g(f(x)) = g\left((F(x))^{\frac{1}{r}}\right) = \left((F(x))^{\frac{1}{r}}\right)^r = F(x)$$

By the given definition, then, F is homothetic on $\mathbb{R}^n_+$.

(b) Let $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ be defined by

$$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2_+$$

Now, f is homogeneous of degree one on $\mathbb{R}^2_+$ because for any $(x_1, x_2) \in \mathbb{R}^2_+$ and any $t > 0$, we have

$$f(tx_1, tx_2) = t^{\alpha} t^{1-\alpha} x_1^{\alpha} x_2^{1-\alpha} = t f(x_1, x_2)$$

Now, let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$g(x) = \frac{x}{1+x} \quad \text{for all } x \geq 0$$

Note that g is increasing on $\mathbb{R}_+$ because for all $x \geq 0$, we have

$$g'(x) = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0$$

Finally, verify that for all $(x_1, x_2) \in \mathbb{R}^2_+$,

$$g(f(x_1, x_2)) = g\left(x_1^{\alpha} x_2^{1-\alpha}\right) = \frac{x_1^{\alpha} x_2^{1-\alpha}}{1 + x_1^{\alpha} x_2^{1-\alpha}} = F(x_1, x_2)$$

By the given definition, then, F is homothetic on $\mathbb{R}^2_+$.

(c) Since f is homogeneous of degree one on $\mathbb{R}^2_{++}$, we have by Theorem 15, that the first-order partial derivatives of f are homogeneous of degree zero on $\mathbb{R}^2_{++}$. That is, for any $x \in \mathbb{R}^2_{++}$ and any $t > 0$, we have

$$D_1 f(tx) = D_1 f(x), \quad D_2 f(tx) = D_2 f(x)$$

Now, using this result after applying the Chain Rule to F, we have that for any $x \in \mathbb{R}^2_{++}$ and any $t > 0$,

$$D_1 F(tx) = g'(f(tx)) D_1 f(tx) = g'(f(tx)) D_1 f(x) \tag{2.1}$$
$$D_2 F(tx) = g'(f(tx)) D_2 f(tx) = g'(f(tx)) D_2 f(x) \tag{2.2}$$

Because we are given that the first order partial derivatives of F are positive, we can divide (2.1) by (2.2) to see that for any $x \in \mathbb{R}^2_{++}$ and any $t > 0$,

$$\frac{D_1 F(tx)}{D_2 F(tx)} = \frac{g'(f(tx))D_1 f(x)}{g'(f(tx))D_2 f(x)} = \frac{D_1 f(x)}{D_2 f(x)} \tag{2.3}$$

Evaluating at $t = 1$, we have that for any $x \in \mathbb{R}^2_{++}$,

$$\frac{D_1 F(x)}{D_2 F(x)} = \frac{D_1 f(x)}{D_2 f(x)} \tag{2.4}$$

Combining (2.3) and (2.4), we have that for any $x \in \mathbb{R}^2_{++}$ and any $t > 0$,

$$\frac{D_1 F(tx)}{D_2 F(tx)} = \frac{D_1 F(x)}{D_2 F(x)}$$

Problem 28 (Inverse Function Theorem)**.**

Suppose g^i (for $i = 1,2$) are continuously differentiable functions from $\mathbb{R}^2_{++}$ to $\mathbb{R}_{++}$, satisfying for all $(x_1,x_2) \in \mathbb{R}^2_{++}$, $D_j g^i(x_1,x_2) > 0$ for $i = 1,2$; $j = 1,2$. Define functions $a_{ij}(x_1,x_2) = D_j g^i(x_1,x_2)$ for $(x_1,x_2) \in \mathbb{R}^2_{++}$. Let $x^0 = (x^0_1, x^0_2) \in \mathbb{R}^2_{++}$; denote $g^i(x^0)$ by y^0_i for $i = 1,2$. Assume that

$$a_{11}(x^0)a_{22}(x^0) > a_{12}(x^0)a_{21}(x^0)$$

Define the functions f^1 and f^2 from $\mathbb{R}^4_{++}$ to $\mathbb{R}_{++}$ as follows:

$$f^1(x_1,x_2,y_1,y_2) = y_1 - g^1(x_1,x_2)$$
$$f^2(x_1,x_2,y_1,y_2) = y_2 - g^2(x_1,x_2)$$

(a) Use the implicit function theorem to show that there exists an open set U containing $y^0 = (y^0_1, y^0_2)$, and an open set V containing x^0, and a unique function $h : U \to V$ such that for all $y = (y_1,y_2) \in U$, $g(h(y)) = y$, and $h(y^0) = x^0$.

(b) Note that h is continuously differentiable on U, and show that $D_1 h^1(y^0) > 0$, and $D_1 h^2(y^0) < 0$.

Solution.

Note that in this problem, x_1 and x_2 are treated as variables and y_1 and y_2 are treated as parameters. This is reversed from the presentation of the Implicit Function Theorem in class.

(a) Note that $y^0 \in \mathbb{R}^2_{++}$ because g^1 and g^2 are functions into $\mathbb{R}_{++}$. We want to apply the Implicit Function Theorem to f^1 and f^2 at the point $(x^0, y^0) \in \mathbb{R}^4_{++}$. We need to check some conditions before applying the theorem:

- Both f^1 and f^2 are defined on the open set $\mathbb{R}^4_{++}$. We would like to use the fact that each of f^1 and f^2 is the sum of two continuously differentiable functions on $\mathbb{R}^4_{++}$ to establish that f^1 and f^2 are continuously differentiable on $\mathbb{R}^4_{++}$, but g^1 and g^2 are defined on $\mathbb{R}^2_{++}$. Therefore, define the following functions, each from $\mathbb{R}^4_{++}$ to $\mathbb{R}_{++}$:

$$\begin{aligned}
\tilde{g}^1(x_1,x_2,y_1,y_2) &= g^1(x_1,x_2) && \text{for all } (x_1,x_2,y_1,y_2) \in \mathbb{R}^4_{++}\\
\tilde{g}^2(x_1,x_2,y_1,y_2) &= g^2(x_1,x_2) && \text{for all } (x_1,x_2,y_1,y_2) \in \mathbb{R}^4_{++}\\
\tilde{g}^3(x_1,x_2,y_1,y_2) &= y_1 && \text{for all } (x_1,x_2,y_1,y_2) \in \mathbb{R}^4_{++}\\
\tilde{g}^4(x_1,x_2,y_1,y_2) &= y_2 && \text{for all } (x_1,x_2,y_1,y_2) \in \mathbb{R}^4_{++}
\end{aligned}$$

  Each of $\tilde{g}^1, \tilde{g}^2, \tilde{g}^3$, and $\tilde{g}^4$ is continuously differentiable on $\mathbb{R}^4_{++}$. We can express each of f^1 and f^2 as sums of these functions:

$$\begin{aligned}
f^1(x_1,x_2,y_1,y_2) &= \tilde{g}^3(x_1,x_2,y_1,y_2) - \tilde{g}^1(x_1,x_2,y_1,y_2)\\
f^2(x_1,x_2,y_1,y_2) &= \tilde{g}^4(x_1,x_2,y_1,y_2) - \tilde{g}^2(x_1,x_2,y_1,y_2)
\end{aligned}$$

  Therefore f^1 and f^2 are continuously differentiable on $\mathbb{R}^4_{++}$.

- We are given that at $(x^0, y^0) \in \mathbb{R}^4_{++}$,

$$\begin{aligned}
f^1(x^0,y^0) &= y_1^0 - g^1(x^0) = y_1^0 - y_1^0 = 0\\
f^2(x^0,y^0) &= y_2^0 - g^2(x^0) = y_2^0 - y_2^0 = 0
\end{aligned}$$

- By the definition of f^1 and f^2 we have that

$$\begin{aligned}
\left(D_j f^i(x^0,y^0)\right) &= \begin{bmatrix} D_1 f^1(x^0,y^0) & D_2 f^1(x^0,y^0) \\ D_1 f^2(x^0,y^0) & D_2 f^2(x^0,y^0) \end{bmatrix}\\
&= \begin{bmatrix} -D_1 g^1(x^0) & -D_2 g^1(x^0) \\ -D_1 g^2(x^0) & -D_2 g^2(x^0) \end{bmatrix}\\
&= \begin{bmatrix} -a_{11}(x^0) & -a_{12}(x^0) \\ -a_{21}(x^0) & -a_{22}(x^0) \end{bmatrix}
\end{aligned}$$

  Then $\det\left(D_j f^i(x^0,y^0)\right) = a_{11}(x^0)a_{22}(x^0) - a_{12}(x^0)a_{21}(x^0) \neq 0$ by the given information.

By the Implicit Function Theorem, then, there is an open set $U \subset \mathbb{R}^2_{++}$ containing y^0, an open set $V \subset \mathbb{R}^2_{++}$ containing x^0, and a unique function $h : U \to V$ such that:

(1) $f(h(y), y) = 0$ for all $y \in U$

(2) $h(y^0) = x^0$

Now, result (1) implies that for each $i = 1, 2$, we have $f^i(h(y), y) = y_i - g^i(h(y)) = 0$ for all $y \in U$. This implies that for all $y \in U$, we have

$$g^1(h^1(y), h^2(y)) = y_1 \tag{3.1}$$

$$g^2(h^1(y), h^2(y)) = y_2 \tag{3.2}$$

This is equivalent to $g(h(y)) = y$ for all $y \in U$, which we were asked to show.

(b) Given the conditions that were checked in part (a), we also have from the Implicit Function Theorem that h is continuously differentiable on U.

Using the Chain Rule to differentiate (3.1) with respect to each of y_1 and y_2, we have that for all $y \in U$,

$$D_1g^1(h^1(y), h^2(y))D_1h^1(y) + D_2g^1(h^1(y), h^2(y))D_1h^2(y) = 1 \tag{3.3}$$

$$D_1g^1(h^1(y), h^2(y))D_2h^1(y) + D_2g^1(h^1(y), h^2(y))D_2h^2(y) = 0 \tag{3.4}$$

Using the Chain Rule to differentiate (3.2) with respect to each of y_1 and y_2, we have that for all $y \in U$,

$$D_1g^2(h^1(y), h^2(y))D_1h^1(y) + D_2g^2(h^1(y), h^2(y))D_1h^2(y) = 0 \tag{3.5}$$

$$D_1g^2(h^1(y), h^2(y))D_2h^1(y) + D_2g^2(h^1(y), h^2(y))D_2h^2(y) = 1 \tag{3.6}$$

Evaluating (3.3), (3.4), (3.5), and (3.6) at y^0 and using the result from part (a) that $h(y^0) = x^0$, we have the four equations

$$D_1g^1(x^0)D_1h^1(y^0) + D_2g^1(x^0)D_1h^2(y^0) = 1$$
$$D_1g^1(x^0)D_2h^1(y^0) + D_2g^1(x^0)D_2h^2(y^0) = 0$$
$$D_1g^2(x^0)D_1h^1(y^0) + D_2g^2(x^0)D_1h^2(y^0) = 0$$
$$D_1g^2(x^0)D_2h^1(y^0) + D_2g^2(x^0)D_2h^2(y^0) = 1$$

This is a system of linear equations that we can express as

$$\begin{bmatrix} a_{11}(x^0) & a_{12}(x^0) \\ a_{21}(x^0) & a_{22}(x^0) \end{bmatrix} \begin{bmatrix} D_1h^1(y^0) & D_2h^1(y^0) \\ D_1h^2(y^0) & D_2h^2(y^0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In particular, we are interested in the system

$$\begin{bmatrix} a_{11}(x^0) & a_{12}(x^0) \\ a_{21}(x^0) & a_{22}(x^0) \end{bmatrix} \begin{bmatrix} D_1h^1(y^0) \\ D_1h^2(y^0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Because we are given that $a_{11}(x^0)a_{22}(x^0) > a_{12}(x^0)a_{21}(x^0)$, $a_{21}(x^0) > 0$, and $a_{22}(x^0) > 0$, we can use Cramer's Rule to solve this system for $D_1h^1(y^0)$ and $D_1h^2(y^0)$, which gives the desired inequalities:

$$D_1h^1(y^0) = \frac{\begin{vmatrix} 1 & a_{12}(x^0) \\ 0 & a_{22}(x^0) \end{vmatrix}}{\begin{vmatrix} a_{11}(x^0) & a_{12}(x^0) \\ a_{21}(x^0) & a_{22}(x^0) \end{vmatrix}} = \frac{a_{22}(x^0)}{a_{11}(x^0)a_{22}(x^0) - a_{12}(x^0)a_{21}(x^0)} > 0$$

$$D_1h^2(y^0) = \frac{\begin{vmatrix} a_{11}(x^0) & 1 \\ a_{21}(x^0) & 0 \end{vmatrix}}{\begin{vmatrix} a_{11}(x^0) & a_{12}(x^0) \\ a_{21}(x^0) & a_{22}(x^0) \end{vmatrix}} = \frac{-a_{21}(x^0)}{a_{11}(x^0)a_{22}(x^0) - a_{12}(x^0)a_{21}(x^0)} < 0$$

Problem 29 (Implicit Function Theorem)**.**

Let X be an open set in $\mathbb{R}^3$, and let f be a continuously differentiable function from X to $\mathbb{R}$. Let $(\overline{x_1}, \overline{x_2}, \overline{x_3})$ be a point in X, such that (i) $f(\overline{x_1}, \overline{x_2}, \overline{x_3}) = 0$, and (ii) for each $i \in \{1,2,3\}$, $D_i f(\overline{x_1}, \overline{x_2}, \overline{x_3}) \neq 0$.

(a) Use the implicit function theorem to obtain three open sets A^1, A^2, A^3, each in $\mathbb{R}^2$, containing $(\overline{x_2}, \overline{x_3}), (\overline{x_1}, \overline{x_3})$, and $(\overline{x_1}, \overline{x_2})$ respectively, and three open sets B^1, B^2, B^3, each in $\mathbb{R}$, containing $\overline{x_1}, \overline{x_2}$, and $\overline{x_3}$ respectively, and unique functions $g^i : A^i \to B^i$ (for $i \in \{1,2,3\}$), such that:

$$\begin{aligned} &(i)\ f(g^1(x_2,x_3),x_2,x_3) = 0 \ \text{ for all } (x_2,x_3) \in A^1 \\ &(ii)\ f(x_1,g^2(x_1,x_3),x_3) = 0 \ \text{ for all } (x_1,x_3) \in A^2 \\ &(iii)\ f(x_1,x_2,g^3(x_1,x_2)) = 0 \ \text{ for all } (x_1,x_2) \in A^3 \end{aligned}$$

and:

$$\overline{x_1} = g^1(\overline{x_2}, \overline{x_3}), \overline{x_2} = g^2(\overline{x_1}, \overline{x_3}), \overline{x_3} = g^3(\overline{x_1}, \overline{x_2})$$

Further, g^i is continuously differentiable on A^i for $i \in \{1,2,3\}$.

(b) Using (a), show that:

$$D_1g^1(\overline{x_2}, \overline{x_3})D_2g^2(\overline{x_1}, \overline{x_3})D_1g^3(\overline{x_1}, \overline{x_2}) = -1$$

Solution.

(a) To solve this problem, we need to apply the Implicit Function Theorem three times. It is enough to work through only one of these three cases and note that the other two are very similar.

Claim: There is an open set $A^1 \subset \mathbb{R}^2$ containing $(\overline{x_2}, \overline{x_3})$, an open set $B^1 \subset \mathbb{R}$ containing $\overline{x_1}$, and a unique function $g^1 : A^1 \to B^1$ such that:

(1) $f(g^1(x_2, x_3), x_2, x_3) = 0$ for all $(x_2, x_3) \in A^1$

(2) $\overline{x_1} = g^1(\overline{x_2}, \overline{x_3})$

Further, g^1 is continuously differentiable on A^1.

Proof: To establish the claim, we want to apply the Implicit Function Theorem to f at the point $(\overline{x_1}, (\overline{x_2}, \overline{x_3})) \in X$. Note that x_1 is treated as a variable and x_2 and x_3 are treated as parameters. We need to check some conditions before applying the theorem:

- The function f is defined on the set $X \subset \mathbb{R}^3$. We are told that X is open and that f is continuously differentiable on X.
- We are given that $f(\overline{x_1}, (\overline{x_2}, \overline{x_3})) = 0$.
- We are given that $D_1 f(\overline{x_1}, (\overline{x_2}, \overline{x_3})) \neq 0$.

By the Implicit Function Theorem, then, the claim above holds.

(b) From part (a) we have that $f(g^1(x_2, x_3), x_2, x_3) = 0$ for all $(x_2, x_3) \in A^1$. Using the Chain Rule to differentiate this with respect to x_2, we have

$$D_1 f(g^1(x_2, x_3), x_2, x_3) D_1 g^1(x_2, x_3) + D_2 f(g^1(x_2, x_3), x_2, x_3)(1) \\ + D_3 f(g^1(x_2, x_3), x_2, x_3)(0) = 0$$

Evaluating at $(\overline{x_1}, \overline{x_2}, \overline{x_3})$ and using the fact that $\overline{x_1} = g^1(\overline{x_2}, \overline{x_3})$, this becomes

$$D_1 f(\overline{x_1}, \overline{x_2}, \overline{x_3}) D_1 g^1(\overline{x_2}, \overline{x_3}) + D_2 f(\overline{x_1}, \overline{x_2}, \overline{x_3}) = 0$$

Because we are given that $D_1 f(\overline{x_1}, \overline{x_2}, \overline{x_3}) \neq 0$, we can rearrange this for

$$D_1 g^1(\overline{x_2}, \overline{x_3}) = -\frac{D_2 f(\overline{x_1}, \overline{x_2}, \overline{x_3})}{D_1 f(\overline{x_1}, \overline{x_2}, \overline{x_3})} \tag{4.1}$$

From part (a) we have that $f(x_1, g^2(x_1, x_3), x_3) = 0$ for all $(x_1, x_3) \in A^2$. Using the Chain Rule to differentiate this with respect to x_3, we have

$$\begin{aligned} D_1 f(x_1, g^2(x_1, x_3), x_3)(0) + D_2 f(x_1, g^2(x_1, x_3), x_3) D_2 g^2(x_1, x_3) \\ + D_3 f(x_1, g^2(x_1, x_3), x_3)(1) = 0 \end{aligned}$$

Evaluating at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and using the fact that $\bar{x}_2 = g^2(\bar{x}_1, \bar{x}_3)$, this becomes

$$D_2 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) D_2 g^2(\bar{x}_1, \bar{x}_3) + D_3 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 0$$

Because we are given that $D_2 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) \neq 0$, we can rearrange this for

$$D_2 g^2(\bar{x}_1, \bar{x}_3) = -\frac{D_3 f(\bar{x}_1, \bar{x}_2, \bar{x}_3)}{D_2 f(\bar{x}_1, \bar{x}_2, \bar{x}_3)} \tag{4.2}$$

From part (a) we have that $f(x_1, x_2, g^3(x_1, x_2)) = 0$ for all $(x_1, x_2) \in A^3$. Using the Chain Rule to differentiate this with respect to x_1, we have

$$\begin{aligned} D_1 f(x_1, x_2, g^3(x_1, x_2))(1) + D_2 f(x_1, x_2, g^3(x_1, x_2))(0) \\ + D_3 f(x_1, x_2, g^3(x_1, x_2)) D_1 g^3(x_1, x_2) = 0 \end{aligned}$$

Evaluating at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and using the fact that $\bar{x}_3 = g^3(\bar{x}_1, \bar{x}_2)$, this becomes

$$D_1 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) + D_3 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) D_1 g^3(\bar{x}_1, \bar{x}_2) = 0$$

Because we are given that $D_3 f(\bar{x}_1, \bar{x}_2, \bar{x}_3) \neq 0$, we can rearrange this for

$$D_1 g^3(\bar{x}_1, \bar{x}_2) = -\frac{D_1 f(\bar{x}_1, \bar{x}_2, \bar{x}_3)}{D_3 f(\bar{x}_1, \bar{x}_2, \bar{x}_3)} \tag{4.3}$$

Combining equations (4.1), (4.2), and (4.3), we have the desired result:

$$D_1 g^1(\bar{x}_2, \bar{x}_3) D_2 g^2(\bar{x}_1, \bar{x}_3) D_1 g^3(\bar{x}_1, \bar{x}_2) = -1$$

References

This material is based on *The Elements of Real Analysis* by *Robert Bartle* (Chapter 7); *Mathematical Analysis: A Modern Approach to Advanced Calculus* by *Tom Apostol* (Chapters 6, 7) and *Calculus on Manifolds* by *Michael Spivak* (Chapter 2).

Chapter 7

Convex Analysis

7.1 Convex Sets

Line Segment

If $x, y \in \mathbb{R}^n$, the *line segment* joining x and y is given by the set of points $\{z \in \mathbb{R}^n : z = \theta x + (1-\theta)y$ for some $0 \le \theta \le 1\}$.

Convex Set

A set $S \subset \mathbb{R}^n$ is a *convex set* if for every $x, y \in S$, the line segment joining x and y is contained in S.

For example, the set of points $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$ is a convex set. The set of points $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is *not* a convex set.

It can be checked that if two sets S_1 and S_2 are convex sets in $\mathbb{R}^n$, then

(i) the intersection of S_1 and S_2 [that is, the set $\{z \in \mathbb{R}^n : z \in S_1$ and $z \in S_2\}$] is a convex set in $\mathbb{R}^n$.

(ii) the sum of S_1 and S_2 [that is, the set $\{z \in \mathbb{R}^n : z = x + y, where\, x \in S_1\, and\, y \in S_2\}$] is a convex set in $\mathbb{R}^n$.

(iii) the (Cartesian) product of S_1 and S_2 [that is the set $\{z \in \mathbb{R}^{2n} : z = (x, y)$, where $x \in S_1$ and $y \in S_2\}$] is a convex set in $\mathbb{R}^{2n}$.

However, if S_1 and S_2 are convex sets in $\mathbb{R}^n$, it does *not* follow that the union of S_1 and S_2 [that is, the set $\{z \in \mathbb{R}^n : z \in S_1$ or $z \in S_2\}$] is a convex set in $\mathbb{R}^n$. For example, the interval $S_1 \equiv [0, 1]$ is a convex set in $\mathbb{R}$, and so is the interval, $S_2 \equiv [3, 4]$. The point 2 is on the line segment joining 1 and 3, but 2 is not in the union of the sets S_1 and S_2. So the union of S_1 and S_2 is not a convex set in $\mathbb{R}$.

A vector $y \in \mathbb{R}^n$ is said to be a *convex combination* of the vectors $x^1, \ldots, x^m \in \mathbb{R}^n$ if there exist m non-negative real numbers $\theta_1, \ldots, \theta_m$ such that

$$(i)\ \sum_{i=1}^{m}\theta_i = 1; \quad and \quad (ii)\ y = \sum_{i=1}^{m}\theta_i x^i$$

A convex set $A \subset \mathbb{R}^n$ can be redefined as a set such that for every *two* vectors in the set A, all convex combinations of these *two* vectors are also in the set A. It can be shown that in the above statement "two" can be replaced by "m" where m is any integer exceeding one.

Proposition 3. *A set $A \subset \mathbb{R}^n$ is convex if and only if for every integer $m > 1$, and every m vectors in A, every convex combination of the m vectors is in A.*

7.2 Separating Hyperplane Theorem for Convex Sets

Hyperplane:

Let $p \in \mathbb{R}^n$ with $p \neq 0$, and let $\alpha \in \mathbb{R}$. The set $H = \{x \in \mathbb{R}^n : px = \alpha\}$ is called a *hyperplane* in $\mathbb{R}^n$ with *normal* p.

A hyperplane H in $\mathbb{R}^n$ divides $\mathbb{R}^n$ into the two sets:

$S_1 \equiv \{x \in \mathbb{R}^n : px \geq \alpha\}$ and $S_2 \equiv \{x \in \mathbb{R}^n : px \leq \alpha\}$

The sets S_1 and S_2 are called the *closed half-spaces* associated with the hyperplane H.

A very important resu1t on convex sets can now be stated.

Theorem 19. *(Minkowski Separation Theorem)*

Let X and Y be non-empty convex sets in $\mathbb{R}^n$, such that X is disjoint from Y. Then there exists $p \in \mathbb{R}^n$, $\|p\| = 1$, and $\alpha \in \mathbb{R}$, such that

$$\begin{aligned} px &\geq \alpha \qquad for\,all\, x \in X \\ py &\leq \alpha \qquad for\,all\, y \in Y \end{aligned}$$

The Minkowski separation theorem can be used to establish a criterion for the existence of *non-negative* solutions to a system of linear equations.

Theorem 20. *(Farkas Lemma)*

Exactly one of the following alternatives holds. Either the equation

$$\underset{(m\times n)}{A}\ \underset{(n\times 1)}{x} = \underset{(m\times 1)}{b} \tag{7.1}$$

has a non-negative solution; or the inequalities

$$\underset{(1\times m)}{y}\ \underset{(m\times n)}{A} \geq 0; \quad \underset{(1\times m)}{y}\ \underset{(m\times 1)}{b} < 0 \tag{7.2}$$

has a solution.

Proof. First, suppose that (7.1) does *not* have a non-negative solution. Define

$$Q = \{q \in \mathbb{R}^n : q = \sum_{i=1}^{n} \lambda_i A^i \qquad for\, some\ (\lambda_1, \ldots, \lambda_n) \geq 0\}$$

It can be checked that Q is a closed convex set. By our hypothesis b is *not* in Q. Since Q is closed, there is an open ball $B(b,r) \subset \sim Q$. That is, Q and $B(b,r)$ are disjoint. Since $B(b,r)$ is clearly convex, we can use Theorem 19 to obtain $p \in \mathbb{R}^n$, $\|p\| = 1$, and $\alpha \in \mathbb{R}$ such that

$$pq \leq \alpha \quad for\, all\ q \in B(b,r) \tag{7.3}$$

and

$$pq \geq \alpha \quad for\, all\ q \in Q \tag{7.4}$$

Note that if $q \in Q$, then $tq \in Q$ for every $t > 0$. Using this in (7.4), for $q \in Q$,

$$pq \geq (\alpha/t) \quad for\, every\ t > 0 \tag{7.5}$$

Clearly (7.5) implies that

$$pq \geq 0 \quad for\, all\ q \in Q \tag{7.6}$$

Since $A^i \in Q$ for $i = 1, \ldots, n$, so (7.6) implies

$$pA \geq 0 \tag{7.7}$$

Using (7.4) again, we note that since $0 \in Q$, we have $\alpha \leq 0$. Using this in (7.3), we have

$$pq \leq 0 \quad for\, all\ q \in B(b,r) \tag{7.8}$$

Define $q^* = b + (r/2)p$. Then $\|q^* - b\| = \|(r/2)p\| = (r/2)\,\|p\| = (r/2)$. So $q^* \in B(b,r)$, and by (7.8),

$$pb + (r/2) = pb + (r/2)\,\|p\|^2 = pq^* \leq 0$$

This implies that

$$pb < 0 \tag{7.9}$$

Now, (7.7) and (7.9) show that we have demonstrated a solution to the inequalities given in (7.2).

To complete the proof of Theorem 20, consider, next, that (7.1) has a non-negative solution, say $x \in \mathbb{R}^n_+$. We have to show that (7.2) does not have a solution. Suppose (7.2) did have a solution, say $y \in \mathbb{R}^n$, then

$$0 \leq (yA)x = y(Ax) = yb < 0$$

which is clearly a contradiction. ∎

7.3 Continuous and Differentiable Functions on Convex Sets

We now provide three very useful theorems on continuous and differentiable functions on convex sets. They are known as the Intermediate Value theorem, the Mean Value theorem and Taylor's theorem.

Theorem 21. *(Intermediate Value Theorem):*

Suppose A is a convex subset of $\mathbb{R}^n$*, and* $f : A \to \mathbb{R}$ *is a continuous function on A. Suppose* x^1 *and* x^2 *are in A, and* $f(x^1) > f(x^2)$*. Then given any* $c \in \mathbb{R}$ *such that* $f(x^1) > c > f(x^2)$*, there is* $0 < \theta < 1$ *such that* $f[\theta x^1 + (1-\theta)x^2] = c$*.*

Example:

Suppose $X \equiv [a,b]$ is a closed interval in $\mathbb{R}$ (with $a < b$). Suppose f is a continuous function on X.

We know, by Weierstrass theorem, that there will exist x^1 and x^2 in X such that $f(x^1) \geq f(x) \geq f(x^2)$ for all $x \in X$. If $f(x^1) = f(x^2)$ [this is the trivial case], then $f(x) = f(x^1)$ for all $x \in X$, and so $f(X)$ is the single point, $f(x^1)$.

If $f(x^1) > f(x^2)$, then using the fact that X is a *convex set*, we can conclude from the Intermediate Value Theorem that every value between $f(x^1)$ and $f(x^2)$ is attained by the function f at some point in X. In other words, $f(X)$ is itself a closed interval.

Theorem 22. *(Mean Value Theorem)*

Suppose A is an open convex subset of $\mathbb{R}^n$*, and* $f : A \to \mathbb{R}$ *is continuously differentiable on A. Suppose* x^1 *and* x^2 *are in A. Then there is* $0 \leq \theta \leq 1$ *such that*

$$f(x^2) - f(x^1) = (x^2 - x^1)\nabla f(\theta x^1 + (1-\theta)x^2)$$

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with the property that $f'(x) > 0$ for all $x \in \mathbb{R}$. Then given any x^1, x^2 in $\mathbb{R}$, with $x^2 > x^1$ we have by the Mean-Value Theorem (since $\mathbb{R}$ is open and convex), the existence of $0 \leq \theta \leq 1$, such that

$$f(x^2) - f(x^1) = (x^2 - x^1) f'(\theta x^1 + (1-\theta)x^2)$$

Now $f'(\theta x^1 + (1-\theta)x^2) > 0$ by assumption, and $x^2 > x^1$ by hypothesis. So $f(x^2) > f(x^1)$. This shows that f is an *increasing function* on $\mathbb{R}$.

A word of caution: a function $f : \mathbb{R} \to \mathbb{R}$ can be increasing without satisfying $f'(x) > 0$ at all $x \in \mathbb{R}$. For example, $f(x) = x^3$ is increasing on $\mathbb{R}$, but $f'(0) = 0$.

Theorem 23. *(Taylor's Expansion up to Second-Order)*

Suppose A is an open, convex subset of $\mathbb{R}^n$*, and* $f : A \to \mathbb{R}$ *is twice continuously differentiable on A. Suppose* x^1 *and* x^2 *are in A. Then there exists* $0 \le \theta \le 1$*, such that*

$$f(x^2) - f(x^1) = (x^2 - x^1)\nabla f(x^1) + \frac{1}{2}(x^2 - x^1)H_f(\theta x^1 + (1-\theta)x^2)(x^2 - x^1)$$

7.4 Concave Functions

Let A be a convex set in $\mathbb{R}^n$. Then $f : A \to \mathbb{R}$ is a *concave function* (on A) if for all x^1, $x^2 \in A$, and for all $0 \le \theta \le 1$,

$$f[\theta x^1 + (1-\theta)x^2] \ge \theta f(x^1) + (1-\theta) f(x^2)$$

The function f is *strictly concave* on A if $f[\theta x^1 + (1-\theta)x^2] > \theta f(x^1) + (1-\theta) f(x^2)$ whenever x^1, $x^2 \in A$, $x^1 \ne x^2$ and $0 < \theta < 1$.

The relation between concave functions and convex sets is given by the following result, which can be proved easily from the definitions of a convex set and a concave function.

Theorem 24. *Suppose A is a convex subset of* $\mathbb{R}^n$ *and* f *is a real-valued function on A. Then f is a concave function if and only if the set* $\{(x,\alpha) \in A \times \mathbb{R} : f(x) \ge \alpha\}$ *is a convex set in* $\mathbb{R}^{n+1}$.

The following result on concave functions is also useful, although it does not provide a characterization of concave functions.

Theorem 25. *Suppose A is a convex subset of* $\mathbb{R}^n$*, and* $f : A \to \mathbb{R}$ *is a concave function. Then, for every* $\alpha \in \mathbb{R}$*, the set*

$$S(\alpha) = \{x \in A : f(x) \ge a\}$$

is a convex set in $\mathbb{R}^n$.

A result on concave functions which parallels Proposition 3 on convex sets can now be noted. It is known as Jensen's inequality, and is a very useful tool in convex analysis.

Proposition 4. *(Jensen's Inequality)*

Let A be a convex subset of $\mathbb{R}^n$*,and f a real-valued function on A. Then a necessary and sufficient condition for f to be concave is that for each integer* $m > 1$,

$$f(\sum_{i=1}^{m}\theta_i x^i) \ge \sum_{i=1}^{m}\theta_i f(x^i)$$

whenever $x^1, \ldots, x^m \in A$, $(\theta_1, \ldots, \theta_m) \in \mathbb{R}^m_+$ *and* $\sum_{i=1}^{m} \theta_i = 1$.

In general, if A is a convex set in $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is concave on A, then f need not be continuous on A. For example, suppose $A = \mathbb{R}_+$, and $f : \mathbb{R}_+ \to \mathbb{R}$ is defined by: $f(x) = 1 + x$ for $x > 0$; $f(x) = 0$ for $x = 0$. Then f is a concave function on A, but f is not continuous at $x = 0$.

If A is an *open* convex set in $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is concave on A, then one can show that f is continuous on A.

Theorem 26. *Suppose A is an open convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is a concave function on A. Then f is a continuous function on A.*

Differentiable Concave Functions:

If $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ is continuously differentiable on A, then we can find a convenient characterization for f to be concave on A in terms of a condition which involves the gradient vector of f. (This is particularly useful in concave programming).

Theorem 27. *Suppose $A \subset \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}$ is continuously differentiable on A. Then f is concave on A if and only if*

$$f(x^2) - f(x^1) \leq \nabla f(x^1)(x^2 - x^1)$$

whenever x^1 and x^2 are in A.

Corollary 4. *Suppose $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ is continuously differentiable on A. Then f is concave on A if and only if*

$$[\nabla f(x^2) - \nabla f(x^1)][x^2 - x^1] \leq 0$$

whenever x^1 and x^2 are in A.

It is interesting to note that a characterization of *strictly* concave functions can be given by replacing the weak inequalities in Theorem 27 and Corollary 4 with strict inequalities (for x^1, x^2 in A with $x^1 \neq x^2$).

Theorem 28. *Suppose $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ is continuously differentiable on A. Then f is strictly concave on A if and only if*

$$f(x^2) - f(x^1) < \nabla f(x^1)(x^2 - x^1)$$

whenever $x^1, x^2 \in A$ and $x^1 \neq x^2$.

Corollary 5. *Suppose $A \subset \mathbb{R}^n$ is an open convex set, and $f : A \to \mathbb{R}$ is continuously differentiable on A. Then f is strictly concave on A if and only if*

$$[\nabla f(x^2) - \nabla f(x^1)][x^2 - x^1] < 0$$

whenever $x^1, x^2 \in A$ and $x^1 \neq x^2$.

Twice-Differentiable Concave Functions

If $A \subset \mathbb{R}^n$ is an open set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A, then we can find a convenient characterization for f to be a concave function in terms of the negative semi-definiteness of the Hessian matrix of f.

Theorem 29. *Suppose $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. Then f is concave on A if and only if $H_f(x)$ is negative semi-definite whenever $x \in A$.*

If the Hessian of f is actually negative definite for all $x \in A$, then f is strictly concave on A; but the converse is not true.

Theorem 30. *Suppose $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. If $H_f(x)$ is negative definite for every $x \in A$, then f is strictly concave on A.*

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be definied by $f(x) = -x^4$ for all $x \in \mathbb{R}$. This is a twice continuously differentiable function on the open, convex set $\mathbb{R}$. It can be checked that f is strictly concave on $\mathbb{R}$, but since $f''(x) = -12x^2$, $f''(0) = 0$. This shows that the converse of Theorem 29 is not valid.

7.5 Quasi-Concave Functions

Let $A \subset \mathbb{R}^n$ be a convex set, and f a real-valued function on A. Then f is quasi-concave on A if

$$f(x^2) \geq f(x^1) \ \textit{implies} \ f[\theta x^1 + (1-\theta)x^2] \geq f(x^1)$$

whenever x^1, $x^2 \in A$, and $0 \leq \theta \leq 1$. The function f is *strictly quasi-concave* on A if $f(x^2) \geq f(x^1)$ implies $f[\theta x^1 + (1-\theta)x^2] > f(x^1)$ whenever x^1, $x^2 \in A$, with $x^1 \neq x^2$, and $0 < \theta < 1$.

While the condition stated in Theorem 25 did not characerize concave functions, it *does* characterize quasi-concave functions.

Theorem 31. *Suppose A is a convex subset of $\mathbb{R}^n$, and f is a real-valued function on A. Then f is quasi-concave on A if and only if for every $\alpha \in \mathbb{R}$, the set*

$$S(\alpha) = \{x \in A : f(x) \geq \alpha\}$$

is a convex set in $\mathbb{R}^n$.

Using the concept of strict quasi-concavity, one can provide the following result on the uniqueness of solutions to constrained maximization problems.

Theorem 32. *(Uniqueness of Solutions)*

Suppose A is a non-empty, compact and convex set in $\mathbb{R}^n$. Suppose $f : A \to \mathbb{R}$ is a continuous, strictly quasi-concave function on A. Then, there exists $x^1 \in A$ such that for all $x \in A$ which are not equal to x^1, we have $f(x) < f(x^1)$.

Proof. By Weierstrass theorem, there is $x^1 \in A$ such that

$$f(x) \leq f(x^1) \quad for\, all\, x \in A \tag{7.10}$$

If the claim of the Theorem were not true, there would exist some $x^2 \in A$, $x^2 \neq x^1$, such that $f(x^2) = f(x^1)$. But then $x^3 \equiv [(1/2)x^1 + (1/2)x^2]$ would belong to A (since A is a convex set), and by strict quasi-concavity of f on A, we would have $f(x^3) = f[(1/2)x^1 + (1/2)x^2] > f(x^1)$, which contradicts (7.10). ■

Differentiable Quasi-Concave Functions:

A characterization of differentiable quasi-concave functions can be given which parallels the characterization of differentiable concave functions stated in Theorem 27. (This is particularly useful in Quasi-Concave Programming).

Theorem 33. *Suppose $A \subset \mathbb{R}^n$ is an open, convex set, and $f : A \to \mathbb{R}$ a continuously differentiable function. Then f is a quasi-concave if and only if*

$$f(x^2) \geq f(x^1) \;\; implies\, (x^2 - x^1)\nabla f(x^1) \geq 0$$

whenever $x^1, x^2 \in A$.

Twice Differentiable Quasi-Concave Functions

An interesting characterization of twice continuously differentiable quasi-concave functions can be given in terms of the "bordered" Hessian matrix associated with the functions.

Let A be an open subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ be a twice continuously differentiable function on A. The *bordered Hessian matrix of f* at $x \in A$ is denoted by $G_f(x)$ and is defined as the following $(n+1) \times (n+1)$ matrix

$$G_f(x) = \begin{bmatrix} 0 & \nabla f(x) \\ \nabla f(x) & H_f(x) \end{bmatrix}$$

We denote the $(k+1)th$ leading principal minor of $G_f(x)$ by $\left|G_f(x;k)\right|$, where $k = 1, \ldots, n$.

Theorem 34. *Let A be an open convex set in $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ be a twice continuously differentiable function on A.*

(i) *If f is quasi-concave on A, then $(-1)^k \left|G_f(x;k)\right| \geq 0$ for $x \in A$, and $k = 1, \ldots, n$.*

(ii) *If $(-1)^k \left|G_f(x;k)\right| > 0$ for $x \in A$, and $k = 1, \ldots, n$, then f is strictly quasi-concave on A.*

7.6 Worked Out Problems on Chapter 7

Problem 30 (Intermediate Value Theorem)**.**

Here is a statement of the Intermediate Value Theorem for continuous real valued functions of a real variable.

Theorem:

Let f be a continuous real valued function on the closed interval $A = [a,b]$. Suppose $x, y \in A$ satisfy $f(x) > f(y)$. Then for every c, satisfying $f(x) > c > f(y)$, there is $z \in A$, such that $f(z) = c$.

We want to use this theorem to prove the following version of the Intermediate Value Theorem for continuous real valued functions of several real variables.

Corollary:

Let B be a convex subset of $\mathbb{R}^n$, and let F be a continuous real valued function on B. Suppose $x^1, x^2 \in B$ satisfy $F(x^1) > F(x^2)$. Then for every c, satisfying $F(x^1) > c > F(x^2)$, there is $v \in B$, such that $F(v) = c$.

Proceed with the following steps.

(a) Define $A = [0,1]$, and for each $t \in A$, define $f(t) = F(tx^1 + (1-t)x^2)$. This function is well defined since B is a convex set, and F is defined on B. Verify that f is a continuous function on A.

(b) Note that $f(1) = F(x^1) > F(x^2) = f(0)$. Use the Theorem (stated above) to obtain $z \in A$ such that $f(z) = c$. This means $F(zx^1 + (1-z)x^2) = c$.

(c) Define $v = zx^1 + (1-z)x^2$, and verify that this proves the Corollary.

Solution.

(a) Note that an arbitrary $x^1 \in B$ and $x^2 \in B$ are given in the hypothesis of the Corollary. Now, let $A = [0,1]$. Define $f : A \to \mathbb{R}$ by

$$f(t) = F(tx^1 + (1-t)x^2) \quad \text{for all } t \in A$$

Note that f is well-defined: since $x^1 \in B$ and $x^2 \in B$, by convexity of B we have that $tx^1 + (1-t)x^2 \in B$ for all $t \in A$.

To show that f is continuous on A, fix some $\bar{t} \in A$ and some $\varepsilon > 0$. We want to show that there is some $\delta > 0$ such that whenever $t \in A$ and $d(t,\bar{t}) < \delta$, we have $|f(t) - f(\bar{t})| < \varepsilon$. Note that since $A \subset \mathbb{R}$, we can write $d(t,\bar{t}) = |t-\bar{t}|$. Also, note that

$$|f(t) - f(\bar{t})| = \left|F(tx^1 + (1-t)x^2) - F(\bar{t}x^1 + (1-\bar{t})x^2)\right|$$

To show that f is continuous at our arbitrary $\bar{t} \in A$, we can use the fact that F is continuous at the point $\bar{t}x^1 + (1-\bar{t})x^2 \in B$. That is, given $\varepsilon > 0$ (note that this is the same ε as above), by continuity of F there is some $\delta' > 0$ such that whenever $x \in B$ and $d(x, \bar{t}x^1 + (1-\bar{t})x^2) < \delta'$, we have $|F(x) - F(\bar{t}x^1 + (1-\bar{t})x^2)| < \varepsilon$.

This is looking similar to the inequality we want to establish. Now, recall that for any $y, z, w \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$, we have that $d(y+w, z+w) = d(y,z)$ and $d(\lambda y, \lambda z) = |\lambda|\ d(y,z)$. With the goal of picking an appropriate δ, consider the following distance:

$$\begin{aligned} d\left(tx^1 + (1-t)x^2, \bar{t}x^1 + (1-\bar{t})x^2\right) &= d\left(tx^1 - \bar{t}x^1, (1-\bar{t})x^2 - (1-t)x^2\right) \\ &= d\left((t-\bar{t})x^1, (t-\bar{t})x^2\right) \\ &= |t-\bar{t}|\ d(x^1, x^2) \end{aligned}$$

Now, let $\delta = \frac{\delta'}{d(x^1,x^2)} > 0$. Whenever $t \in A$ and $|t-\bar{t}| < \delta$, we have that $tx^1 + (1-t)x^2 \in B$ and

$$\begin{aligned} d\left(tx^1 + (1-t)x^2, \bar{t}x^1 + (1-\bar{t})x^2\right) &= |t-\bar{t}|\ d(x^1, x^2) \\ &< \delta\ d(x^1, x^2) \\ &= \delta' \end{aligned}$$

Then by continuity of F we have that $|F(tx^1 + (1-t)x^2) - F(\bar{t}x^1 + (1-\bar{t})x^2)| < \varepsilon$, which is equivalent to $|f(t) - f(\bar{t})| < \varepsilon$. Since $\bar{t}$ was chosen arbitrarily, this shows that f is continuous on A.

(b) By the definition of f, $f(1) = F(x^1)$ and $f(0) = F(x^2)$. By hypothesis in the Corollary, $F(x^1) > F(x^2)$, which is equivalent to $f(1) > f(0)$. Let some scalar c satisfy $f(1) > c > f(0)$. Then since f is continuous on the closed interval A, we have by the stated Theorem that there is some $z \in A$ such that $c = f(z) = F(zx^1 + (1-z)x^2)$.

(c) Define $v = zx^1 + (1-z)x^2$. Since B is convex and $z \in A = [0,1]$, we have that $v \in B$. Using part (b), we have that for any c satisfying $F(x^1) > c > F(x^2)$, there is some $v \in B$ such that $F(v) = c$. This proves the Corollary.

Problem 31 (Mean Value Theorem)**.**

Here is a statement of the Mean Value Theorem for real valued functions of a real variable.

Theorem:

Let f be a continuous real valued function on the closed interval $A = [a,b]$. Suppose f is differentiable for all $x \in (a,b)$. Then there is $c \in (a,b)$ such that:

$$f(b) - f(a) = (b-a)f'(c)$$

Use this theorem to prove the following version of the Mean Value Theorem for real valued functions of several real variables.

Corollary:

Let B be an open convex subset of $\mathbb{R}^n$, and let F be a continuously differentiable real valued function on B. Suppose $x^1, x^2 \in B$. Then there is $\theta \in (0,1)$, satisfying:

$$F(x^2) - F(x^1) = (x^2 - x^1)\nabla F(\theta x^1 + (1-\theta)x^2)$$

Solution.

We can use the same function f that we used in Problem 1 to help us establish this result. Note that an arbitrary $x^1 \in B$ and $x^2 \in B$ are given in the hypothesis of the Corollary. Now, let $A = [0,1]$. Define $f : A \to \mathbb{R}$ by

$$f(t) = F(tx^1 + (1-t)x^2) \quad \text{for all } t \in A$$

We showed in part (a) of problem 1 that f is well-defined and continuous on A. Now, because we have made the additional assumptions in this problem that B is open and F is continuously differentiable on B, we can use the Chain Rule to differentiate f on the open interval $(0,1)$. Remember that x_i^1 denotes the ith component of the vector x^1, and x_i^2 denotes the ith component of the vector x^2. For all $t \in (0,1)$, we have

$$\begin{aligned} f'(t) &= \sum_{i=1}^{n}\left[D_iF(tx^1 + (1-t)x^2)\ \frac{\partial}{\partial t}(tx_i^1 + (1-t)x_i^2)\right] \\ &= \sum_{i=1}^{n}\left[(x_i^1 - x_i^2)\ D_iF(tx^1 + (1-t)x^2)\right] \\ &= (x^1 - x^2)\ \nabla F(tx^1 + (1-t)x^2) \end{aligned}$$

Since F is continuously differentiable on B and $tx^1 + (1-t)x^2 \in B$ whenever $t \in (0,1)$, we have that $f'(t)$ is continuous on $(0,1)$. By the stated Theorem, there is $\theta \in (0,1)$ such that $f(1) - f(0) = (1-0)f'(\theta)$. And by the definition of f, we have that $f(1) = F(x^1)$ and $f(0) = F(x^2)$. Using the above expression for f', then, we have that for some $\theta \in (0,1)$,

$$F(x^1) - F(x^2) = (x^1 - x^2)\ \nabla F(\theta x^1 + (1-\theta)x^2)$$

Multiplying each side of this equation by -1 gives the desired result.

Problem 32 (Jensen's Inequality for Concave Functions)**.**

Suppose A is a convex subset of $\mathbb{R}^n$, and f a concave function on A. Let $x^1, x^2, x^3 \in A$, and let $\theta_1, \theta_2, \theta_3$ be real numbers satisfying $\theta_i \geq 0$ for $i = 1,2,3$, and $(\theta_1 + \theta_2 + \theta_3) = 1$.

(a) Using the definition of a convex set, verify that:

$$(\theta_1 x^1 + \theta_2 x^2 + \theta_3 x^3) \in A$$

(b) Using the definition of a concave function, show that:

$$f(\theta_1 x^1 + \theta_2 x^2 + \theta_3 x^3) \geq \theta_1 f(x^1) + \theta_2 f(x^2) + \theta_3 f(x^3)$$

Solution.

(a) If $\theta_1 = 0$, $\theta_2 = 0$, or $\theta_3 = 0$ the problem is trivial, because $\theta_1 x^1 + \theta_2 x^2 + \theta_3 x^3$ becomes a convex combination of just two vectors in A, and we know that A is convex. So, assume that $\theta_1 \neq 0$, $\theta_2 \neq 0$, and $\theta_3 \neq 0$. Then we can write

$$\theta_1 x^1 + \theta_2 x^2 + \theta_3 x^3 = (\theta_1 + \theta_2)\frac{\theta_1}{\theta_1 + \theta_2} x^1 + (\theta_1 + \theta_2)\frac{\theta_2}{\theta_1 + \theta_2} x^2 + \theta_3 x^3$$

We know that $\frac{\theta_1}{\theta_1+\theta_2} \geq 0$, $\frac{\theta_2}{\theta_1+\theta_2} \geq 0$, and $\frac{\theta_1}{\theta_1+\theta_2} + \frac{\theta_2}{\theta_1+\theta_2} = 1$. Then since A is convex,

$$\tilde{x} = \frac{\theta_1}{\theta_1 + \theta_2} x^1 + \frac{\theta_2}{\theta_1 + \theta_2} x^2 \in A$$

Again, since $\theta_1 + \theta_2 \geq 0$, $\theta_3 \geq 0$, and $\theta_1 + \theta_2 + \theta_3 = 1$, convexity of A implies that

$$(\theta_1 + \theta_2)\tilde{x} + \theta_3 x^3 \in A$$

This is equivalent to $\theta_1 x^1 + \theta_2 x^2 + \theta_3 x^3 \in A$, which is what we wanted to show.

(b) As in part (a), if $\theta_1 = 0$, $\theta_2 = 0$, or $\theta_3 = 0$ the result will follow directly from the concavity of f on A. So, assume that $\theta_1 \neq 0$, $\theta_2 \neq 0$, and $\theta_3 \neq 0$. We can get the result by twice applying the concavity of f on A. Note that the exact conditions we need to verify before applying concavity have been shown in part (a) to be satisfied.

$$\begin{aligned}
f\left((\theta_1 + \theta_2)\tilde{x} + \theta_3 x^3\right) &\geq (\theta_1 + \theta_2) f(\tilde{x}) + \theta_3 f(x^3) \\
&= (\theta_1 + \theta_2)\ f\left(\frac{\theta_1}{\theta_1 + \theta_2} x^1 + \frac{\theta_2}{\theta_1 + \theta_2} x^2\right) + \theta_3 f(x^3) \\
&\geq (\theta_1 + \theta_2)\left[\frac{\theta_1}{\theta_1 + \theta_2} f(x^1) + \frac{\theta_2}{\theta_1 + \theta_2} f(x^2)\right] + \theta_3 f(x^3) \\
&= \theta_1 f(x^1) + \theta_2 f(x^2) + \theta_3 f(x^3)
\end{aligned}$$

Problem 33 (Test for Concavity)**.**

Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by:

$$f(x,y) = Ax^a y^b \text{ for all } (x,y) \in \mathbb{R}^2_+$$

where A, a and b are positive parameters. [It is known that f is continuous on $\mathbb{R}^2_+$ and twice continuously differentiable on $\mathbb{R}^2_{++}$.]

(a) Show that f is concave on $\mathbb{R}^2_+$ if $(a+b) \le 1$.

(b) Show that f is not concave on $\mathbb{R}^2_+$ if $(a+b) > 1$.

Solution.

(a) Testing for concavity by testing for negative semi-definiteness of the Hessian of f only works on open sets, but we want to show that f is concave on $\mathbb{R}^2_+$, which is not open. Showing that f is concave on some open set that includes $\mathbb{R}^2_+$ will not work, as we will see from the calculations below. So, consider two cases.

Case 1: Since f is twice continuously differentiable on the open and convex set $\mathbb{R}^2_{++}$, we can show that f is concave on $\mathbb{R}^2_{++}$ by showing that the Hessian of f is negative semi-definite on $\mathbb{R}^2_{++}$. The Hessian of f at any $(x,y) \in \mathbb{R}^2_{++}$ is

$$H_f(x,y) = \begin{bmatrix} a(a-1)Ax^{a-2}y^b & abAx^{a-1}y^{b-1} \\ abAx^{a-1}y^{b-1} & b(b-1)Ax^a y^{b-2} \end{bmatrix}$$

Now, $a(a-1)Ax^{a-2}y^b < 0$ and $b(b-1)Ax^a y^{b-2} < 0$ for all $(x,y) \in \mathbb{R}^2_{++}$ because $a \in (0,1)$, $b \in (0,1)$, and $A > 0$. The determinant of the Hessian of f at any $(x,y) \in \mathbb{R}^2_{++}$ is

$$\begin{aligned} \det H_f(x,y) &= ab(a-1)(b-1)A^2x^{2a-2}y^{2b-2} - a^2b^2A^2x^{2a-2}y^{2b-2} \\ &= A^2x^{2a-2}y^{2b-2}ab((a-1)(b-1)-ab) \\ &= A^2x^{2a-2}y^{2b-2}ab(1-a-b) \\ &\ge 0 \end{aligned}$$

since $a \in (0,1)$, $b \in (0,1)$, and $a+b \le 1$. This shows that the Hessian of f is negative semi-definite on $\mathbb{R}^2_{++}$, which shows that f is concave on $\mathbb{R}^2_{++}$.

Case 2: Consider $(x_1,y_1) \in \mathbb{R}^2_+$, $(x_2,y_2) \in \mathbb{R}^2_+$, and $\theta \in [0,1]$. Because $\mathbb{R}^2_+$ is convex, we know that $\theta(x_1,y_1)+(1-\theta)(x_2,y_2) \in \mathbb{R}^2_+$. Assume that either $x_1 = 0$ or $y_1 = 0$, so that (x_1, y_1) is on the boundary of the set $\mathbb{R}^2_+$. Given the specific f we are considering, this means that $f(x_1, y_1) = 0$. Now,

$$\begin{aligned}
& f(\theta(x_1,y_1) + (1-\theta)(x_2,y_2)) \\
&= f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\
&= A(\theta x_1 + (1-\theta)x_2)^a (\theta y_1 + (1-\theta)y_2)^b \\
&\geq A((1-\theta)x_2)^a ((1-\theta)y_2)^b && (\text{since } a > 0 \text{ and } b > 0) \\
&= A(1-\theta)^{a+b} x_2^a y_2^b \\
&= (1-\theta)^{a+b} f(x_2,y_2) \\
&\geq (1-\theta) f(x_2,y_2) && (\text{since } 1-\theta \in [0,1] \text{ and } a+b \leq 1) \\
&= \theta f(x_1,y_1) + (1-\theta) f(x_2,y_2) && (\text{since } f(x_1,y_1) = 0)
\end{aligned}$$

Combining the two cases, we have that f is concave on $\mathbb{R}^2_+$.

(b) Recall from part (a) that the determinant of the Hessian of f at any $(x,y) \in \mathbb{R}^2_{++}$ is

$$\det H_f(x,y) = A^2 x^{2a-2} y^{2b-2} ab(1-a-b) < 0$$

since $a > 0$, $b > 0$, and $a + b > 1$. This shows that the Hessian of f cannot be negative semi-definite at any point $(x,y) \in \mathbb{R}^2_{++}$. Therefore f is not concave on $\mathbb{R}^2_{++}$, so f is not concave on $\mathbb{R}^2_+$.

Problem 34 (Characterization of Quasi-Concave Functions)**.**

Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a real valued function.

(a) Show that if f is quasi-concave on $\mathbb{R}^n_+$, then for every $\alpha \in \mathbb{R}$, the set:

$$U(\alpha) = \{x \in \mathbb{R}^n_+ : f(x) \geq \alpha\}$$

is a convex set.

(b) Show that if for every $\alpha \in \mathbb{R}$, the set:

$$U(\alpha) = \{x \in \mathbb{R}^n_+ : f(x) \geq \alpha\}$$

is a convex set, then f is quasi-concave on $\mathbb{R}^n_+$.

Solution.

(a) For all α such that $U(\alpha)$ is the empty set or has only one element, the result holds trivially. So, pick some $\alpha \in \mathbb{R}$ such that $U(\alpha)$ has at least two distinct elements. Let $x^1, x^2 \in U(\alpha)$ and $\theta \in [0,1]$. We have by the definition of $U(\alpha)$ that $f(x^1) \geq \alpha$ and $f(x^2) \geq \alpha$. Suppose without loss of generality that $f(x^1) \geq f(x^2)$. Since f is quasi-concave on $\mathbb{R}^n_+$, we have that $f(\theta x^1 + (1-\theta)x^2) \geq f(x^2) \geq \alpha$, so $\theta x^1 + (1-\theta)x^2 \in U(\alpha)$. Thus $U(\alpha)$ is a convex set.

(b) Let $x^1, x^2 \in \mathbb{R}^n_+$, $\theta \in [0,1]$, and assume without loss of generality that $f(x^1) \geq f(x^2)$. Since $f(x^2) \in \mathbb{R}$ and $U(\alpha)$ is convex for all $\alpha \in \mathbb{R}$, the set

$$U(f(x^2)) = \left\{x \in \mathbb{R}^n_+ \mid f(x) \geq f(x^2)\right\}$$

is convex. Now, since $x^1 \in U(f(x^2))$, $x^2 \in U(f(x^2))$, and $U(f(x^2))$ is convex, we have that $\theta x^1 + (1-\theta)x^2 \in U(f(x^2))$. By the definition of $U(f(x^2))$, this is equivalent to $f(\theta x^1 + (1-\theta)x^2) \geq f(x^2)$, which shows that f is quasi-concave on $\mathbb{R}^n_+$.

Problem 35 (Testing for Quasi-Concavity)**.**

(a) Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by:

$$f(x,y) = Ax^a y^b \text{ for all } (x,y) \in \mathbb{R}^2_+$$

where A, a and b are positive parameters. [It is known that f is continuous on $\mathbb{R}^2_+$ and twice continuously differentiable on $\mathbb{R}^2_{++}$.] Show that f is quasi-concave on $\mathbb{R}^2_+$.

(b) Let $g : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by:

$$g(x,y) = x^a + y^b \text{ for all } (x,y) \in \mathbb{R}^2_+$$

where a and b are parameters, with $a > 1$ and $b > 1$. Show that g is increasing on $\mathbb{R}^2_+$ but it is not quasi-concave on $\mathbb{R}^2_+$.

Solution.

(a) As in Problem 4, part (a), we must consider two cases, since $\mathbb{R}^2_+$ is not open and we will not be able to show that f is quasi-concave on some open set that contains $\mathbb{R}^2_+$.

Case 1: Since f is twice continuously differentiable on the open and convex set $\mathbb{R}^2_{++}$, we can show that f is quasi-concave on $\mathbb{R}^2_{++}$ by showing that for all $(x,y) \in \mathbb{R}^2_{++}$, the second leading principal minor of the bordered Hessian of f at (x,y) is strictly negative and the determinant of the bordered Hessian of f at (x,y) is strictly positive. The bordered Hessian of f at any $(x,y) \in \mathbb{R}^2_{++}$ is

$$B_f(x,y) = \begin{bmatrix} 0 & aAx^{a-1}y^b & bAx^ay^{b-1} \\ aAx^{a-1}y^b & a(a-1)Ax^{a-2}y^b & abAx^{a-1}y^{b-1} \\ bAx^ay^{b-1} & abAx^{a-1}y^{b-1} & b(b-1)Ax^ay^{b-2} \end{bmatrix}$$

Now, for all $(x,y) \in \mathbb{R}^2_{++}$ the second leading principal minor of this matrix is

$$\begin{vmatrix} 0 & aAx^{a-1}y^b \\ aAx^{a-1}y^b & a(a-1)Ax^{a-2}y^b \end{vmatrix} = -a^2A^2x^{2a-2}y^{2b} < 0$$

since $a > 0$ and $A > 0$. The determinant of the bordered Hessian of f at any $(x,y) \in \mathbb{R}^2_{++}$ is

$$\begin{aligned} \det B_f(x,y) &= -aAx^{a-1}y^b\left[ab(b-1)A^2x^{2a-1}y^{2b-2} - ab^2A^2x^{2a-1}y^{2b-2}\right] \\ &\quad + bAx^ay^{b-1}\left[a^2bA^2x^{2a-2}y^{2b-1} - ab(a-1)A^2x^{2a-2}y^{2b-1}\right] \\ &= -a^2bA^3x^{3a-2}y^{3b-2}(b-1-b) + ab^2A^3x^{3a-2}y^{3b-2}(a-a+1) \\ &= a^2bA^3x^{3a-2}y^{3b-2} + ab^2A^3x^{3a-2}y^{3b-2} \\ &= abA^3x^{3a-2}y^{3b-2}(a+b) \\ &> 0 \end{aligned}$$

since $a > 0$, $b > 0$, and $A > 0$. This shows that f is quasi-concave on $\mathbb{R}^2_{++}$.

Case 2: Consider $(x_1,y_1) \in \mathbb{R}^2_+$, $(x_2,y_2) \in \mathbb{R}^2_+$, and $\theta \in [0,1]$. Because $\mathbb{R}^2_+$ is convex, we know that $\theta(x_1,y_1) + (1-\theta)(x_2,y_2) \in \mathbb{R}^2_+$. Assume that either $x_1 = 0$ or $y_1 = 0$, so that (x_1,y_1) is on the boundary of the set $\mathbb{R}^2_+$. Given the specific f we are considering, this means that $f(x_2,y_2) \geq f(x_1,y_1) = 0$. Now,

$$f(\theta(x_1,y_1) + (1-\theta)(x_2,y_2)) \geq 0 = f(x_1,y_1)$$

Combining the two cases, we have that f is quasi-concave on $\mathbb{R}^2_+$.

(b) Note that since g is defined on $\mathbb{R}^2_+$, which is not open, we can only consider derivatives on the open set $\mathbb{R}^2_{++}$. For all $(x,y) \in \mathbb{R}^2_{++}$, we have that $D_1 g(x,y) = ax^{a-1} > 0$ since $a > 1$ and $D_2 g(x,y) = by^{b-1} > 0$ since $b > 1$. This shows that g is increasing on $\mathbb{R}^2_{++}$. We can check directly that g is increasing on the boundary. For all $x > 0$ and $\bar{y} \geq 0$, we have that $g(x,\bar{y}) - g(0,\bar{y}) = x^a > 0$. For all $\bar{x} \geq 0$ and $y > 0$, we have that $g(\bar{x},y) - g(\bar{x},0) = y^b > 0$. Combined with the result that g is increasing on $\mathbb{R}^2_{++}$, we have that g is increasing on $\mathbb{R}^2_+$.

Since g is twice continuously differentiable on the open and convex set $\mathbb{R}^2_{++}$, quasi-concavity of g on $\mathbb{R}^2_{++}$ would imply that the determinant of the bordered Hessian of g at any $(x,y) \in \mathbb{R}^2_{++}$ is non-negative. But we have that at any $(x,y) \in \mathbb{R}^2_{++}$,

$$\begin{aligned}
\det B_g(x,y) &= \begin{vmatrix} 0 & ax^{a-1} & by^{b-1} \\ ax^{a-1} & a(a-1)x^{a-2} & 0 \\ by^{b-1} & 0 & b(b-1)y^{b-2} \end{vmatrix} \\
&= -ax^{a-1}\left[ab(b-1)x^{a-1}y^{b-2}\right] + by^{b-1}\left[-ab(a-1)x^{a-2}y^{b-1}\right] \\
&= -a^2b(b-1)x^{2a-2}y^{b-2} - ab^2(a-1)x^{a-2}y^{2b-2} \\
&< 0
\end{aligned}$$

since $a > 1$ and $b > 1$. Therefore g is not quasi-concave on $\mathbb{R}^2_{++}$, so g is not quasi-concave on $\mathbb{R}^2_{+}$.

References

This material is based on *Convex Structures and Economic Theory* by *H. Nikaido* (Chapter 1); *Mathematical Economics* by *A. Takayama* (Chapters 0,1); *Mathematical Analysis* by *T. Apostol (Chapters 4,6); The Elements of Real Analysis* by *Robert Bartle* (Chapters 4,7); and *Non-Linear Programming* by *O.L. Mangasarian* (Chapters 2, 3, 4, 6 and 9).

Part III

Classical Optimization Theory

Chapter 8

Unconstrained Optimization

8.1 Preliminaries

We will present below the elements of "classical optimization theory". We will concentrate on characterizing points of *maximum* of a function of several variables; the theory which characterizes points of *minimum* of such a function can be inferred without too much difficulty.

Our first task will be to look at the theory of *unconstrained* maximization, and discuss the relevant necessary and sufficient conditions for such an unconstrained maximum to occur. This is the subject matter of this chapter. Our second task will be to present the theory of *constrained* maximization, where the only constraints are *equality* constraints. This is the theory involving the well-known "Lagrange multiplier method", and is taken up in Chapter 9.

Unconstrained Maximization Theory

Our framework is the following. There is a set $A \subset \mathbb{R}^n$; there is a function $f : A \to \mathbb{R}$. We are interested in identifying points in A at which the function attains a (local or global) maximum.

Local and Global Maximum

Let $A \subset \mathbb{R}^n$, and let f be a function from A to $\mathbb{R}$. A point $c \in A$ is said to be a *point of local maximum* of f if there exists $\delta > 0$, such that $f(c) \geq f(x)$ for all $x \in A$ which satisfy $\|x - c\| < \delta$. It is said to be a *point of global maximum* of f if $f(c) \geq f(x)$ for all $x \in A$.

8.2 Necessary Conditions for a Local Maximum

We will present two necessary conditions for a local maximum. One is a condition on the first-order partial derivatives of the relevant function (called “first-order conditions”); the other is a condition on the second-order partial derivatives of the relevant function (called “second-order necessary conditions”).

Theorem 35. *Let A be an open set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a continuously differentiable function on A. If $c \in A$ is a point of local maximum of f, then*

$$\nabla f(c) = 0 \tag{8.1}$$

Remark: The n equations given by (8.1) are called the first-order conditions for a local maximum.

Theorem 36. *Let A be an open set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a twice continuously differentiable function on A. If $c \in A$ is a point of local maximum of f, then*

$$H_f(c) \text{ is negative semi-definite} \tag{8.2}$$

Remark: The condition (8.2) is called the second-order necessary condition for a local maximum.

Necessary conditions like (8.1) and (8.2) stated above are useful because they help us to rule out points where a local maximum cannot occur, thereby narrowing our search for points where a local maximum does occur. The following two examples illustrate this point.

Examples:

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1 - x^2$ for all $x \in \mathbb{R}$. Then $\mathbb{R}$ is an open set, and f a continuously differentiable function on $\mathbb{R}$. Consider the point $c = 1$. We calculate $f'(c) = f'(1) = -2(1) = -2$. By Theorem 35, we can therefore conclude that $c = 1$ is *not* a point of local maximum of f.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1 - 2x + x^2$ for all $x \in \mathbb{R}$. Then $\mathbb{R}$ is an open set, and f a twice continuously differentiable function on $\mathbb{R}$. Consider the point $c = 1$. We can calculate $f'(c) = f'(1) = -2 + 2(1) = 0$, so condition (8.1) of Theorem 35 is satisfied. *Notice that Theorem 35, by itself, fails to be of any help at this point.* We cannot conclude from Theorem 35 that $c = 1$ is a point of local maximum; we cannot conclude from Theorem 35 that $c = 1$ is not a point of local maximum. Theorem 36, however, is useful at this point. We can calculate $f''(c) = f''(1) = 2 > 0$, and so condition (8.2) of Theorem 36 is violated. Consequently, by Theorem 36, we can conclude that $c = 1$ is *not* a point of local maximum of f.

8.3 Sufficient Conditions for a Local Maximum

We present below a set of sufficient conditions for a local maximum.

Theorem 37. *Let A be an open set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a twice continuously differentiable function on A. If $c \in A$, satisfies*

$$\nabla f(c) = 0 \tag{8.3}$$

and

$$H_f(c) \text{ is negative definite} \tag{8.4}$$

then c is a point of local maximum of f.

Remark: Condition (8.3) of Theorem 37 is called the second-order sufficient condition for a local maximum.

It should be noted that condition (8.4) cannot be weakened to condition (8.2) in the statement of Theorem 37. The following example illustrates this point.

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$ for all $x \in \mathbb{R}$. Then $\mathbb{R}$ is an open set, and f is a twice continuously differentiable function on $\mathbb{R}$. At $c = 0$, $f'(c) = f'(0) = 0$, and $f''(c) = f''(0) = 0$, so condition (8.3) and condition (8.2) are satisfied. But c is clearly not a point of local maximum of f since f is an increasing function on $\mathbb{R}$.

It may also be observed that condition (8.2) cannot be strengthened to condition (8.4) in the statement of Theorem 36. The following example illustrates this point.

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x^4$ for all $x \in \mathbb{R}$. Then $\mathbb{R}$ is an open set, and f is a twice continuously differentiable function on $\mathbb{R}$. Clearly, $c = 0$ is a point of local maximum of f [since $f(0) = 0$, while $f(x) < 0$ for all $x \neq 0$]. One can calculate that $f'(c) = f'(0) = 0$, and $f''(c) = f''(0) = 0$. Thus conditions (8.1) and (8.2) are satisfied, but (8.4) is violated.

The outcome of this discussion is the following: the second-order necessary conditions for a local maximum are different from (weaker than) the second-order sufficient conditions for a local maximum. This simply reflects the fact that, in general, the first and second derivatives of a function at a point do not capture all aspects relevant to the occurrence of a local maximum of the function at that point.

The sufficient conditions of Theorem 37 enable us to find points of local maximum of a function, as the following example shows.

Example:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = 2x_1x_2 - 2x_1^2 - x_2^2$ for all $(x_1, x_2) \in \mathbb{R}^2$. Then $\mathbb{R}^2$ is an open set, and f is a twice continuously differentiable function. If we write down the condition (8.1) [or (8.3)] at a point $(c_1, c_2) \in \mathbb{R}^2$, we get

$$D_1 f(c_1, c_2) = 2c_2 - 4c_1 = 0$$

$$D_2 f(c_1, c_2) = 2c_1 - 2c_2 = 0$$

Thus condition (8.3) is satisfied if and only if $c_1 = c_2 = 0$.

Next, we can find the Hessian of f at $(c_1, c_2) \in \mathbb{R}^2$:

$$H_f(c_1, c_2) = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}$$

Now, $-4 < 0$, and $(-4)(-2) - (2)(2) = 8 - 4 > 0$. So $H_f(c_1, c_2)$ is negative definite for each $(c_1, c_2) \in \mathbb{R}^2$. Thus, condition (8.4) is clearly satisfied. It follows from Theorem 37 that $(c_1, c_2) = (0, 0)$ is a point of local maximum of f.

8.4 Sufficient Conditions for a Global Maximum

While several sets of sufficient conditioins for a global maximum can be developed, the following two are among the most useful.

Theorem 38. *Let A be an open convex set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a continuously differentiable function on A. If $c \in A$ satisfies*

$$\nabla f(c) = 0 \tag{8.5}$$

and f is a concave function on A, then c is a point of global maximum of f.

To see this note that for all $x \in A$,

$$f(x) - f(c) \le (x - c)\nabla f(c) \tag{8.6}$$

since f is concave and continuously differentiable on A [See Theorem 9 of Chapter 7 on "Convex Analysis"]. Using (8.5) in (8.6), we get $f(x) \le f(c)$ for all $x \in A$, so c is a point of global maximum of f.

Theorem 39. *Let A be an open convex set in $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a twice continuously differentiable function on A. If $c \in A$ satisfies*

$$\nabla f(c) = 0 \tag{8.7}$$

and

$$H_f(x) \text{ is negative semi-definite for all } x \in A \tag{8.8}$$

then c is a point of global maximum of f.

To see this, note that (8.8) ensures that f is concave on A; so the result follows readily from Theorem 38.

It is worth noting that Theorem 38 (or 39) might be applicable in cases where Theorem 37 is not applicable. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x^4$. Here, one notes that $f'(0) = 0$ and $f''(x) = -12x^2 \leq 0$ for *all* x. Thus applying Theorem 38 (or 39), we can conclude that $x = 0$ is a point of global maximum, and hence of local maximum. But the conclusion that $x = 0$ is a point of local maximum cannot be derived from Theorem 37, since $f''(0) = 0$.

8.5 The Method of Least Squares

Suppose we are given n points (x_i, y_i), $i = 1, \ldots, n$ in $\mathbb{R}^2$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$ for all $x \in \mathbb{R}$. We wish to find a function f (that is, we want to choose $a \in \mathbb{R}$ and $b \in \mathbb{R}$) such that the quantity

$$\sum_{i=1}^{n} [f(x_i) - y_i]^2$$

is minimized.

We can set up the problem as an *unconstrained maximization problem* as follows. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(a,b) = -\sum_{i=1}^{n} [ax_i + b - y_i]^2$$

The maximization problem then is

$$\underset{(a,b)}{Max} F(a,b)$$

F is twice continuously differentible on $\mathbb{R}^2$, and we can calculate

$$\begin{aligned}
D_1F &= -2\sum_{i=1}^{n}[ax_i + b - y_i]x_i = -2\sum_{i=1}^{n}[ax_i^2 + bx_i - x_iy_i] \\
D_2F &= -2\sum_{i=1}^{n}[ax_i + b - y_i] \\
D_{11}F &= -2\sum_{i=1}^{n}x_i^2; \qquad D_{12}F = -2\sum_{i=1}^{n}x_i \\
D_{21}F &= -2\sum_{i=1}^{n}x_i; \qquad D_{22}F = -2n
\end{aligned}$$

Thus, the determinant of the Hessian of F is

$$\det(H_F(a,b)) = 4n\sum_{i=1}^{n}x_i^2 - 4[\sum_{i=1}^{n}x_i]^2$$

Now, by the Cauchy-Schwarz inequality,

$$\left|\sum_{i=1}^{n} x_i\right| \leq [\sum x_i^2]^{1/2} n^{1/2}$$

so

$$[\sum_{i=1}^{n} x_i]^2 \leq n \sum x_i^2$$

and consequently, $\det(H_F(a,b)) \geq 0$. Since $D_{11}F(a,b) \leq 0$, $D_{22}F(a,b) \leq 0$, and $\det(H_F(a,b)) \geq 0$, $H_F(a,b)$ is negative semi-definite. Consequently, if (a^*, b^*) satisfies the first-order conditions, then (a^*, b^*) is a point of global maximum of f by Theorem 39. The first-order conditions are

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i \tag{8.9}$$

$$a\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i \tag{8.10}$$

Denoting $(1/n)\sum_{i=1}^{n} x_i$ by $\overline{x}$ and $(1/n)\sum_{i=1}^{n} y_i$ by $\overline{y}$ [the means of x and y respectively], we get from (8.10) that

$$a\overline{x} + b = \overline{y} \tag{8.11}$$

Using this in (8.9) leads to

$$a\sum_{i=1}^{n} x_i^2 + (\overline{y} - a\overline{x})n\overline{x} = \sum_{i=1}^{n} x_i y_i \tag{8.12}$$

Thus,

$$\begin{aligned} \frac{(1/n)\sum_{i=1}^{n} x_i y_i - \overline{x}\,\overline{y}}{(1/n)\sum_{i=1}^{n} x_i^2 - \overline{x}^2} &= a \\ \overline{y} - a\overline{x} &= b \end{aligned}$$

solves the problem, provided not all the x_i are the same.

8.6 The Envelope Theorem

Let X be an open convex set of $\mathbb{R}^n$, and A be an open subset of $\mathbb{R}^m$. Let f be a twice continuously differentiable function on $X \times A$ which is quasi-concave on X, given $a \in A$. We interpret $(x_1, \ldots, x_n)$ as the "variables" and $(a_1, \ldots, a_m)$ as the "parameters".

Given $a \in A$, we can formulate the following maximization problem:

$$\underset{x \in X}{Max} f(x;a) \tag{P}$$

Let $a^* \in A$. Suppose $\bar{x} \in X$ is a point such that

$$\nabla f(\bar{x};a^*) = 0$$

$$H_f(\bar{x};a^*) \textit{is negative definite}$$

Using Theorem 37, we then know that $\bar{x}$ is a point of local maximum of f, given a^*. Given $a^* \in A$, f is quasi-concave on X, so $\bar{x}$ can be shown to be a solution of problem (P), given a^*. (Can you show this?) In fact, $\hat{x}$ can be shown to be the unique solution to (P), given a^*.

Notice that by the implict function theorem, there is an open set B containing a^*, and an open set C containing $\bar{x}$, and a unique function $g: B \to C$, such that

(i) $\bar{x} = g(a^*)$

(ii) $\nabla f[g(a);a] = 0 \quad for\, a \in B$

Also, g is continuously differentiable on B. Furthermore, one can choose the open set B so as to ensure

(iii) $H_f[g(a;a]$ is negative definite for $a \in B$

For $\hat{a} \in B$, [$\hat{a}$ not necessarily equal to a^*], we then note by (ii) and (iii), that $g(\hat{a})$ is a point of local maximum of f, given $\hat{a}$. And since f is quasi-concave on X [given $\hat{a} \in B$], $g(\hat{a})$ is the unique solution to problem (P), given $\hat{a}$.

We, can now define for problem (P), the *value*, $V: B \to \mathbb{R}$ by

$$V(a) = \underset{x \in X}{Max} f(x;a)$$

Furthermore, we can define the *maximizer* $h: B \to X$ by

$$h(a) = \{z \in X : f(z;a) \geq f(x;a) \;\; for\, all\, x \in X\}$$

What we have just established in the previous paragraph is that

$$V(a) = f[g(a);a] \quad for\, a \in B \tag{8.13}$$

$$h(a) = g(a) \qquad for\, a \in B \tag{8.14}$$

Two questions which now arise quite naturally are:
(1) If a changes a little from a^*, how will the maximized value of f change?
(2) If a changes a little from a^*, how will the maximizer change?

Suppose, for concreteness, we want to answer these questions for a parameter change, where only a_1 changes. Then question (1) can be answered by finding out $D_1V(a^*)$; and question (2) can be answered by finding out the vector $[D_1g^1(a^*), D_1g^2(a^*), ..., D_1g^n(a^*)]$.

Using (8.13), and the chain-rule,

$$D_1V(a^*) = \sum_{i=1}^{n} D_i f[g(a^*); a^*] D_1 g^i(a^*) + D_{n+1} f[g(a^*); a^*]$$

Using (ii), we have $D_i f[g(a^*); a^*] = 0$ for $i = 1, \ldots, n$. Hence,

$$D_1V(a^*) = D_{n+1} f[\bar{x}; a^*] \tag{8.15}$$

This result is known as the "envelope theorem", and it answers question (1) above.

To answer question (2), use (ii) to obtain (employing again the chain-rule),

$$\sum_{i=1}^{n} D_{1i} f[g(a^*); a^*] D_1 g^i(a^*) + D_{1n+1} f[g(a^*); a^*] = 0$$
$$-------------------------$$
$$\sum_{i=1}^{n} D_{ni} f[g(a^*); a^*] D_1 g^i(a^*) + D_{nn+1} f[g(a^*); a^*] = 0$$

Then using (iii), we can employ Cramer's Rule to obtain the vector $[D_1g^1(a^*), \ldots, D_1g^n(a^*)]$. This answers question (2).

As an application, consider $X \equiv \mathbb{R}_{++}$, and $A = \mathbb{R}^2_{++}$, with $f : X \times A \to \mathbb{R}$ given by

$$f(x; p, w) = p\phi(x) - wx$$

We interpret $\phi : \mathbb{R}_+ \to \mathbb{R}$ as the production function [with x as the input level and $\phi(x)$ the corresponding output level], p as the output price and w as the input price. Thus, $f(x; p, w)$ is the profit if x of input is employed when the output price is p, and input price is w.

Assuming $\phi(x) = 2x^{1/2}$ for $x \geq 0$, we observe that f is twice continuously differentiable on $X \times A$, and given $(p, w) \in A$, f is quasi-concave on X, since ϕ is concave on X.

Let $(p^*, w^*) \in A$. Then, we note that

$$0 = D_1 f(\bar{x}; p^*, w^*) = (p^*/\bar{x}^{1/2}) - w^*$$

when $\bar{x} = (p^*/w^*)^2$. Furthermore,

$$D_{11} f(\bar{x}; p^*, w^*) = -[p^*/2\bar{x}^{3/2}] < 0$$

Then applying the implicit function theorem, we can obtain an open set B containing (p^*, w^*) and an open set C containing $\bar{x}$, and a unique function $g : B \to C$, such that

(i) $\bar{x} = g(p^*, w^*)$

(ii) $D_1 f[g(p,w); p,w] = 0 \; for \; (p,w) \in B$

Further, g is continuously differentiable on B.

The *value* for this problem is known as the "profit function", and by (8.13), it is given by

$$\pi(p,w) = f[g(p,w); p,w]$$

The maximizer is known as the (input) *demand function*, and by (8.14), it is given by

$$x(p,w) = g(p,w)$$

If we answer question (1) in this framework, we get, from (8.15),

$$D_1\pi(p^*, w^*) = \phi(g(p^*, w^*)) = \phi(x(p^*, w^*))$$

$$D_2\pi(p^*, w^*) = -g(p^*, w^*) = -x(p^*, w^*)$$

This is generally known as "Hotelling result" in the theory of the firm.

8.7 Worked Out Problems on Chapter 8

Problem 36 (Unconstrained Optimization: First-Order Conditions)**.**

Here is a statement of the first-order condition for a maximum of real valued functions of a real variable.

Theorem:

Let f be a continuously differentiable real valued function on the interval $A = (a, b)$. If $c \in A$ satisfies $f(c) \geq f(t)$ for all $t \in A$, then $f'(c) = 0$.

We want to use this theorem to prove the following version of the first-order condition for a local maximum of real valued functions of several real variables.

Corollary:

Let C be an open subset of $\mathbb{R}^n$, and let F be a continuously differentiable real valued function on C. If $\bar{x} \in C$ is a point of local maximum of F, then $\nabla F(\bar{x}) = 0$.

Proceed with the following steps.

(a) Since $\bar{x} \in C$ is a point of local maximum of F, we can find $r > 0$ such that $B \equiv B(\bar{x}, 2r) \subset C$, and $F(\bar{x}) \geq F(x)$ for all $x \in B$.

(b) Pick any $k \in \{1, \cdots, n\}$, and define $a(k) = \bar{x} - re^k, b(k) = \bar{x} + re^k$, where e^k is the k-th unit vector in $\mathbb{R}^n$. Then, by definition of B, we have $a(k) \in B$ and $b(k) \in B$. And, since B is a convex set, we have $[tb(k) + (1-t)a(k)] \in B$ for all $t \in I \equiv [0, 1]$. For $t \in I$, define $f(t) = F(tb(k) + (1-t)a(k))$, and note that this function is well defined, since F is defined on C (which contains the set B). Define $A = (0, 1)$, and verify that f is continuously differentiable on A.

(c) Show that $f(1/2) = F(\bar{x})$, and $f(1/2) \geq f(t)$ for all $t \in A$.

(d) Use the Theorem (stated above) to obtain $f'(1/2) = (2r)D_k F(\bar{x}) = 0$. Verify that this proves the Corollary.

Solution.

(a) Since $\bar{x} \in C$ is a point of local maximum of F, there is some $\beta > 0$ such that for all $x \in C$ satisfying $d(x, \bar{x}) < \beta$, we have $F(x) \leq F(\bar{x})$. And since C is open, there is some $\gamma > 0$ such that for all $x \in \mathbb{R}^n$ satisfying $d(x, \bar{x}) < \gamma$, we have $x \in C$. Now, let $r = \frac{1}{2}\min\{\beta, \gamma\}$, so that $B \equiv B(\bar{x}, 2r) = \{x \in \mathbb{R}^n \mid d(x, \bar{x}) < 2r\} \subset C$. Then for all $x \in B$, we have that $F(\bar{x}) \geq F(x)$.

(b) For any $k \in \{1, \ldots, n\}$, define

$$a(k) = \bar{x} - re^k, \quad b(k) = \bar{x} + re^k$$

Now, $a(k) \in B$ and $b(k) \in B$ because $d(a(k), \bar{x}) = d(b(k), \bar{x}) = d(re^k, 0) = r < 2r$. Since the open ball B is convex, we have that $tb(k) + (1-t)a(k) \in B$ for all $t \in I = [0,1]$. Define $f: I \to \mathbb{R}$ by

$$f(t) = F\big(tb(k) + (1-t)a(k)\big) \quad \text{for all } t \in I$$

Since $tb(k) + (1-t)a(k) \in B \subset C$ for all $t \in I$ and F is defined on C, the function f is well-defined. Let $A = (0,1)$. We showed in problem 1 of Problem Set 8 that f is continuous on $[0,1]$, so it follows that f is continuous on A. Since F is continuously differentiable on the open set C, we can use the Chain Rule to differentiate f on A. For all $t \in A$, we have

$$\begin{aligned}
f'(t) &= \sum_{i=1}^{n} \left[D_i F\big(tb(k) + (1-t)a(k)\big) \frac{\partial}{\partial t}\big(tb_i(k) + (1-t)a_i(k)\big) \right] \\
&= \sum_{i=1}^{n} \left[\big(b_i(k) - a_i(k)\big) D_i F\big(tb(k) + (1-t)a(k)\big) \right] \\
&= \big(b(k) - a(k)\big) \nabla F\big(tb(k) + (1-t)a(k)\big) \\
&= (2re^k) \nabla F\big(\bar{x} + (2t-1)re^k\big) \\
&= 2rD_k F\big(\bar{x} + (2t-1)re^k\big)
\end{aligned}$$

Since F is continuously differentiable on C and $tb(k) + (1-t)a(k) = \bar{x} + (2t-1)re^k \in C$ whenever $t \in A$, we have that $f'(t)$ is continuous on A. So f is continuously differentiable on A.

(c) Since $tb(k) + (1-t)a(k) = \bar{x} + (2t-1)re^k$, we have $f\left(\frac{1}{2}\right) = F(\bar{x})$. And since $tb(k) + (1-t)a(k) \in B$ for all $t \in I$ and $F(\bar{x}) \geq F(x)$ for all $x \in B$, we have by the definition of f that $f\left(\frac{1}{2}\right) \geq f(t)$ for all $t \in I$, and thus for all $t \in A$.

(d) Using the result in part (c), we have by the stated Theorem that $f'\left(\frac{1}{2}\right) = 0$. And from part (b), we have that $f'\left(\frac{1}{2}\right) = 2rD_k F(\bar{x})$. Since $r > 0$, these two equations imply that $D_k F(\bar{x}) = 0$. Because $k \in \{1, \dots, n\}$ was chosen arbitrarily, we have $\nabla F(\bar{x}) = 0$. This proves the Corollary.

Problem 37 (Unconstrained Optimization: Sufficient Conditions for a Global Maximum)**.**

Suppose A is an open convex set in $\mathbb{R}^n$, and $f: A \to \mathbb{R}$ is twice continuously differentiable and quasi-concave on A. Suppose there is $\bar{x}$ in A satisfying:

(i) $\nabla f(\bar{x}) = 0$, and (ii) $H_f(\bar{x})$ is negative definite.

Show that $\bar{x}$ is the unique point of global maximum of f on A.

Solution.

We will prove this by verifying a series of claims. First, we will show that the Hessian of f is negative definite in some neighborhood around $\bar{x}$. Second, we will use Taylor's Theorem to show that $\bar{x}$ is a point of strict local maximum of f. Finally, we will use the quasi-concavity of f on A to show that $\bar{x}$ is the unique point of global maximum of f on A.
Claim 1: There is some $\delta > 0$ such that for all $x \in B(\bar{x}, \delta) = \{x \in \mathbb{R}^n \mid d(x, \bar{x}) < \delta\}$, we have that $x \in A$ and $H_f(x)$ is negative definite.
Proof: For each $r = 1, \ldots, n$, let $F^r : A \to \mathbb{R}$ denote the rth leading principal minor of the Hessian of f. Since F^r is constructed by summing products of second derivatives of f, which are continuous on A, it follows that each F^r is continuous on A.

Since $H_f(\bar{x})$ is negative definite, we know that $F^1(\bar{x}) = D_{11} f(\bar{x}) < 0$. By the continuity of F^1, for $\varepsilon^1 = -F^1(\bar{x}) > 0$ there is some $\delta^1 > 0$ such that whenever $x \in A$ and $d(x, \bar{x}) < \delta^1$, we have $|F^1(x) - F^1(\bar{x})| < \varepsilon^1$, which implies that $F^1(x) < 0$.

We can construct $\delta^2, \ldots, \delta^n$ similarly. Then for each $r = 1, \ldots, n$ we have that for all $x \in A$ satisfying $d(x, \bar{x}) < \delta^r$, it follows that $F^r(x)$ takes the appropriate sign (strictly negative for odd r and strictly positive for even r).

Since A is open, there is some $\gamma > 0$ such that for all $x \in \mathbb{R}^n$ satisfying $d(x, \bar{x}) < \gamma$, we have $x \in A$.

Now, define $\delta = \min\{\delta^1, \ldots, \delta^n, \gamma\} > 0$. Then for all $x \in \mathbb{R}^n$ satisfying $d(x, \bar{x}) < \delta$, we have that $x \in A$ and $H_f(x)$ is negative definite. This proves the claim.
Claim 2: $\bar{x}$ is a point of strict local maximum of f.
Proof: Note that by Theorem 37, the conditions in the problem imply that $\bar{x}$ is a point of local maximum of f. We want to use Claim 1 to argue that we can strengthen this result to show that $\bar{x}$ is a point of strict local maximum of f.

Consider some x' such that $x' \neq \bar{x}$ and $d(x', \bar{x}) < \delta$. By Claim 1, we have that $x' \in A$, so f is well-defined at x'. We want to show that $f(\bar{x}) > f(x')$. Since A is open and convex and $f : A \to \mathbb{R}$ is twice continuously differentiable on A, we can apply Taylor's Theorem, which says that there is $\theta \in (0, 1)$ such that

$$f(x') - f(\bar{x}) = (x' - \bar{x}) \nabla f(\bar{x}) + \frac{1}{2}\left[(x' - \bar{x})' H_f\big(\theta x' + (1-\theta)\bar{x}\big)(x' - \bar{x})\right]$$

We are given that $\nabla f(\bar{x}) = 0$. Also, since $B(\bar{x}, \delta)$ is a convex set and $x', \bar{x} \in B(\bar{x}, \delta)$, we have that $\theta x' + (1-\theta)\bar{x} \in B(\bar{x}, \delta)$, so it follows by Claim 1 that $H_f\big(\theta x' + (1-\theta)\bar{x}\big)$ is negative definite. Because $x' - \bar{x} \neq 0$, this negative definiteness means we can use the above equation to write

$$f(x') - f(\bar{x}) = \frac{1}{2}\left[(x' - \bar{x})' H_f\big(\theta x' + (1-\theta)\bar{x}\big)(x' - \bar{x})\right] < 0$$

This implies $f(x') < f(\bar{x})$, which shows that $\bar{x}$ is a point of strict local maximum of f.
Claim 3: $\bar{x}$ is the unique point of global maximum of f on A.

Proof: Seeking contradiction, suppose the claim does not hold. Then there is some $\tilde{x} \in A$ such that $\tilde{x} \neq \overline{x}$ and $f(\tilde{x}) \geq f(\overline{x})$. Now, we can take some $\lambda \in (0,1)$ sufficiently close to zero such that $x'' = \lambda\tilde{x} + (1-\lambda)\overline{x} \in B(\overline{x}, \delta)$. It follows by Claim 2 that $f(\overline{x}) > f(x'')$. But since $f(\tilde{x}) \geq f(\overline{x})$ and f is quasi-concave on A, we have that $f(x'') = f\big(\lambda\tilde{x} + (1-\lambda)\overline{x}\big) \geq f(\overline{x})$. This is a contradiction, so the claim holds.

References:

This material is based on *The Elements of Real Analysis* by *R. Bartle* (Chapter 7); *Mathematical Analysis* by *T. Apostol* (Chapter 6); and *Mathematical Economics* by *A. Takayama* (Chapter 1).

Chapter 9

Constrained Optimization

9.1 Preliminaries

Let A be a subset of $\mathbb{R}^n$, and f, g be real-valued functions on A. Define the *constraint set*, $C \equiv \{x \in A : g(x) = 0\}$. A point $x^* \in C$ is *a point of local maximum of* f *subject to the constraint* $g(x) = 0$, if there is $\delta > 0$, such that $x \in C \cap B(x^*, \delta)$ implies $f(x) \leq f(x^*)$. A point $x^* \in C$ is *a point of global maximum of f subject to the constraint g(x)=0*, if x^* solves the problem:

$$\begin{aligned} Max\, f(x) \\ subject\, to\, x \in C \end{aligned}$$

9.2 Necessary Conditions for a Constrained Local Maximum

The basic necessary condition for a constrained local maximum is provided by Lagrange's theorem.

Theorem 40. *(Lagrange)*

Let $A \subset \mathbb{R}^n$ be open, and $f : A \to \mathbb{R}, g : A \to \mathbb{R}$ be continuously differentiable functions on A. Suppose x^* is a point of local maximum of f subject to the constraint $g(x) = 0$. Suppose, further, that $\nabla g(x^*) \neq 0$. Then there is $\lambda^* \in \mathbb{R}$ such that

[First-Order Condition] $\qquad \nabla f(x^*) = \lambda^* \nabla g(x^*)$

Remark: There is an easy way of remembering the conclusion of the theorem. We write

$$L(x; \lambda) = f(x) - \lambda g(x)$$

where $L : A\times\mathbb{R} \to \mathbb{R}$. L is known as the "Lagrangian", and λ as the "Lagrange multiplier". Consider now the problem of finding the local maximum in an *unconstrained maximization problem* in which L is the function to be maximized. The first-order conditions are

$$D_i L(x,\lambda) = 0 \qquad for\, i = 1,\ldots,n+1$$

This yields

$$D_i f(x) = \lambda D_i g(x) \qquad i = 1,\ldots,n$$

and

$$g(x) = 0$$

The first n equations can be written as

$$\nabla f(x) = \lambda \nabla g(x)$$

The method described above is known as the "Lagrange multiplier method".

The Constraint Qualification

It is particularly important to check the condition $\nabla g(x^*) \neq 0$, before applying the conclusion of Lagrange's theorem. This condition is known as the *constraint qualification*. Without this condition, the conclusion of Lagrange's theorem would not be valid, as the following example shows.

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = 2x_1 + 3x_2$ for all $(x_1, x_2) \in \mathbb{R}^2$; let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x_1, x_2) = x_1^2 + x_2^2$. Consider the constraint set $C = \{(x_1, x_2) \in \mathbb{R}^2 : g(x_1,x_2) = 0\}$. The only element of this set is (0,0), so $(x_1^*, x_2^*) = (0,0)$ is a point of local maximum of f subject to the constraint $g(x) = 0$.

The conclusion of Lagrange's theorem does not hold here. For, if it did, there would exist $\lambda^* \in \mathbb{R}$ such that

$$\nabla f(0,0) = \lambda^* \nabla g(0,0)$$

But this means that

$$(2,3) = (0,0)$$

which is clearly a contradiction. The problem here is that $\nabla g(x_1^*, x_2^*) = \nabla g(0,0) = (0,0)$, so the constraint qualification is violated.

Theorem 41. *Let $A \subset \mathbb{R}^n$ be open, and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be twice continuously differentiable functions on A. Suppose x^* is a point of local maximum of f subject to the constraint $g(x) = 0$. Suppose, further, that $\nabla g(x^*) \neq 0$. Then there is $\lambda^* \in \mathbb{R}$ such that*

[First-Order Condition} $\nabla f(x^*) = \lambda^* \nabla g(x^*)$
[Second-Order Necessary Condition] $yH_L(x^*, \lambda^*)y \leq 0$ *for all* y satisfying

$$y \nabla g(x^*) = 0$$

where $L(x;\lambda^*) = f(x) - \lambda^* g(x)$ *for all* $x \in A$, *and* H_L *is the* $n \times n$ *Hessian matrix of* L *with respect to* $(x_1, \ldots, x_n)$.

9.3 The Arithmetic Mean-Geometric Mean Inequality

Consider the constrained maximization problem

$$\left.\begin{array}{c} Max \prod_{i=1}^{n} x_i \\ subject\,to \sum_{i=1}^{n} x_i = n \\ and\, x_i \geq 0 \qquad i = 1, \ldots, n \end{array}\right\} \tag{P}$$

Define $C = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1)$. Then C is a non-empty, closed and bounded set in $\mathbb{R}^n$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x_1, \ldots., x_n) = \prod_{i=1}^{n} x_i$. Then F is a continuous function on $\mathbb{R}^n$. By Weierstrass' Theorem, there is $x^* \in C$, such that $F(x) \leq F(x^*)$ for all $x \in C$. That is, x^* solves (P). Clearly, $x_i^* > 0$ for all i. We can therefore conclude that x^* also solves the following problem:

$$\left.\begin{array}{c} Max \prod_{i=1}^{n} x_i \\ subject\,to \sum_{i=1}^{n} x_i = n \\ and\, x_i > 0 \qquad i = 1, \ldots, n \end{array}\right\} \tag{Q}$$

Define $A = \mathbb{R}^n_{++}$; then A is an open subset of $\mathbb{R}^n$. Define $g : \mathbb{R}^n_{++} \to \mathbb{R}$ by $g(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i - n$, and $f : \mathbb{R}^n_{++} \to \mathbb{R}$ by $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$. Then x^* solves (Q), so x^* is a point of local maximum of f subject to the constraint $g(x) = 0$. Also, $\nabla g(x^*) = (1, 1, \ldots, 1) \neq 0$. So, by the Lagrange theorem, there is $\lambda \in \mathbb{R}$, such that

$$\nabla f(x^*) = \lambda \nabla g(x^*)$$

Defining $y^* \equiv \prod_{i=1}^{n} x_i^*$, we obtain

$$[(y^*/x_1^*), \ldots, (y^*/x_n^*)] = [\lambda, \ldots., \lambda]$$

Thus $\lambda > 0$, and $x_i^* = (y^*/\lambda)$ for all i, so that $x_1^* = x_2^* = \ldots = x_n^*$. Since $\sum_{i=1}^{n} x_i^* = n$, we get $x_i^* = 1$ for all i. Thus $\prod_{i=1}^{n} x_i^* = 1$. To summarize, we have now demonstrated that if $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$, and $\sum_{i=1}^{n} x_i = n$, then $\prod_{i=1}^{n} x_i \leq 1$.

Let $a_1, \ldots, a_n$ be n positive numbers. Define $\alpha = \sum_{i=1}^{n} a_i$, and $x_i = [na_i/\alpha]$ for $i = 1, \ldots, n$. Then $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^{n} x_i = n$. So by our conclusion in the preceding paragraph, we have $\prod_{i=1}^{n} x_i \leq 1$. This means

$$\prod_{i=1}^{n} (na_i/\alpha) \leq 1$$

so,

$$\left(\frac{n}{a}\right)^n \prod_{i=1}^{n} a_i \leq 1$$

and,

$$\prod_{i=1}^{n} a_i \leq \left(\frac{\alpha}{n}\right)^n = \left[\frac{\sum_{i=1}^{n} a_i}{n}\right]^n$$

This yields finally

$$(\prod_{i=1}^{n} a_i)^{1/n} \leq \frac{\sum_{i=1}^{n} a_i}{n}$$

which is the Arithmetic Mean - Geometric Mean inequality.

9.4 Sufficient Conditions for a Constrained Local Maximum

Theorem 42. *Let $A \subset \mathbb{R}^n$ be open, and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be twice continuously differentiable functions on A. Suppose $(x^*, \lambda^*) \in C \times \mathbb{R}$ and*

[First-Order Condition] $\nabla f(x^*) = \lambda^* \nabla g(x^*)$
[Second-Order Sufficient Condition] $yH_L(x^*, \lambda^*)y < 0$ *for all* $y \neq 0$ *satisfying* $y \nabla g(x^*) = 0$

where $L(x,\lambda^*) = f(x) - \lambda^* g(x)$ for all $x \in A$ and H_L is the $n \times n$ Hessian matrix of L with respect to $(x_1, ..., x_n)$. Then, x^* is a point of local maximum of f subject to the constraint $g(x) = 0$.

There is a convenient method of checking the *second-order sufficient condition* stated in the above theorem, by checking the signs of the leading principal minors of the relevant "bordered" matrix. This method is stated in the following Proposition.

Proposition 5. *Let B be an $n \times n$ symmetric matrix, and a be an n-vector with $a_1 \neq 0$. Define the $(n+1) \times (n+1)$ matrix S by*

$$S = \begin{bmatrix} 0 & a \\ a & B \end{bmatrix}$$

and let $|S(k)|$ be the $(k+1)^{th}$ leading principal minor of S for $k = 1, \ldots, n$. Then the following two statements are equivalent:
(i) $yBy < 0$ *for all* $y \neq 0$ *such that* $ya = 0$
(ii) $(-1)^k |S(k)| > 0$ *for* $k = 1, \ldots, n$.

9.5 Sufficient Conditions for a Global Maximum

Theorem 43. *Let $A \subset \mathbb{R}^n$ be an open convex set, and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be continuously differentiable functions on A. Suppose $(x^*, \lambda^*) \in C \times \mathbb{R}$ satisfies*
[First-Order Condition] $\nabla f(x^*) = \lambda^* \nabla g(x^*)$
If $L(x,\lambda^*) \equiv f(x) - \lambda^* g(x)$ is concave in x on A, then x^* is a point of global maximum of f subject to the constraint $g(x) = 0$.

To see this, let $x \in C$. Then,

$$L(x,\lambda^*) - L(x^*,\lambda^*) \leq (x - x^*)[\nabla f(x^*) - \lambda^* \nabla g(x^*)]$$

by concavity of L in x on A. Using the first-order condition, we get

$$f(x) - \lambda^* g(x) = L(x,\lambda^*) \leq L(x^*,\lambda^*) = f(x^*) - \lambda^* g(x^*).$$

Since $x \in C$, and $x^* \in C$, we have $g(x) = g(x^*) = 0$. Thus, $f(x) \leq f(x^*)$, and so x^* is a point of global maximum of f subject to the constraint $g(x) = 0$.
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (1 - x^2 - y^2)$; $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = x + 4y - 2$.

To apply the sufficient conditions, set up the Lagrangian

$$L(x,y,\lambda) = (1 - x^2 - y^2) - \lambda[x + 4y - 2]$$

The first-order conditions yield

$$-(2x^*, 2y^*) = \lambda^*[1,4] \tag{9.1}$$

Also, to satisfy the constraint,

$$x^* + 4y^* = 2 \tag{9.2}$$

Thus, from (9.1), $x^* = -(\lambda^*/2)$, $y^* = -2\lambda^*$. Using this in (9.2),

$$-(\lambda^*/2) - 8\lambda^* = 2 \tag{9.3}$$

which yields $\lambda^* = -(4/17)$, $x^* = (2/17)$, $y^* = (8/17)$.

To check the second-order sufficient condition, we write the relevant bordered matrix

$$S = \begin{bmatrix} 0 & \nabla g(x^*,y^*) \\ \nabla g(x^*,y^*) & H_L(x^*,y^*,\lambda^*) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & -2 & 0 \\ 4 & 0 & -2 \end{bmatrix}$$

$$|S(1)| = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 < 0$$

$$\begin{aligned} |S(2)| &= (-1)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 4 & 0 \end{vmatrix} \\ &= 2 + 32 = 34 > 0 \end{aligned}$$

Thus $(-1)^k |S(k)| > 0$ for $k = 1,2$; by Proposition 5, $zH_L(x^*,y^*;\lambda^*)z < 0$ for all $z \neq 0$ such that $z\nabla g(x^*,y^*) = 0$. Thus, the second-order condition of Theorem 42 is satisfied, and so $(x^*,y^*) = (2/17, 8/17)$ is a point of local maximum of f subject to the constraints $g(x,y) = 0$.

Notice that the Hessian of L with respect to (x,y) is

$$H_L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

which is negative definite. Since $A = \mathbb{R}^2$ is a convex set, $L(x,y,\lambda^*)$ is concave in (x,y) on A. Hence, using Theorem 43, $(x^*,y^*) = (2/17, 8/17)$ is a point of global maximum of f subject to the constraint $g(x,y) = 0$.

9.6 Worked Out Problems on Chapter 9

Problem 38 (Constrained Optimization: Proof of the Lagrange Theorem, using the Implicit Function Theorem)**.**

Let A be an open subset of $\mathbb{R}^n$, and let f and g be continuously differentiable functions from A to $\mathbb{R}$. Define $C = \{x \in A : g(x) = 0\}$. Assume that there is $\bar{x} \in C$ such that:

$$\left.\begin{array}{cc} f(x) \le f(\bar{x}) & \text{for all } x \in C \\ \text{and} & D_n g(\bar{x}) \ne 0 \end{array}\right\} \tag{1}$$

(a) Use the implicit function theorem to show that there is an open set $X \subset \mathbb{R}^{n-1}$ which contains $(\bar{x}_1, \cdots, \bar{x}_{n-1})$ and an open set $Y \subset \mathbb{R}$ which contains $\bar{x}_n$, and a unique function $h : X \to Y$ such that $(x_1, \cdots, x_{n-1}, h(x_1, \cdots, x_{n-1})) \in A$ for all $(x_1, \cdots, x_{n-1}) \in X$, and:

$$\left.\begin{array}{l} (i)\ g(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})) = 0 \quad \text{for all } (x_1, \ldots, x_{n-1}) \in X \\ (ii)\ h(\bar{x}_1, \ldots, \bar{x}_{n-1}) = \bar{x}_n \end{array}\right\} \tag{2}$$

Further, h is continuously differentiable on X.

(b) Using (2), we have $(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})) \in C$ for all $(x_1, \ldots, x_{n-1}) \in X$, and so by (1) we have, for all $(x_1, \ldots, x_{n-1}) \in X$:

$$f(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})) \le f(\bar{x}_1, \ldots, \bar{x}_{n-1},\ h(\bar{x}_1, \ldots, \bar{x}_{n-1})) \tag{3}$$

Use (2) and (3), and the Corollary in problem 1 above, to prove that if we define $\lambda = D_n f(\bar{x}) / D_n g(\bar{x})$, then:

$$\nabla f(\bar{x}) = \lambda \nabla g(\bar{x}) \tag{4}$$

Solution.

(a) We want to show the existence of a Lagrange multiplier, given the knowledge that $\bar{x} \in C$ solves the constrained optimization problem. We are told that

$$f(x) \le f(\bar{x}) \text{ for all } x \in C \text{ and } D_n g(\bar{x}) \ne 0 \tag{1}$$

Treating x_n as the variable and $x_1, \ldots, x_{n-1}$ as parameters, we want to apply the Implicit Function Theorem to g at the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n) \in A$. We need to check the following conditions:

- The function g is defined and continuously differentiable on the open set $A \subset \mathbb{R}^n$.

- We are told that $\bar{x} \in C$, which implies $g(\bar{x}) = 0$.
- We are given that $D_n g(\bar{x}) \neq 0$. Since x_n is the only variable, this is the only derivative condition we need to check.

By the Implicit Function Theorem, then, there is an open set $X \subset \mathbb{R}^{n-1}$ containing $(\bar{x}_1, \ldots, \bar{x}_{n-1})$, an open set $Y \subset \mathbb{R}$ containing $\bar{x}_n$, and a unique function $h: X \to Y$ such that $\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \in A$ for all $(x_1, \ldots, x_{n-1}) \in X$ and

$$\left.\begin{array}{ll} \text{(i)} & g\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) = 0 \text{ for all } (x_1, \ldots, x_{n-1}) \in X \\ \text{(ii)} & \bar{x}_n = h(\bar{x}_1, \ldots, \bar{x}_{n-1}) \end{array}\right\} \tag{2}$$

Further, h is continuously differentiable on X.

(b) By result (2)(i) above, we have that $\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \in C$ whenever $(x_1, \ldots, x_{n-1}) \in X$. Then since $f(x) \leq f(\bar{x})$ for all $x \in C$, we have that for all $(x_1, \ldots, x_{n-1}) \in X$,

$$f\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \leq f\big(\bar{x}_1, \ldots, \bar{x}_{n-1}, h(\bar{x}_1, \ldots, \bar{x}_{n-1})\big) \tag{3}$$

Since $D_n g(\bar{x}) \neq 0$ we can define $\lambda = \frac{D_n f(\bar{x})}{D_n g(\bar{x})}$. We want to show that

$$\nabla f(\bar{x}) = \lambda \nabla g(\bar{x}) \tag{4}$$

That is, we want to show that $D_i f(\bar{x}) = \lambda D_i g(\bar{x})$ for all $i = 1, \ldots, n-1, n$. Note that the last of these n equations, $D_n f(\bar{x}) = \lambda D_n g(\bar{x})$, holds by the definition of λ. Now, define the function $F: X \to \mathbb{R}$ by

$$F(x_1, \ldots, x_{n-1}) = f\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \quad \text{for all } (x_1, \ldots, x_{n-1}) \in X$$

Since $\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \in C \subset A$ whenever $(x_1, \ldots, x_{n-1}) \in X$ and since f is defined on A, the function F is well-defined. We know that h is continuously differentiable on X. Also, we found in part (a) that whenever $(x_1, \ldots, x_{n-1}) \in X$ we have $\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \in A$, and we know that f is continuously differentiable on A. So we can use the Chain Rule to find the derivative of F on the open set X. We can also use the Chain Rule to differentiate result (2)(i) above. For all $i = 1, \ldots, n-1$ and all $(x_1, \ldots, x_{n-1}) \in X$, we have

$$\begin{aligned} D_i F(x_1, \ldots, x_{n-1}) = {} & D_i f\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \\ & + D_n f\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) D_i h(x_1, \ldots, x_{n-1}) \end{aligned} \tag{5}$$

$$\begin{aligned} 0 = {} & D_i g\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) \\ & + D_n g\big(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})\big) D_i h(x_1, \ldots, x_{n-1}) \end{aligned} \tag{6}$$

We have by (3) that $(\overline{x}_1,\ldots,\overline{x}_{n-1})$ is a point of local maximum of F. Now, because $(\overline{x}_1,\ldots,\overline{x}_{n-1})\in X$, X is open, and F is continuously differentiable on X, we have by the Corollary in problem 1 that $\nabla F(\overline{x}_1,\ldots,\overline{x}_{n-1})=0$. That is, $D_iF(\overline{x}_1,\ldots,\overline{x}_{n-1})=0$ for all $i=1,\ldots,n-1$. We also know from (2)(ii) that $\overline{x}_n=h(\overline{x}_1,\ldots,\overline{x}_{n-1})$. Evaluating (5) and (6) at $(\overline{x}_1,\ldots,\overline{x}_{n-1})$, then, for all $i=1,\ldots,n-1$ we have

$$D_if(\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x}_n)+D_nf(\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x}_n)D_ih(\overline{x}_1,\ldots,\overline{x}_{n-1})=0 \quad (7)$$
$$D_ig(\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x}_n)+D_ng(\overline{x}_1,\ldots,\overline{x}_{n-1},\overline{x}_n)D_ih(\overline{x}_1,\ldots,\overline{x}_{n-1})=0 \quad (8)$$

Now, since $D_ng(\overline{x})\neq 0$ we can solve (8) for $D_ih(\overline{x}_1,\ldots,\overline{x}_{n-1})$ and plug this expression into (7) to obtain that for all $i=1,\ldots,n-1$,

$$D_if(\overline{x})-D_nf(\overline{x})\frac{D_ig(\overline{x})}{D_ng(\overline{x})}=0 \quad (9)$$

Using the fact that $\lambda=\frac{D_nf(\overline{x})}{D_ng(\overline{x})}$, we have from (9) that for all $i=1,\ldots,n-1$,

$$D_if(\overline{x})=\lambda D_ig(\overline{x}) \quad (10)$$

Since $D_nf(\overline{x})=\lambda D_ng(\overline{x})$ holds by the definition of λ, we have by (10) that (4) holds, which is what we wanted to show.

Problem 39 (Constrained Optimization: Sufficient Conditions for a Global Maximum)**.**

Suppose A is an open convex set in $\mathbb{R}^n$, $f:A\to\mathbb{R}$ is continuously differentiable and concave on A, and $g:A\to\mathbb{R}$ is a linear function on A.

Suppose there is $\overline{x}$ in A and $\lambda\in\mathbb{R}$ which satisfy the following conditions:

$$\left.\begin{array}{l}(i)\ g(\overline{x})=0\\(ii)\ \nabla f(\overline{x})=\lambda\nabla g(\overline{x})\end{array}\right\}(FOC)$$

Show that $\overline{x}$ solves the problem:

$$\left.\begin{array}{ll}Max & f(x)\\ subject\ to & g(x)=0\\ and & x\in A\end{array}\right\}(P)$$

Solution.

Given the λ in the hypothesis of the theorem, define the function $L:A \to \mathbb{R}$ by

$$L(x) = f(x) - \lambda g(x) \quad \text{for all } x \in A$$

Because f is concave on A, g is linear on A, and λ is fixed, we have that L is concave on A. Moreover, L is continuously differentiable on A because f and g are continuously differentiable on A. In addition, we are given that A is open and convex. So we can apply Theorem 27 to see that for all $x \in A$ satisfying the constraint $g(x) = 0$, we have

$$\begin{gathered} L(x) - L(\bar{x}) \le (x - \bar{x})\nabla L(\bar{x}) \\ f(x) - f(\bar{x}) - \lambda\big(g(x) - g(\bar{x})\big) \le (x - \bar{x})\big(\nabla f(\bar{x}) - \lambda \nabla g(\bar{x})\big) \\ f(x) - f(\bar{x}) \le 0 \end{gathered}$$

This shows that $\bar{x}$ solves the problem (P).

Problem 40 (Sufficient Conditions for Constrained Maximization: Application)**.**

We want to solve the problem:

$$\left.\begin{array}{ll} Max & \prod_{i=1}^{n} x_i \\ subject\ to & \sum_{i=1}^{n} x_i = n \\ and & x \in \mathbb{R}^n_+ \end{array}\right\} (Q)$$

by using the sufficient conditions for constrained maximization, developed in problem 4 above.

Instead of solving (Q) directly, we look at the following problem:

$$\left.\begin{array}{ll} Max & \sum_{i=1}^{n} \ln x_i \\ subject\ to & \sum_{i=1}^{n} x_i - n = 0 \\ and & x \in \mathbb{R}^n_{++} \end{array}\right\} (R)$$

(a) Define $A = \mathbb{R}^n_{++}$, $f : A \to \mathbb{R}$ by $f(x) = \sum_{i=1}^{n} \ln x_i$ for all $x \in A$, and $g : A \to \mathbb{R}$ by $g(x) = \sum_{i=1}^{n} x_i - n$ for all $x \in A$. Verify that A, f and g satisfy all the hypotheses stated in problem 4.

(b) Show that $\bar{x} = (1,1,\ldots,1)$ and $\lambda = 1$ satisfy the (FOC) of problem 4. Use the result of problem 4 to infer that $\bar{x} = (1,1,\ldots,1)$ solves problem (R).

(c) Verify, using (b), that $\bar{x} = (1,1,\ldots,1)$ solves:

$$\left.\begin{array}{ll} Max & \prod_{i=1}^{n} x_i \\ subject\ to & \sum_{i=1}^{n} x_i - n = 0 \\ and & x \in \mathbb{R}^n_{++} \end{array}\right\} (R')$$

(d) Conclude, using (c), that $\bar{x} = (1,1,\ldots,1)$ solves (Q).

Solution.

(a) Define $A = \mathbb{R}^n_{++}$ and note that A is an open and convex set in $\mathbb{R}^n$. Now, define $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{n} \ln x_i \quad \text{for all } x \in A, \quad g(x) = \sum_{i=1}^{n} x_i - n \quad \text{for all } x \in A$$

The function f can be expressed as the sum of functions that are continuously differentiable and concave on A, so f is also continuously differentiable and concave on A. Also, g is linear on A. Therefore A, f, and g satisfy all the hypotheses stated in problem 4.

(b) It is straightforward to check that when $\bar{x} = (1, \ldots, 1) \in A$ and $\lambda = 1$, we have $g(\bar{x}) = 0$ and $\nabla f(\bar{x}) = \lambda \nabla g(\bar{x})$. Therefore by the sufficiency result in problem 4, $\bar{x}$ solves the problem (R).

(c) Define the function $h : A \to \mathbb{R}$ by

$$h(x) = \prod_{i=1}^{n} x_i \quad \text{for all } x \in A$$

Then for all $x \in A$, we have $\ln h(x) = f(x)$, which is equivalent to $h(x) = e^{f(x)}$. Seeking contradiction, suppose that $\bar{x}$ solves (R) but does not solve (R'). Then there is $x' \in A$, $x' \neq \bar{x}$ satisfying $g(x') = 0$ and $h(x') > h(\bar{x})$, which is equivalent to $e^{f(x')} > e^{f(\bar{x})}$. Since the exponential function is strictly increasing on $\mathbb{R}$, this implies that $f(x') > f(\bar{x})$. But that contradicts $\bar{x}$ solving (R), so it must be that $\bar{x}$ solves both (R) and (R').

(d) We have from part (c) that $h(\bar{x}) \geq h(x)$ for all $x \in A = \mathbb{R}^n_{++}$ satisfying $g(x) = 0$. Consider the set

$$S = \mathbb{R}^n_{+} \setminus \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n_{+} \mid x_i = 0 \text{ for some } i \in \{1, \ldots, n\}\}$$

For all $x \in S$, we have that $h(x) = 0 < 1 = h(\bar{x})$. Therefore $\bar{x}$ solves (Q).

References

This material is based on *The Elements of Real Analysis* by *R. Bartle* (Chapter 7); *Mathematical Analysis* by *T. Apostol* (Chapter 6); and *Mathematical Ecoomics* by *A. Takayama* (Chapter 1).

The theory of maximum of a function of several variables is a useful tool to prove many important inequalities. For more on this, see *Inequalities* by *G.H. Hardy, J.E. Littlewood and G. Polya* (Chapter 4). A proof of the important Proposition 5 can be found in H.B. Mann: "Quadratic Forms with Linear Constraints", *American Mathematical Monthly* (1943), 430-433, and also in G. Debreu "Definite and Semidefinite Quadratic Forms", *Econometrica* (1952), 295-300.

Part IV

Modern Optimization Theory

Chapter 10

Concave Programming

10.1 Preliminaries

We will present below the elements of "modern optimization theory" as formulated by Kuhn and Tucker, and a number of authors who have followed their general approach. As in our exposition of "classical optimization theory", we will concentrate on characterizing points of *maximum* of a function of several variables (subject to certain constraints). The theory which characterizes points of *minimum* of a function of several variables (subject to certain constraints) can be obtained analogously.

Modern constrained maximization theory is concerned with the following problem:

$$\left.\begin{array}{lll} & Max & f(x) \\ Subject\,to & & g^j(x) \geq 0 \;\; for\; j = 1, \ldots, m \\ & and & x \in X \end{array}\right\} \qquad \text{(P)}$$

where X is a non-empty subset of $\mathbb{R}^n$, and $f, g^j\,(j = 1, \ldots, m)$ are functions from X to $\mathbb{R}$.

We define the *constraint set*, C as follows:

$$C = \{x \in X : g(x) \geq 0\}$$

where, as usual, $g(x) = [g^1(x), \ldots, g^m(x)]$.

An element $\hat{x} \in X$ is *a point of constrained global maximum* if $\hat{x}$ solves the problem (P). A pair $(\hat{x}, \hat{\lambda}) \in (X \times \mathbb{R}^m_+)$ is a *saddle point* if

$$\phi(x, \hat{\lambda}) \leq \phi(\hat{x}, \hat{\lambda}) \leq \phi(\hat{x}, \lambda)$$

for all $x \in X$ and all $\lambda \in \mathbb{R}^m_+$, where $\phi(x, \lambda) = f(x) + \lambda g(x)$ for $(x, \lambda) \in X \times \mathbb{R}^m_+$.

10.2 Constrained Global Maxima and Saddle Points

A major part of modern optimization theory is concerned with establishing (under suitable conditions) an equivalence result between a point of constrained global maximum and a saddle point. We explore this theory in what follows.

Theorem 44. *If* $(\hat{x},\hat{\lambda}) \in (X \times \mathbb{R}^m_+)$ *is a saddle point, then (i)* $\hat{\lambda}g(\hat{x}) = 0$*, (ii)* $g(\hat{x}) \geq 0$*, and (iii)* $\hat{x}$ *is a point of constrained global maximum.*

Proof. Since $(\hat{x},\hat{\lambda})$ is a saddle point, we have for all $\lambda \in \mathbb{R}^m_+$,

$$f(\hat{x}) + \hat{\lambda}g(\hat{x}) \leq f(\hat{x}) + \lambda g(\hat{x})$$

That is, we have

$$\hat{\lambda}g(\hat{x}) \leq \lambda g(\hat{x}) \quad for\ all\ \lambda \in \mathbb{R}^m_+ \tag{10.1}$$

Choosing $\lambda = 0$ in (10.1), we get $\hat{\lambda}g(\hat{x}) \leq 0$. Choosing $\lambda = 2\hat{\lambda}$ in (10.1), we get $\hat{\lambda}g(\hat{x}) \geq 0$. Thus $\hat{\lambda}g(\hat{x}) = 0$, which proves (i).

Using (i) in (10.1), we get

$$0 \leq \lambda g(\hat{x}) \qquad for\,all\,\lambda \in \mathbb{R}^m_+ \tag{10.2}$$

Choosing λ in turn to be the m unit vectors in $\mathbb{R}^m$ in (10.2), we get

$$0 \leq g(\hat{x}) \tag{10.3}$$

which proves (ii). Thus $\hat{x}$ is in the constraint set, C. Now, let x be an arbitrary element of C. Since $(\hat{x},\hat{\lambda})$ is a saddle point, we have

$$f(\hat{x}) + \hat{\lambda}g(\hat{x}) \geq f(x) + \hat{\lambda}g(x) \tag{10.4}$$

Using (i), and the fact that $\hat{\lambda} \in \mathbb{R}^m_+$ in (10.4), we obtain

$$f(\hat{x}) \geq f(x)$$

Thus, $\hat{x}$ solves (P), proving (iii).// ∎

A converse of Theorem 44 can be proved if X is a convex set, $f, g^j\,(j = 1,\ldots,m)$ are concave functions on X, and a condition on the constraints generally known as "Slater's condition" is satisfied. [Notice that none of these conditions are needed for the validity of Theorem 44].

Given the problem (P), we will say that *Slater's condition* holds if there is $\bar{x} \in X$, such that $g^j(\bar{x}) > 0$ for $j = 1,\ldots,m$.

Theorem 45. *(Kuhn-Tucker) Suppose $\hat{x} \in X$ is a point of constrained global maximum. If X is a convex set, $f, g^j (j = 1, \ldots, m)$ are concave functions on X, and Slater's condition holds, then there is $\hat{\lambda} \in \mathbb{R}^m_+$ such that (i) $\hat{\lambda} g(\hat{x}) = 0$, and (ii) $(\hat{x}, \hat{\lambda})$ is a saddle point.*

Proof. Define the sets A and B as follows:

$A = \{(\alpha, \beta) \in \mathbb{R}^{m+1} : f(x) - f(\hat{x}) \geq \alpha$ and $g(x) \geq \beta$ for some $x \in X\}$

$B = \{(\alpha, \beta) \in \mathbb{R}^{m+1} : \alpha > 0$ and $\beta >> 0\}$

Then B is clearly a non-empty, convex set. And A is a non-empty, convex set, since X is convex, and $f, g^j (j = 1, \ldots, m)$ are concave functions. Since x solves (P), A and B are disjoint. Using the Minkowski Separation theorem, we have $(\mu, \nu) \in \mathbb{R}^{m+1}$, $(\mu, \nu) \neq 0$, and $\theta \in \mathbb{R}$, such that

$$\mu\alpha + \nu\beta \leq \theta \quad for\, all\, (\alpha, \beta) \in A \tag{10.5}$$

$$\mu\alpha + \nu\beta \geq \theta \quad for\, all\, (\alpha, \beta) \in B \tag{10.6}$$

Using (10.5), $\theta \geq 0$, while using (10.6), $\theta \leq 0$. Thus $\theta = 0$. Also, using (10.6), $\mu \geq 0$ and $\nu \geq 0$. Summarizing, we have $(\mu, \nu) \in \mathbb{R}^{m+1}_+, (\mu, \nu) \neq 0$, such that

$$\mu[f(x) - f(\hat{x})] + \nu g(x) \leq 0 \tag{10.7}$$

for all $x \in X$.

We claim, now, that $\mu \neq 0$. For if $\mu = 0$, then $\nu \neq 0$, and (10.7) yields

$$\nu g(x) \leq 0 \quad for\, all\, x \in X \tag{10.8}$$

By Slater's Condition, there is $\bar{x} \in X$ with $g(\bar{x}) >> 0$. Since $\nu \geq 0$ and $\nu \neq 0$, so $\nu g(\bar{x}) > 0$, which contradicts (10.8). Thus, $\mu \neq 0$; that is, $\mu > 0$. Define $\hat{\lambda} = (\nu/\mu)$. Then $\hat{\lambda} \in \mathbb{R}^m_+$ and (10.7) yields

$$f(x) + \hat{\lambda} g(x) \leq f(\hat{x}) \qquad for\, all\, x \in X \tag{10.9}$$

Putting $x = \hat{x}$ in (10.9), we get

$$\hat{\lambda} g(\hat{x}) \leq 0 \tag{10.10}$$

Also $g(\hat{x}) \geq 0$, and $\hat{\lambda} \in \mathbb{R}^m_+$ implies

$$\hat{\lambda} g(\hat{x}) \geq 0 \tag{10.11}$$

Clearly (10.10) and (10.11) imply $\hat{\lambda} g(\hat{x}) = 0$, which proves (i).

Using (i) in (10.9), we get

$$f(x) + \hat{\lambda} g(x) \leq f(\hat{x}) + \hat{\lambda} g(\hat{x}) \qquad for\, all\, x \in X \tag{10.12}$$

If $\lambda \in \mathbb{R}^m_+$, then $\lambda g(\hat{x}) \geq 0$, since $g(\hat{x}) \geq 0$. Thus, using (i) again, we have

$$f(\hat{x}) + \hat{\lambda} g(\hat{x}) \leq f(\hat{x}) + \lambda g(\hat{x}) \qquad for\, all\, \lambda \in \mathbb{R}^m_+ \tag{10.13}$$

Combining (10.12) and (10.13), one can conclude that $(\hat{x}, \hat{\lambda})$ is a saddle point, which establishes (ii). ■

The following examples demonstrate why the assumptions of Theorem 45 are needed for the conclusion to be valid.

Example: Let $X = \mathbb{R}_+$; let $f : X \to \mathbb{R}$ be given by $f(x) = x$, and $g : X \to \mathbb{R}$ be given by $g(x) = -x^2$. Then X is convex, f and g are concave. But Slater's Condition is clearly violated. It is easily checked that $\hat{x} = 0$ is a point of constrained global maximum. But there is no $\hat{\lambda} \in \mathbb{R}_+$ such that $(\hat{x}, \hat{\lambda})$ is a saddle point. For if there were such a $\hat{\lambda}$, then

$$x - \hat{\lambda}x^2 \leq \hat{x}^2 - \hat{\lambda}\hat{x}^2 = 0$$

for all $x \in X$. But by choosing $x > 0$ and x sufficiently close to zero, this inequality is clearly violated.

Example: Let $X = \mathbb{R}_+$; let $f : X \to \mathbb{R}$ be given by $f(x) = x^2$, and $g : X \to \mathbb{R}$ be given by $g(x) = 1 - x$. Here, X is convex, g is concave, and Slater's condition is satisfied with (for instance) $\bar{x} = (1/2)$. Buf f is not concave on X. It is easily checked that $\hat{x} = 1$ is a point of constrained global maximum. But there is no $\hat{\lambda} \in \mathbb{R}_+$ such that $(\hat{x}, \hat{\lambda})$ is a saddle point. For if there were such a $\hat{\lambda}$, then

$$x^2 + \hat{\lambda}(1 - x) \leq \hat{x}^2 + \hat{\lambda}(1 - \hat{x}) = 1$$

for all $x \in X$. But by choosing $x > 0$ and x sufficiently large, this inequality is clearly violated.

Example: Let $X = \mathbb{R}_+$; let $f : X \to \mathbb{R}$ be given by $f(x) = x$, and $g : X \to \mathbb{R}$ be given by $g(x) = 1 - x^{1/2}$. Then X is convex, f is concave, and Slater's condition is satisfied with (for instance) $\bar{x} = (1/4)$. But g is not concave on X. It can be checked that $\hat{x} = 1$ is a point of constrained global maximum. But there is no $\hat{\lambda} \in \mathbb{R}_+$, such that $(\hat{x}, \hat{\lambda})$ is a saddle point. For, if there were such a $\hat{\lambda}$, then

$$x + \hat{\lambda}(1 - x^{1/2}) \leq \hat{x} + \hat{\lambda}(1 - \hat{x}^{1/2}) = 1$$

for all $x \in X$. But, by choosing $x > 0$ and x sufficiently large, this inequality is clearly violated.

10.3 The Kuhn-Tucker Conditions and Saddle Points

Let X be an open set in $\mathbb{R}^n$, and $f, g^j\,(j = 1, \ldots, m)$ be continuously differentiable on X. A pair $(\hat{x}, \hat{\lambda})$ in $X \times \mathbb{R}^m_+$ satisfies the *Kuhn-Tucker conditions* if

$$(i) \qquad D_i f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j D_i g^j(\hat{x}) = 0 \qquad i = 1, \ldots, n$$

$$(ii) \qquad g(\hat{x}) \geq 0 \qquad and \;\; \hat{\lambda} g(\hat{x}) = 0$$

A part of modern optimization theory is concerned with establishing the equivalence (under some suitable conditions) between a saddle point and a point where the Kuhn-Tucker conditions are satisfied. We examine this theory in what follows.

Theorem 46. *Let X be an open set in $\mathbb{R}^n$, and $f, g^j\,(j = 1, \ldots, m)$ be continuously differentiable on X. Suppose a pair $(\hat{x}, \hat{\lambda}) \in X \times \mathbb{R}^m_+$ satisfies the Kuhn-Tucker conditions. If X is convex and $f, g^j\,[j = 1, \ldots, m]$ are concave on X, then (i) $(\hat{x}, \hat{\lambda})$ is a saddle point, and (ii) $\hat{x}$ is a point of constrained global maximum.*

Proof. Define, as usual, $\phi(x, \lambda) = f(x) + \lambda g(x)$ for $(x, \lambda) \in X \times \mathbb{R}^m_+$. Given $\hat{\lambda}$, $\phi(x, \hat{\lambda})$ is concave in x, since f and $g^j\,(j = 1, \ldots, n)$ are concave in x. Thus for $x \in X$ we have

$$\phi(x, \hat{\lambda}) - \phi(\hat{x}, \hat{\lambda}) \leq (x - \hat{x}) \nabla \phi(\hat{x}, \hat{\lambda})$$

Using the Kuhn-Tucker conditions, we have $\nabla \phi(\hat{x}, \hat{\lambda}) = 0$, so

$$\phi(x, \hat{\lambda}) \leq \phi(\hat{x}, \hat{\lambda}) \qquad for\,all\,x \in X \tag{10.14}$$

Also, by the Kuhn-Tucker conditions, $g(\hat{x}) \geq 0$ and $\hat{\lambda} g(\hat{x}) = 0$. So, for all $\lambda \in \mathbb{R}^m_+$, $\phi(\hat{x}, \lambda) = f(\hat{x}) + \lambda g(\hat{x}) \geq f(\hat{x}) = f(\hat{x}) + \hat{\lambda} g(\hat{x}) = \phi(\hat{x}, \hat{\lambda})$. Thus, we have

$$\phi(\hat{x}, \hat{\lambda}) \leq \phi(\hat{x}, \lambda) \qquad for\,all\,\lambda \in \mathbb{R}^m_+ \tag{10.15}$$

Using (10.14) and (10.15), $(\hat{x}, \hat{\lambda})$ is a saddle point, which proves (i). Using Theorem 44, $\hat{x}$ solves (P) which proves (ii). ■

Theorem 47. *Let X be an open set in $\mathbb{R}^n$, and $f, g^j\,(j = 1, \ldots, m)$ be continuously differentiable on X. Suppose a pair $(\hat{x}, \hat{\lambda}) \in X \times \mathbb{R}^m_+$ is a saddle point. Then $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions.*

Proof. Since $(\hat{x}, \hat{\lambda})$ is a saddle point, we have

$$\phi(x, \hat{\lambda}) \leq \phi(\hat{x}, \hat{\lambda}) \qquad for\, all\, x \in X$$

Thus, given $\hat{\lambda}$, $\phi(x, \hat{\lambda})$ attains a maximum at $\hat{x} \in X$. Since X is open, and ϕ is continuously differentiable on X, we have

$$\nabla\phi(\hat{x}, \hat{\lambda}) = 0$$

Thus, $D_i f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j D_i g^j(\hat{x}) = 0$ for $i = 1, \ldots, n$. Also, by Theorem 44, we know that $g(\hat{x}) \geq 0$ and $\hat{\lambda} g(\hat{x}) = 0$. Thus $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions. ■

10.4 The Kuhn-Tucker Conditions and Constrained Local Maxima

An element $\hat{x} \in X$ is a *point of constrained local maximum* if $\hat{x} \in C$ and there is $\delta > 0$ such that for all $x \in B(\hat{x}, \delta) \cap C$, $f(\hat{x}) \geq f(x)$.

Let X be an open set in $\mathbb{R}^n$, and f, $g^j (j = 1, \ldots, m)$ be continuously differentiable on X. We now establish the useful result (corresponding to the classical Lagrange theorem) that if $\hat{x} \in X$ is a point of constrained local maximum then under suitable conditions, there is $\lambda \in \mathbb{R}^m_+$ such that $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions. The important condition needed for the development of this theory is called a "constraint qualification" (just as it is in classical theory). While there are several versions of this condition, the following one, due to Arrow, Hurwicz and Uzawa, appears to be the most useful.

Let $\bar{x}$ be a point in the constraint set C. Let $E(\bar{x}) \subset \{1, \ldots, m\}$ be the set of indices, j, for which the constraints are *binding* at $\bar{x}$; that is $g^j(\bar{x}) = 0$ for $j \in E(\bar{x})$. Then $\bar{x}$ satisfies the *Arrow-Hurwicz-Uzawa (AHU) constraint qualification* if at least one of the following three conditions is satisfied:

(a) $E(\bar{x})$ is empty

(b) X is a convex set and g^j is a convex function for each $j \in E(\bar{x})$

(c) X is a convex set and there is $h \in \mathbb{R}^n$ such that $h\nabla g^j(\bar{x}) > 0$ for all $j \in E(\bar{x})$.

Theorem 48. *(Arrow-Hurwicz-Uzawa) Let X be an open set in $\mathbb{R}^n$, and f, $g^j (j = 1, \ldots, m)$ be continuously differentiable on X. Suppose $\hat{x} \in X$ is a point of constrained local maximum. Suppose, further, that $\hat{x}$ satisfies the Arrow-Hurwicz-Uzawa constraint qualification condition, then there is $\hat{\lambda} \in \mathbb{R}^m_+$ such that $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions.*

Proof. If condition (a) of the AHU constraint qualification is satisfied, then $g^j(\hat{x}) > 0$ for $j = 1, \ldots, m$. Define $Y = \{x \in X : g^j(x) > 0$ for $j = 1, \ldots, m\}$. Then $\hat{x} \in Y$ is a point of local maximum of f on the open set Y. So, by the theory of unconstrained maximum,

$$\nabla f(\hat{x}) = 0$$

Define $\hat{\lambda}_j = 0$ for $j = 1, \ldots, m$. Then, $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions.

If condition (a) is not satisfied, let $E \equiv E(\hat{x})$ be the set of indices, j, for which $g^j(\hat{x}) = 0$. (This is the set of effective constraints at the point of constrained local maximum). Since $\hat{x}$ is a point of constrained local maximum, there is $\delta > 0$, such that $x \in B(\hat{x}, \delta) \cap C$ implies $f(x) \le f(\hat{x})$. Let $z \in \mathbb{R}^n$ be an arbitrary vector satisfying $z \nabla g^j(\hat{x}) \ge 0$ for $j \in E$. We will now show that

$$z \nabla f(\hat{x}) \le 0 \tag{10.16}$$

if either condition (b) or condition (c) of the AHU constraint qualification is satisfied.

First, suppose condition (b) is satisfied. By defining $h = tz$ with $\hat{t} > t > 0$ and $\hat{t}$ sufficiently close to zero, we have $h \in \mathbb{R}^n$, $(\hat{x} + h) \in B(\hat{x}, \delta)$, $g^j(\hat{x} + h) > 0$ for $j \in\sim E$, and $h \nabla g^j(\hat{x}) \ge 0$ for $j \in E$. By convexity of X and of g^j for $j \in E$, we have $g^j(\hat{x} + h) - g^j(\hat{x}) \ge h \nabla g^j(\hat{x}) \ge 0$ for $j \in E$. Thus, $g^j(\hat{x} + h) \ge g^j(\hat{x}) \ge 0$ for $j \in E$. So $(\hat{x} + h) \in B(\hat{x}, \delta) \cap C$. Since $\hat{x}$ is a point of constrained local maximum, $f(\hat{x} + h) \le f(\hat{x})$. Thus, by applying the Mean Value theorem, we get $0 \ge f(\hat{x} + h) - f(\hat{x}) = tz \nabla f(\xi)$ where ξ is a convex combination of $\hat{x}$ and $(\hat{x} + tz)$. Thus, $z \nabla f(\xi) \le 0$. Letting $t \to 0$, we get $\xi \to \hat{x}$, and $\nabla f(\xi) \to \nabla f(\hat{x})$. So, we obtain $z \nabla f(\hat{x}) \le 0$, which is (10.16).

Next, suppose condition (c) is satisfied. Define $y = \lambda h + z$, where $\lambda \in (0, 1)$, and h is given by condition (c). Then, we have $y \nabla g^j(\hat{x}) > 0$ for all $j \in E$, since $z \nabla g^j(\hat{x}) \ge 0$ and $h \nabla g^j(\hat{x}) > 0$ for each $j \in E$.Defining $b = ty$ with $\hat{t} > t > 0$,and $\hat{t}$ sufficiently close to zero, we have $b \in \mathbb{R}^n$, $(\hat{x} + b) \in B(\hat{x}, \delta)$, $g^j(\hat{x} + b) > 0$ for $j \in\sim E$, and for all $\theta \in [0, 1]$, $b \nabla g^j(\hat{x} + \theta b) > 0$ for $j \in E$. Now, for $j \in E$, by the Mean-Value theorem, $g^j(\hat{x} + b) - g^j(\hat{x}) = b \nabla g^j(\hat{x} + \theta_j b)$ for some $\theta_j \in [0, 1]$. So, for $j \in E$, $g^j(\hat{x} + b) - g^j(\hat{x}) > 0$,and consequently $g^j(\hat{x} + b) > 0$. Thus, $(\hat{x} + b) \in B(\hat{x}, \delta) \cap C$. Since $\hat{x}$ is a point of constrained local maximum, we have $f(\hat{x} + b) \le f(\hat{x})$. Now, following the argument used in the previous paragraph, we get $y \nabla f(\hat{x}) \le 0$. Thus, we have shown that for every $\lambda \in (0, 1)$, $y = \lambda h + z$ satisfies $y \nabla f(\hat{x}) \le 0$;that is, $z \nabla f(\hat{x}) + \lambda h \nabla f(\hat{x}) \le 0$. Letting $\lambda \to 0$,we obtain $z \nabla f(\hat{x}) \le 0$, as we had claimed.

We have now established that if $z \in \mathbb{R}^n$ is an arbitrary vector which satisfies $z \nabla g^j(\hat{x}) \ge 0$ for $j \in E$, then $z[-\nabla f(\hat{x})] \ge 0$. By the Farkas Lemma [see Chapter 7 on "Convex Analysis"], there exist $\hat{\lambda}_j \ge 0$ for $j \in E$, such that for $i = 1, \ldots, n$

$$-D_i f(\hat{x}) = \sum \hat{\lambda}_j D_i g^j(\hat{x})$$

Define $\hat{\lambda}_j = 0$ for $j \in\sim E$. Then $\hat{\lambda} \in \mathbb{R}^m_+$, and for $i = 1, \ldots, n$,

$$D_i f(\hat{x}) + \sum_{j=1}^{m} \lambda_j D_i g^j(\hat{x}) = 0$$

Since $\hat{x} \in C$, we get $g(\hat{x}) \geq 0$. Since $g^j(\hat{x}) = 0$ for $j \in E$, and $\hat{\lambda}_j = 0$ for $j \in \sim E$, we obtain $\hat{\lambda} g(\hat{x}) = 0$. Thus, $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions. ■

Corollary 6. *Suppose X is an open, convex set in $\mathbb{R}^n$, and $f, g^j (j = 1, \ldots, m)$ are continuously differentiable on X. Suppose $\hat{x} \in X$ is a point of constrained local maximum. Suppose, further, that at least one of the following two conditions is satisfied:*

(i) g^j *is convex for all* $j = 1, \ldots, m$

(ii) g^j *is concave for all* $j = 1, \ldots, m$*, and Slater's condition holds.*

Then, there is $\hat{\lambda} \in \mathbb{R}^m_+$ such that $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions.

Proof. Let $E \equiv E(\hat{x})$ be the set of indices for which $g^j(\hat{x}) = 0$. If $E(\hat{x})$ is empty, then condition (a) of the AHU constraint qualification is satisfied and the result follows from Theorem 48. If E is non-empty, and condition (i) is satisfied, then clearly condition (b) of the AHU constraint qualification is satisfied and, again, the result follows from Theorem 48.

Suppose E is non-empty, and condition (ii) is satisfied. Then, there is $\bar{x} \in C$, such that $g^j(\bar{x}) > 0$ for $j = 1, \ldots, m$. Thus, for $j \in E$, we have $g^j(\bar{x}) - g^j(\hat{x}) \leq (\bar{x} - \hat{x}) \nabla g^j(\hat{x})$, by concavity of g^j. Since $g^j(\hat{x}) = 0$ and $g^j(\bar{x}) > 0$ for each $j \in E$, we obtain $(\bar{x} - \hat{x}) \nabla g^j(\hat{x}) > 0$ for $j \in E$. Defining $h = (\bar{x} - \hat{x}) \in \mathbb{R}^n$, we have $h \nabla g^j(\hat{x}) > 0$ for all $j \in E$, and so condition (c) of the AHU constraint qualification is satisfied, and the result follows from Theorem 48. ■

Corollary 7. *Suppose $c \in \mathbb{R}^n$, $b \in \mathbb{R}^k$ and A is a $k \times n$ matrix. Consider the following maximization problem:*

$$\left.\begin{array}{ll} Max & cx \\ Subject\, to & Ax \leq b \\ and & x \in \mathbb{R}^n_+ \end{array}\right\} \qquad (P)$$

Suppose $\hat{x}$ solves (P), then there is $\hat{\mu} \in \mathbb{R}^k_+$ such that for all $x \in \mathbb{R}^n_+$, and all $\mu \in \mathbb{R}^k_+$

(i) $cx + \hat{\mu}(b - Ax) \leq c\hat{x} + \hat{\mu}(b - A\hat{x}) \leq c\hat{x} + \mu(b - A\hat{x})$

(ii) $\hat{\mu}$ *solves the following minimization problem:*

$$\left.\begin{array}{ll} Min & \mu b \\ Subject\, to & \mu A \geq c \\ and & \mu \in \mathbb{R}^k_+ \end{array}\right\} \qquad (Q)$$

Proof. Define $X = \mathbb{R}^n$; then X is an open, convex set in $\mathbb{R}^n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = cx$; define $g^j : \mathbb{R}^n \to \mathbb{R}$ by $g^j(x) = x_j$ for $j = 1, \ldots, n$; $g^j(x) = (b - Ax)_{(j-n)}$ for $j = n+1, \ldots, n+k$. Define $m = n + k$. Then $f, g^j (j = 1, \ldots, m)$ are continuously differentiable on X. Clearly g^j

is convex for all $j = 1,\ldots,m$. Thus, there is $\hat{\lambda} \in \mathbb{R}^m_+$ such that $(\hat{x},\hat{\lambda})$ satisfies the Kuhn-Tucker conditions, by applying Corollary 6. We can write $\hat{\lambda} = (\hat{\nu},\hat{\mu})$ where $\hat{\nu} \in \mathbb{R}^n_+$ and $\hat{\mu} = \mathbb{R}^k_+$. Then, we have for $i = 1,\ldots,n$

$$c_i + \hat{\nu}_i - \sum_{j=1}^{k} \hat{\mu}_j a_{ji} = 0$$

That is,

$$c + \hat{\nu} - \hat{\mu}A = 0 \tag{10.17}$$

and $\hat{x} \geq 0$, $(b - A\hat{x}) \geq 0, \hat{\nu}\hat{x} + \hat{\mu}(b - A\hat{x}) = 0$. Clearly, then,

$$\hat{\nu}\hat{x} = 0 \; and \; \hat{\mu}(b - A\hat{x}) = 0 \tag{10.18}$$

Using (10.17), we get for all $x \in \mathbb{R}^n_+$, $cx + \hat{\mu}(b - Ax) = cx + \hat{\nu}x - \hat{\mu}Ax + \hat{\mu}b - \hat{\nu}x \leq \hat{\mu}b$. Also, using (10.17) and (10.18), we get

$$c\hat{x} + \hat{\mu}(b - A\hat{x}) = c\hat{x} + \hat{\nu}\hat{x} - \hat{\mu}A\hat{x} + \hat{\mu}b - \hat{\nu}\hat{x} = \hat{\mu}b \tag{10.19}$$

Thus, for all $x \in \mathbb{R}^n_+$

$$cx + \hat{\mu}(b - Ax) \leq c\hat{x} + \hat{\mu}(b - A\hat{x}) \tag{10.20}$$

For all $\mu \in \mathbb{R}^k_+$, we have $c\hat{x} + \mu(b - A\hat{x}) \geq c\hat{x}$, since $(b - A\hat{x}) \geq 0$. Also $c\hat{x} + \hat{\mu}(b - A\hat{x}) = c\hat{x}$, since $\hat{\mu}(b - A\hat{x}) = 0$. Thus, for all $\mu \in \mathbb{R}^k_+$,

$$c\hat{x} + \mu(b - A\hat{x}) \geq c\hat{x} + \hat{\mu}(b - A\hat{x} \tag{10.21}$$

Combining (10.20) and (10.21) establishes the result (i).

To prove (ii), we can proceed as follows. We get from (10.18), (10.19) that

$$c\hat{x} = \hat{\mu}b \tag{10.22}$$

Using this in (10.20), we get for all $x \in \mathbb{R}^n_+$, $(c - \hat{\mu}A)x + \hat{\mu}b \leq \hat{\mu}b + \hat{\mu}(b - A\hat{x})$ and using (10.18) in the above inequality

$$(c - \hat{\mu}A)x \leq 0 \qquad for\,all\, x \in \mathbb{R}^n_+ \tag{10.23}$$

Choosing x in turn to be the n unit vectors in $\mathbb{R}^n$, we get from (10.23),

$$\hat{\mu}A \geq c \tag{10.24}$$

Since $\hat{\mu} \in \mathbb{R}^k_+$, $\hat{\mu}$ is in the constraint set of problem (Q).

Now, consider an arbitrary $\mu \in \mathbb{R}^k_+$ such that $\mu A \geq c$. Then $(c - \mu A)\hat{x} \leq 0$ since $\hat{x} \in \mathbb{R}^m_+$, and using this in (10.21) yields

$$\mu b \geq c\hat{x} + \hat{\mu}(b - A\hat{x}) \tag{10.25}$$

Using (10.18) and (10.22) in (10.25) implies that $\mu b \geq \hat{\mu}b$. This proves that $\hat{\mu}$ solves (Q).
■

10.5 Constrained Local and Global Maxima

It is clear that if $\hat{x}$ is a point of constrained global maximum, then $\hat{x}$ is also a point of constrained local maximum. The circumstances under which the converse is true are given by the following theorem.

Theorem 49. *Let X be a convex set in $\mathbb{R}^n$. Let $f, g^j\,(j = 1, \ldots, m)$ be concave functions on X. Suppose $\hat{x}$ is a point of constrained local maximum. Then, $\hat{x}$ is a point of constrained global maximum.*

Proof. Since $\hat{x}$ is a point of constrained local maximum, there is $\delta > 0$, such that for all $x \in B(\hat{x}, \delta) \cap C$, we have $f(x) \leq f(\hat{x})$.

Now, if x is *not* a point of constrained global maximum, then there is some $\bar{x} \in C$, such that $f(\bar{x}) > f(\hat{x})$. One can choose $0 < \theta < 1$ with θ sufficiently close to zero, such that $\tilde{x} \equiv [\theta\bar{x} + (1-\theta)\hat{x}] \in B(\hat{x}, \delta)$. Since X is convex and $g^j\,(j = 1, \ldots, m)$ are concave, C is a convex set, and $\tilde{x} \equiv [\theta\bar{x} + (1-\theta)\hat{x}] \in C$. Thus $\tilde{x} \equiv [\theta\bar{x} + (1-\theta)\hat{x}] \in B(\hat{x}, \delta) \cap C$. Also, since f is concave, $f(\tilde{x}) = f(\theta\bar{x} + (1-\theta)\hat{x}) \geq \theta f(\bar{x}) + (1-\theta) f(\hat{x}) > \theta f(\hat{x}) + (1-\theta) f(\hat{x}) = f(\hat{x})$. But this contradicts the fact that $\hat{x}$ is a point of constrained local maximum. ■

10.6 Appendix: On the Kuhn-Tucker Conditions

Let X be an open set in $\mathbb{R}^n$, containing $\mathbb{R}^n_+$, and $f, G^j (j = 1, \ldots, r)$ be continuously differentiable on X. We consider the optimization problem:

$$\left.\begin{array}{rll} Max & f(x) & \\ subject\ to & G^j(x) \geq 0 & for\ \ j = 1, \ldots, r \\ and & x \in \mathbb{R}^n_+ & \end{array}\right\} (P)$$

We rewrite problem (P) in its equivalent form, given by:

$$\left.\begin{array}{lll} Max & f(x) & \\ subject\ to & g^j(x) \geq 0 & for\ \ j = 1, \ldots, m \\ and & x \in X & \end{array}\right\} (Q)$$

where $m = r + n$, and:

$$\begin{array}{c} g^j(x) = G^j(x) \ \ for\ j = 1, \ldots, r \\ g^j(x) = x_{j-r} \ \ for\ j = r+1, \ldots, r+n \end{array} \tag{1}$$

We can now write down the Kuhn-Tucker conditions for problem (Q). A pair $(\hat{x}, \hat{\lambda})$ in $X \times \mathbb{R}^m_+$ satisfies the *Kuhn-Tucker conditions* for problem (Q) if:

$$\left.\begin{array}{lll} (i) & D_i f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j D_i g^j(\hat{x}) = 0 & i = 1, \ldots, n \\ (ii) & g(\hat{x}) \geq 0 \qquad and\ \ \hat{\lambda} g(\hat{x}) = 0 & \end{array}\right\} (KT\ I) \tag{2}$$

Denoting $\hat{\lambda}_{r+i}$ by $\hat{\mu}_i$ for $i = 1, \ldots, n$, and $\hat{\lambda}_j = \hat{\nu}_j$ for $j = 1, \ldots, r$, we see that (2) can be written as:

$$\begin{array}{lll} (i) & D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x}) + \hat{\mu}_i = 0 & i = 1, \ldots, n \\ (ii) & G(\hat{x}) \geq 0,\ \hat{x} \geq 0\ and\ \ \hat{\nu} G(\hat{x}) = 0, \hat{\mu}\hat{x} = 0 & \end{array} \tag{3}$$

Since $\hat{\mu}_i \geq 0$ and $\hat{\mu}_i \hat{x}_i = 0$ for $i = 1, \ldots, n$ by (3), we obtain $(\hat{x}, \hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$:

$$\left.\begin{array}{l} (i)(a) \qquad D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x}) \leq 0 \qquad i = 1, \ldots, n \\ (i)(b)\ [D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x})]\hat{x}_i = 0\ \ i = 1, \ldots, n \\ (ii)(a) G^j(\hat{x}) \geq 0\ for\ j = 1, \ldots r \\ (ii)(b)\ \sum_{j=1}^{r} \hat{\nu}_j G^j(\hat{x}) = 0 \end{array}\right\} (KT\ II) \tag{4}$$

Conversely, given problem (P), suppose we can obtain $(\hat{x},\hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$ satisfying $(KT\ II)$. Then we can define:

$$\hat{\mu}_i = -[D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x})] \; for \; i = 1,\ldots,n \tag{5}$$

and:

$$\hat{\lambda}_j = \hat{\nu}_j \; for \; j = 1,\ldots,r, \hat{\lambda}_{r+i} = \hat{\mu}_i \; for \; i = 1,\ldots,n \tag{6}$$

Then denoting $(r+n)$ by m, we obtain $\hat{\mu} \in \mathbb{R}^n_+$ by $(KT\ II(i)(a))$ and (5) and $\hat{\lambda} \in \mathbb{R}^m_+$ by (6).

Further, defining:

$$\begin{aligned} g^j(x) &= G^j(x) \;\; for \; j = 1,\ldots,r \\ g^j(x) &= x_{j-r} \;\; for \; j = r+1,\ldots,r+n \end{aligned} \tag{7}$$

we obtain:

$$g^j(\hat{x}) \geq 0 \; for \; j = 1,\ldots m \tag{8}$$

since $\hat{x} \in \mathbb{R}^n_+$ and $(KT\ II(ii)(a))$ holds.

Finally, using $(KT\ II(i)(b))$ and $(KT\ II(ii)(b))$, one obtains:

$$\hat{\nu}G(\hat{x}) = 0, \hat{\mu}\hat{x} = 0 \tag{9}$$

Combining (5)-(9), we obtain $(\hat{x},\hat{\lambda})$ in $X \times \mathbb{R}^m_+$ such that:

$$\left.\begin{array}{lll} (i) & D_i f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j D_i g^j(\hat{x}) = 0 & i = 1,\ldots,n \\ (ii) & g(\hat{x}) \geq 0 \quad and \;\; \hat{\lambda} g(\hat{x}) = 0 & \end{array}\right\} (KT\ I) \tag{10}$$

Thus $(KT\ I)$ and $(KT\ II)$ are equivalent ways of writing the Kuhn-Tucker conditions, given problem (P).

10.7 Worked Out Problems on Chapter 10

Problem 41 (Using Kuhn-Tucker Sufficiency Theory by Contracting the Domain)**.**

Let p be an arbitrary positive real number. Consider the following constrained optimization problem:

$$\left.\begin{array}{ll} Maximize & x_1^{0.5}+x_2^{0.5} \\ subject\ to & px_1+x_2 \le px_3+x_4 \\ & (x_3)^2+(x_4)^2 \le 1 \\ & (x_1,x_2,x_3,x_4) \in \mathbb{R}_+^4 \end{array}\right\}(R)$$

(a) To solve problem (R), first solve problem (S) given below:

$$\left.\begin{array}{ll} Maximize & x_1^{0.5}+x_2^{0.5} \\ subject\ to & px_1+x_2 \le px_3+x_4 \\ & (x_3)^2+(x_4)^2 \le 1 \\ & (x_1,x_2,x_3,x_4) \in \mathbb{R}_{++}^4 \end{array}\right\}(S)$$

Define $X=\mathbb{R}_{++}^4$, $f(x)=x_1^{0.5}+x_2^{0.5}$, $g^1(x)=px_3+x_4-px_1-x_2$, $g^2(x)=1-[(x_3)^2+(x_4)^2]$, where $x=(x_1,x_2,x_3,x_4)\in X$. Write down and solve the Kuhn-Tucker conditions for problem (S), and denote the solution of the Kuhn-Tucker conditions by $(\bar{x},\bar{\lambda})\in X\times\mathbb{R}_+^2$.

(b) Show that $\bar{x}$ solves problem (S), and $(\bar{x},\bar{\lambda})$ satisfies:

$$f(x)+\bar{\lambda}g(x)\le f(\bar{x})+\bar{\lambda}g(\bar{x}) \text{ for all } x\in X$$

(c) Use (b) and the continuity of f , g^1 and g^2 on $\mathbb{R}_+^4$ to establish that $\bar{x}$ solves (R).

Solution.

(a) Define the set $X=\mathbb{R}_{++}^4$ and define the functions f, g^1, and g^2, each from X to $\mathbb{R}$, by

$$\begin{aligned} f(x) &= x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}} && \text{for all } x\in X \\ g^1(x) &= px_3+x_4-px_1-x_2 && \text{for all } x\in X \\ g^2(x) &= 1-x_3^2-x_4^2 && \text{for all } x\in X \end{aligned}$$

A pair $(\bar{x},\bar{\lambda})\in X\times\mathbb{R}_+^2$ satisfies the Kuhn-Tucker conditions for problem (S) if it satisfies the following:

$$\begin{aligned} \frac{1}{2}\bar{x}_1^{-\frac{1}{2}}+\bar{\lambda}_1(-p) &= 0 \quad (1.1) & p\bar{x}_3+\bar{x}_4-p\bar{x}_1-\bar{x}_2 &\ge 0 \quad (1.5) \\ \frac{1}{2}\bar{x}_2^{-\frac{1}{2}}+\bar{\lambda}_1(-1) &= 0 \quad (1.2) & 1-\bar{x}_3^2-\bar{x}_4^2 &\ge 0 \quad (1.6) \\ \bar{\lambda}_1(p)+\bar{\lambda}_2(-2\bar{x}_3) &= 0 \quad (1.3) & \bar{\lambda}_1\left(p\bar{x}_3+\bar{x}_4-p\bar{x}_1-\bar{x}_2\right) &= 0 \quad (1.7) \\ \bar{\lambda}_1+\bar{\lambda}_2(-2\bar{x}_4) &= 0 \quad (1.4) & \bar{\lambda}_2\left(1-\bar{x}_3^2-\bar{x}_4^2\right) &= 0 \quad (1.8) \end{aligned}$$

Since the objective function is increasing in x_1 and x_2, we expect that the first constraint in problem (S) will hold with equality at any solution. In addition, the greater is the right hand side of the first constraint, the greater is the value of the objective function that we can achieve. So we expect the second constraint in problem (S) to hold with equality, also.

Seeking contradiction, suppose that $\bar{\lambda}_1 = 0$. Then (1.1) and (1.2) cannot hold, so we have a contradiction. Therefore $\bar{\lambda}_1 > 0$. By (1.7), then, we have

$$p\bar{x}_3 + \bar{x}_4 - p\bar{x}_1 - \bar{x}_2 = 0 \tag{1.9}$$

Again seeking contradiction, suppose that $\bar{\lambda}_2 = 0$. But then (1.4) implies that $\bar{\lambda}_1 = 0$, which contradicts what we have just shown. Therefore $\bar{\lambda}_2 > 0$, so by (1.8) we have

$$1 - \bar{x}_3^2 - \bar{x}_4^2 = 0 \tag{1.10}$$

From (1.3) and (1.4), we have that $p = \frac{\bar{x}_3}{\bar{x}_4}$. Using this in (1.10), we have that $(p\bar{x}_4)^2 + \bar{x}_4^2 = 1$, which gives

$$\bar{x}_4 = \frac{1}{\sqrt{1+p^2}}, \qquad \bar{x}_3 = p\bar{x}_4 = \frac{p}{\sqrt{1+p^2}}$$

Now, using (1.1) and (1.2) to eliminate $\bar{\lambda}_1$, we have $p = \sqrt{\frac{\bar{x}_2}{\bar{x}_1}}$, or $\bar{x}_2 = p^2\bar{x}_1$. Then we can use (1.9) and $p = \frac{\bar{x}_3}{\bar{x}_4}$ to write $p^2\bar{x}_4 + \bar{x}_4 - p\bar{x}_1 - p^2\bar{x}_1 = 0$. This implies that

$$\bar{x}_1 = \frac{1+p^2}{p(1+p)}\bar{x}_4 = \frac{\sqrt{1+p^2}}{p(1+p)}, \qquad \bar{x}_2 = p^2\bar{x}_1 = \frac{p\sqrt{1+p^2}}{1+p}$$

From (1.2) and then (1.4), we have

$$\bar{\lambda}_1 = \frac{1}{2}\bar{x}_2^{-\frac{1}{2}} = \frac{(1+p)^{\frac{1}{2}}}{2p^{\frac{1}{2}}(1+p^2)^{\frac{1}{4}}}, \qquad \bar{\lambda}_2 = \frac{\bar{\lambda}_1}{2\bar{x}_4} = \frac{(1+p)^{\frac{1}{2}}(1+p^2)^{\frac{1}{4}}}{4p^{\frac{1}{2}}}$$

The pair $(\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}^2_+$ given above is the unique solution to the Kuhn-Tucker conditions for problem (S).

(b) We need to verify that the conditions of the Kuhn-Tucker Sufficiency Theorem are satisfied before we can conclude that $\bar{x}$ solves problem (S).

The set X is open and convex in $\mathbb{R}^4$. The functions f, g^1, and g^2 are continuously differentiable on X.

The function f can be expressed as the sum of functions that are concave on X, so f is concave on X. The function g^1 is linear on X, so it is concave on X. Now, the functions $h^1(x) = x_3^2$ and $h^3(x) = x_4^2$ are convex on X, so $-h^1$ and $-h^2$ are concave on X. Since we can write $g^2(x) = 1 - h^1(x) - h^2(x)$ for all $x \in X$, we can express g^2 as the sum of functions that are concave on X. This means that g^2 is concave on X.

So all the conditions of the Kuhn-Tucker Sufficiency Theorem are met. Since $(\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}^2_+$ satisfies the Kuhn-Tucker conditions for problem (S), then, $\bar{x}$ solves problem (S).

Theorem 46, Chapter 10, which is the Kuhn-Tucker Sufficiency Theorem, includes the result that $(\bar{x}, \bar{\lambda})$ is a saddle point. By the definition of a saddle point, then, we have that $f(x) + \bar{\lambda}g(x) \le f(\bar{x}) + \bar{\lambda}g(\bar{x})$ for all $x \in X$.

(c) Let x be an arbitrary point in $\mathbb{R}^4_+$, and define the sequence $\{x^n\}_{n=1}^{\infty}$ by

$$x^n = \left(x_1 + \frac{1}{n}, x_2 + \frac{1}{n}, x_3 + \frac{1}{n}, x_4 + \frac{1}{n}\right) \text{ for all } n = 1, 2, \ldots$$

Then $x^n \in \mathbb{R}^4_{++}$ for all $n = 1, 2, \ldots$, and the sequence $\{x^n\}_{n=1}^{\infty}$ converges to x. Since f, g^1, and g^2 are continuous on $\mathbb{R}^4_+$, the sequence $\{f(x^n)\}_{n=1}^{\infty}$ converges to $f(x)$, the sequence $\{g^1(x^n)\}_{n=1}^{\infty}$ converges to $g^1(x)$, and the sequence $\{g^2(x^n)\}_{n=1}^{\infty}$ converges to $g^2(x)$. That is, the sequence $\{f(x^n) + \bar{\lambda}g(x^n)\}_{n=1}^{\infty}$ converges to $f(x) + \bar{\lambda}g(x)$.

Using the result in part (b) and the fact that $x^n \in X = \mathbb{R}^4_{++}$ for all $n = 1, 2, \ldots$, we have that $f(x^n) + \bar{\lambda}g(x^n) \le f(\bar{x}) + \bar{\lambda}g(\bar{x})$ for all $n = 1, 2, \ldots$. Since weak inequalities are preserved in the limit and the sequence $\{f(x^n) + \bar{\lambda}g(x^n)\}_{n=1}^{\infty}$ converges to $f(x) + \bar{\lambda}g(x)$, we have that $f(x) + \bar{\lambda}g(x) \le f(\bar{x}) + \bar{\lambda}g(\bar{x})$ for our arbitrary $x \in \mathbb{R}^4_+$, and thus for all $x \in \mathbb{R}^4_+$.

Now, in part (b) we said that $(\bar{x}, \bar{\lambda})$ is a saddle point for problem (S), which includes the inequality $f(\bar{x}) + \bar{\lambda}g(\bar{x}) \le f(\bar{x}) + \lambda g(\bar{x})$ for all $\lambda \in \mathbb{R}^2_+$. So, from this and the previous paragraph, we have that

$$f(x) + \bar{\lambda}g(x) \le f(\bar{x}) + \bar{\lambda}g(\bar{x}) \le f(\bar{x}) + \lambda g(\bar{x}) \text{ for all } (x, \lambda) \in \mathbb{R}^4_+ \times \mathbb{R}^2_+$$

That is, $(\bar{x}, \bar{\lambda})$ is a saddle point for problem (R). By Theorem 44, then, $\bar{x}$ solves problem (R).

Problem 42 (Using Kuhn-Tucker Sufficiency Theory by Expanding the Domain)**.**

Let a,b be arbitrary positive numbers, satisfying $a > b > 1$. Consider the following constrained maximization problem:

$$\left.\begin{array}{cc} Maximize & a\ln(1+x_1)+b\ln(1+x_2)+x_3 \\ subject\quad to & x_1+x_2+x_3 \leq 1 \\ and & (x_1,x_2,x_3) \in \mathbb{R}^3_+ \end{array}\right\}(R)$$

(a) Define $X = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_i > -1 \text{ for all } i \in \{1,2,3\}\}$. Now, define $f(x_1,x_2,x_3) = a\ln(1+x_1)+b\ln(1+x_2)+x_3$, $g^1(x_1,x_2,x_3) = 1-(x_1+x_2+x_3)$, $g^2(x_1,x_2,x_3) = x_1$, $g^3(x_1,x_2,x_3) = x_2$, $g^4(x_1,x_2,x_3) = x_3$ for all $(x_1,x_2,x_3) \in X$. Write down the appropriate Kuhn-Tucker conditions for problem (R).

(b) Solve the Kuhn-Tucker conditions in each of the following three cases: (i) $a \geq 2b$; (ii) $a < 2b$ and $(a+b) \geq 3$; (iii) $a < 2b$ and $(a+b) < 3$.

(c) Use your solutions to the Kuhn-Tucker conditions to obtain solutions to (R) in each of the three cases specified in (b) above.

Solution.

(a) Define the set

$$X = \left\{ (x_1,x_2,x_3) \in \mathbb{R}^3 \,\middle|\, x_i > -1 \text{ for all } i = 1,2,3 \right\}$$

Now, define the following functions, each from X to $\mathbb{R}$:

$$\begin{array}{ll} f(x_1,x_2,x_3) = a\ln(1+x_1)+b\ln(1+x_2)+x_3 & \text{for all } (x_1,x_2,x_3) \in X \\ g^1(x_1,x_2,x_3) = 1-(x_1+x_2+x_3) & \text{for all } (x_1,x_2,x_3) \in X \\ g^2(x_1,x_2,x_3) = x_1 & \text{for all } (x_1,x_2,x_3) \in X \\ g^3(x_1,x_2,x_3) = x_2 & \text{for all } (x_1,x_2,x_3) \in X \\ g^4(x_1,x_2,x_3) = x_3 & \text{for all } (x_1,x_2,x_3) \in X \end{array}$$

A pair $(\hat{x},\hat{\lambda}) \in X \times \mathbb{R}^4_+$ satisfies the Kuhn-Tucker conditions for problem (R) if it satisfies the following:

$$\frac{a}{1+\hat{x}_1} - \hat{\lambda}_1 + \hat{\lambda}_2 = 0 \qquad (2.1)$$
$$\frac{b}{1+\hat{x}_2} - \hat{\lambda}_1 + \hat{\lambda}_3 = 0 \qquad (2.2)$$
$$1 - \hat{\lambda}_1 + \hat{\lambda}_4 = 0 \qquad (2.3)$$
$$1 - (\hat{x}_1 + \hat{x}_2 + \hat{x}_3) \geq 0 \qquad (2.4)$$
$$\hat{x}_1 \geq 0 \qquad (2.5)$$
$$\hat{x}_2 \geq 0 \qquad (2.6)$$
$$\hat{x}_3 \geq 0 \qquad (2.7)$$
$$\hat{\lambda}_1\left[1 - (\hat{x}_1 + \hat{x}_2 + \hat{x}_3)\right] = 0 \qquad (2.8)$$
$$\hat{\lambda}_2 \hat{x}_1 = 0 \qquad (2.9)$$
$$\hat{\lambda}_3 \hat{x}_2 = 0 \qquad (2.10)$$
$$\hat{\lambda}_4 \hat{x}_3 = 0 \qquad (2.11)$$

(b) We are looking for $(\hat{x}, \hat{\lambda}) \in X \times \mathbb{R}^4_+$ satisfying (2.1)–(2.11). The motivation for the following strategy was discussed in the final section meeting of the course.

First, we want to show that the budget constraint, (2.4), holds with equality. Seeking contradiction, suppose that $\hat{\lambda}_1 = 0$. Then (2.3) gives $\hat{\lambda}_4 = -1 \ngeq 0$, which is a contradiction. So $\hat{\lambda}_1 > 0$. Then by (2.8), we have

$$1 - (\hat{x}_1 + \hat{x}_2 + \hat{x}_3) = 0 \qquad (2.12)$$

Next, we want to show that $\hat{x}_1 > 0$. Seeking contradiction, suppose $\hat{x}_1 = 0$. Then by (2.1), $\hat{\lambda}_1 = a + \hat{\lambda}_2 \geq a > 1$. Then (2.3) implies that $\hat{\lambda}_4 = \hat{\lambda}_1 - 1 > 0$, so (2.11) implies that $\hat{x}_3 = 0$. Then from (2.12) we have $\hat{x}_2 = 1$, so (2.10) gives $\hat{\lambda}_3 = 0$. By (2.2), then, $\hat{\lambda}_1 = \frac{b}{2}$, so by (2.1) we have $\hat{\lambda}_2 = \frac{b}{2} - a$. But we require $\hat{\lambda}_2 \geq 0$, which can only hold if $b \geq 2a$. This contradicts the given information that $a > b > 1$, so we conclude that $\hat{x}_1 > 0$.

At this point we need to split the problem into cases.

Case 1: $\hat{x}_1 > 0$, $\hat{x}_2 = 0$. By (2.9), we have that $\hat{\lambda}_2 = 0$. Since $\hat{x}_2 = 0$, it follows from (2.2) that $b - \hat{\lambda}_1 + \hat{\lambda}_3 = 0$. Then (2.3) gives $\hat{\lambda}_4 = \hat{\lambda}_1 - 1 = b + \hat{\lambda}_3 - 1 > 0$ since $b > 1$, so (2.11) implies $\hat{x}_3 = 0$. From (2.12), then, we have $\hat{x}_1 = 1$. Plugging this into (2.1) and using the fact that $\hat{\lambda}_2 = 0$, we have $\hat{\lambda}_1 = \frac{a}{2}$. Then (2.2) yields $\hat{\lambda}_3 = \frac{a}{2} - b$. Since we require $\hat{\lambda}_3 \geq 0$, we must have $a \geq 2b$. From (2.3) we have $\hat{\lambda}_4 = \frac{a}{2} - 1$. When $a \geq 2b$ we have that $a > 2$, so $\hat{\lambda}_4 > 0$.

Case 2: $\hat{x}_1 > 0$, $\hat{x}_2 > 0$, $\hat{x}_3 = 0$. By (2.9) and (2.10), we have that $\hat{\lambda}_2 = 0$ and $\hat{\lambda}_3 = 0$. By (2.12), we have $\hat{x}_1 + \hat{x}_2 = 1$. Using this after solving (2.1) and (2.2) for $\hat{\lambda}_1$ and equating the two expressions, we have $\frac{a}{2-\hat{x}_2} = \frac{b}{1+\hat{x}_2}$. This gives $\hat{x}_2 = \frac{2b-a}{a+b}$. Since we must have $\hat{x}_2 > 0$, this case requires $a < 2b$. Then $\hat{x}_1 = 1 - \hat{x}_2 = \frac{2a-b}{a+b} > 0$. Next, we can use (2.1) to solve for $\hat{\lambda}_1 = \frac{a}{1+\hat{x}_1} = \frac{a+b}{3}$. Then from (2.3) we have $\hat{\lambda}_4 = \hat{\lambda}_1 - 1 = \frac{a+b}{3} - 1$. Since we must have $\hat{\lambda}_4 \geq 0$, a further requirement for this case is $a + b \geq 3$.

Case 3: $\hat{x}_1 > 0$, $\hat{x}_2 > 0$, $\hat{x}_3 > 0$. By (2.9), (2.10), and (2.11), we have that $\hat{\lambda}_2 = 0$, $\hat{\lambda}_3 = 0$, and $\hat{\lambda}_4 = 0$. Then (2.3) yields $\hat{\lambda}_1 = 1$. Using this in (2.1) and (2.2), we have $\hat{x}_1 = a - 1 > 0$ and $\hat{x}_2 = b - 1 > 0$. Then (2.12) implies $\hat{x}_3 = 1 - \hat{x}_1 - \hat{x}_2 = 3 - (a+b)$, so $\hat{x}_3 > 0$ requires $a + b < 3$. Now, if $a + b < 3$ and $a > b > 1$, it must also be that $a < 2b$.

To summarize:

(i) If $a \geq 2b$, the unique solution to the Kuhn-Tucker conditions is

$$\left(\hat{x}, \hat{\lambda}\right) = \left((1,0,0), \left(\frac{a}{2}, 0, \frac{a}{2} - b, \frac{a}{2} - 1\right)\right)$$

(ii) If $a < 2b$ and $a + b \geq 3$, the unique solution to the Kuhn-Tucker conditions is

$$\left(\hat{x}, \hat{\lambda}\right) = \left(\left(\frac{2a-b}{a+b}, \frac{2b-a}{a+b}, 0\right), \left(\frac{a+b}{3}, 0, 0, \frac{a+b}{3} - 1\right)\right)$$

(iii) If $a < 2b$ and $a + b < 3$, the unique solution to the Kuhn-Tucker conditions is

$$\left(\hat{x}, \hat{\lambda}\right) = \left(\left(a-1, b-1, 3-(a+b)\right), \left(1,0,0,0\right)\right)$$

(c) We need to verify that the conditions of the Kuhn-Tucker Sufficiency Theorem are satisfied before we can conclude that the various $\hat{x}$ we found in part (b) solve problem (R) under the associated parameter restrictions.

The set X, defined in part (a), is open and convex in $\mathbb{R}^3$. The functions f, g^1, g^2, g^3, and g^4 are continuously differentiable on X.

For all $x \in X$, the Hessian of f at x is

$$H_f(x) = \begin{bmatrix} -\frac{a}{(1+x_1)^2} & 0 & 0 \\ 0 & -\frac{b}{(1+x_2)^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first order principal minors of $H_f(x)$ are $-\frac{a}{(1+x_1)^2}$, $-\frac{b}{(1+x_2)^2}$, and 0, which are each less than or equal to zero for all $x \in X$. The second order principal minors of $H_f(x)$ are $\frac{a}{(1+x_1)^2}\frac{b}{(1+x_2)^2}$ and 0, which are both greater than or equal to zero for all

$x \in X$. The determinant of $H_f(x)$ is zero for all $x \in X$. So $H_f(x)$ is negative semi-definite for all $x \in X$. Therefore, since X is an open set, we have that f is concave on X.

Since the functions g^1, g^2, g^3, and g^4 are linear on X, they are concave on X.

By the Kuhn-Tucker Sufficiency Theorem, if a pair $(\hat{x}, \hat{\lambda}) \in X \times \mathbb{R}^4_+$ satisfies the Kuhn-Tucker conditions given by (2.1)–(2.11), then $\hat{x}$ solves problem (R). So from part (b) we have the following solutions to problem (R), depending on the values of the parameters a and b:

(i) If $a \geq 2b$, then $\hat{x} = (1, 0, 0)$ solves (R).

(ii) If $a < 2b$ and $a + b \geq 3$, then $\hat{x} = \left(\frac{2a-b}{a+b}, \frac{2b-a}{a+b}, 0\right)$ solves (R).

(iii) If $a < 2b$ and $a + b < 3$, then $\hat{x} = \big(a-1, b-1, 3-(a+b)\big)$ solves (R).

Problem 43 (Using Kuhn-Tucker Sufficiency Theory for a Non-Differentiable Objective Function)**.**

Let a_1, a_2, p_1, p_2, w be arbitrary positive numbers. Consider the constrained optimization problem:

$$\left.\begin{array}{cc} \textit{Maximize} & \min\{a_1x_1, a_2x_2\} \\ \textit{subject to} & p_1x_1 + p_2x_2 \leq w \\ & (x_1, x_2) \geq 0 \end{array}\right\} (Q)$$

Here the objective function is not differentiable (the indifference curves are L-shaped). Instead of solving (Q), we consider the following constrained optimization problem:

$$\left.\begin{array}{cc} \textit{Maximize} & a_1x_1 \\ \textit{subject to} & a_2x_2 \geq a_1x_1 \\ & p_1x_1 + p_2x_2 \leq w \\ & (x_1, x_2) \geq 0 \end{array}\right\} (Q')$$

(a) Solve problem (Q') by using the Kuhn-Tucker sufficiency theorem.
(b) Show that any solution of (Q') is also a solution of (Q).

Solution.

(a) To use the Kuhn-Tucker Sufficiency Theorem, we need to specify an open set X. The set $\mathbb{R}^2_{++}$ would work, but we would have to go through the trouble of ruling out corner solutions. On the other hand, if we think about solving the utility maximization problem on all of $\mathbb{R}^2$, it is clear that the solution will be somewhere in $\mathbb{R}^2_{++}$. So it suffices to use $X = \mathbb{R}^2$, and we don't even have to include the constraints $x_1 \geq 0$ and $x_2 \geq 0$ as long as we find a solution $\hat{x} \in \mathbb{R}^2_+$ for all positive parameter values, as required by the problem.

Define the following functions, each from $X = \mathbb{R}^2$ to $\mathbb{R}$:

$$\begin{aligned} f(x_1,x_2) &= a_1x_1 && \text{for all } (x_1,x_2) \in X \\ g^1(x_1,x_2) &= a_2x_2 - a_1x_1 && \text{for all } (x_1,x_2) \in X \\ g^2(x_1,x_2) &= w - p_1x_1 - p_2x_2 && \text{for all } (x_1,x_2) \in X \end{aligned}$$

Now, X is open and convex in $\mathbb{R}^2$. The functions f, g^1, and g^2 are each linear on X, so they are each continuously differentiable and concave on X. Therefore all the conditions of the Kuhn-Tucker Sufficiency Theorem are met. A pair $(\hat{x},\hat{\lambda}) \in X \times \mathbb{R}^2_+$ satisfies the Kuhn-Tucker conditions for problem (Q') if it satisfies the following:

$$a_1 - \hat{\lambda}_1 a_1 - \hat{\lambda}_2 p_1 = 0 \tag{3.1}$$

$$\hat{\lambda}_1 a_2 - \hat{\lambda}_2 p_2 = 0 \tag{3.2}$$

$$a_2\hat{x}_2 - a_1\hat{x}_1 \geq 0 \tag{3.3}$$

$$w - p_1\hat{x}_1 - p_2\hat{x}_2 \geq 0 \tag{3.4}$$

$$\hat{\lambda}_1(a_2\hat{x}_2 - a_1\hat{x}_1) = 0 \tag{3.5}$$

$$\hat{\lambda}_2(w - p_1\hat{x}_1 - p_2\hat{x}_2) = 0 \tag{3.6}$$

First, note that (3.1) and (3.2) form two equations in two unknowns. Without dividing by $\hat{\lambda}_1$ or $\hat{\lambda}_2$, we can solve them for

$$\hat{\lambda}_1 = \frac{a_1p_2}{a_2p_1 + a_1p_2}, \qquad \hat{\lambda}_2 = \frac{a_1a_2}{a_2p_1 + a_1p_2}$$

Since $\hat{\lambda}_1 > 0$ and $\hat{\lambda}_2 > 0$, from (3.5) and (3.6) we have that $a_2\hat{x}_2 - a_1\hat{x}_1 = 0$ and $w - p_1\hat{x}_1 - p_2\hat{x}_2 = 0$. This is again two equations in two unknowns, and without dividing by $\hat{x}_1$ or $\hat{x}_2$ we can solve them for

$$\hat{x}_1 = \frac{a_2w}{a_2p_1 + a_1p_2} > 0, \qquad \hat{x}_2 = \frac{a_1w}{a_2p_1 + a_1p_2} > 0$$

By the Kuhn-Tucker Sufficiency Theorem, $(\hat{x}_1,\hat{x}_2)$ solves problem (Q').

(b) Consider the problem

$$\left.\begin{array}{ll} \text{maximize} & a_2x_2 \\ \text{subject to} & a_1x_1 \geq a_2x_2 \\ & p_1x_1 + p_2x_2 \leq w \\ & (x_1, x_2) \geq 0 \end{array}\right\}(Q'')$$

Following steps symmetric to those in part (a), we have that the solution $\hat{x}$ also solves problem (Q'').

Let $\bar{x}$ by a solution to problem (Q'). Then $\bar{x}$ also solves problem (Q''). Note that $a_1\bar{x}_1 = a_2\bar{x}_2$. Seeking contradiction, suppose that $\bar{x}$ is not a solution to (Q). Then there is some $x' \in \mathbb{R}^2_+$, $x' \neq \bar{x}$ such that $p_1x'_1 + p_2x'_2 \leq w$ and $\min\{a_1x'_1, a_2x'_2\} > a_1\bar{x}_1 = a_2\bar{x}_2$. We can consider two cases.

Case 1: $\min\{a_1x'_1, a_2x'_2\} = a_1x'_1$. Then we have $a_1x'_1 \leq a_2x'_2$, so that x' is in the constraint set of problem (Q'), and $a_1x'_1 > a_1\bar{x}_1$. This contradicts $\bar{x}$ solving problem (Q').

Case 2: $\min\{a_1x'_1, a_2x'_2\} = a_2x'_2$. Then we have $a_2x'_2 \leq a_1x'_1$, so that x' is in the constraint set of problem (Q''), and $a_2x'_2 > a_2\bar{x}_2$. This contradicts $\bar{x}$ solving problem (Q'').

In either of these two collectively exhaustive cases, we arrive at a contradiction. So it must be that if $\bar{x}$ solves (Q'), then $\bar{x}$ also solves (Q).

Problem 44 (Applying the Arrow-Hurwicz-Uzawa Necessity Theorem)**.**

Let a, b, c and p, q, r be arbitrary positive numbers, satisfying: $1 > (p/r) \geq (a+b)c$. Consider the following two problems of constrained optimization:

$$\left.\begin{array}{ll} Maximize & x_1^{ac}x_2^{bc} - px_1 - qx_2 \\ subject\ to & (x_1, x_2) \in \mathbb{R}^2_+ \end{array}\right\}(Q)$$

$$\left.\begin{array}{ll} Maximize & x_1^{ac}x_2^{bc} - px_1 - qx_2 \\ subject\ to & rx_1 + qx_2 - x_1^{ac}x_2^{bc} \geq 0 \\ & (x_1, x_2) \in \mathbb{R}^2_+ \end{array}\right\}(R)$$

(a) Using an appropriately modified version of Weierstrass theorem, establish that there is a solution $\bar{x}$ to problem (Q), and a solution $\hat{x}$ to problem (R).

(b) Show that $\bar{x}_i > 0$ for $i = 1, 2$, and $\hat{x}_i > 0$ for $i = 1, 2$.

(c) Use the Arrow-Hurwicz-Uzawa necessity theorem to compare the solutions $\bar{x}$ and $\hat{x}$, and show that:

$$(\hat{x}_1 / \hat{x}_2) > (\bar{x}_1 / \bar{x}_2)$$

explaining your procedure clearly.[Hint: you might want to use the fact that $x_1^{ac}x_2^{bc}$ is homogeneous of degree $(a+b)c$ in (x_1,x_2)].

Solution.

(a) Define $f:\mathbb{R}_+^2 \to \mathbb{R}$ and $g:\mathbb{R}_+^2 \to \mathbb{R}$ by

$$\begin{aligned} f(x_1,x_2) &= x_1^{ac}x_2^{bc} - px_1 - qx_2 \quad \text{for all } (x_1,x_2) \in \mathbb{R}_+^2 \\ g(x_1,x_2) &= rx_1 + qx_2 - x_1^{ac}x_2^{bc} \quad \text{for all } (x_1,x_2) \in \mathbb{R}_+^2 \end{aligned}$$

Note that f and g are continuous on $\mathbb{R}_+^2$. Define the constraint sets

$$C_Q = \mathbb{R}_+^2, \qquad C_R = \{\, (x_1,x_2) \in \mathbb{R}_+^2 \mid g(x_1,x_2) \geq 0 \,\}$$

Now, C_Q is closed in $\mathbb{R}^2$ since $\mathbb{R}_+^2$ is closed in $\mathbb{R}^2$, and C_Q is nonempty since $(0,0) \in C_Q$. Also, C_R is closed in $\mathbb{R}^2$ since g is continuous on $\mathbb{R}_+^2$, and C_R is nonempty since $(0,0) \in C_R$. But C_Q and C_R are both unbounded sets in $\mathbb{R}^2$.

Because $(a+b)c < 1$, for large enough x_1 or x_2, we will have $f(x_1,x_2) < 0$. So we can use $(0,0)$, with $f(0,0) = 0$, as our comparison point in order to apply the Extension of Weierstrass Theorem. Here is a formal argument of this. First, define $k = \max\{x_1,x_2\}$ and $\pi = \min\{p,q\}$. Then we have

$$\begin{aligned} f(x_1,x_2) &= x_1^{ac}x_2^{bc} - px_1 - qx_2 \\ &\leq [\max\{x_1,x_2\}]^{(a+b)c} - [\min\{p,q\}]\,(x_1+x_2) \\ &= k^{(a+b)c} - \pi(x_1+x_2) \\ &\leq k^{(a+b)c} - \pi\,[\max\{x_1,x_2\}] \\ &= k^{(a+b)c} - \pi k \end{aligned}$$

Define the function $e:\mathbb{R}_+ \to \mathbb{R}$ by

$$e(k) = k^{(a+b)c} - \pi k \quad \text{for all } k \geq 0$$

Our goal is to find some $\bar{k} > 0$ such that $e(k) < 0$ whenever $k > \bar{k}$. That will imply that $f(x_1,x_2) < 0$ whenever $\max\{x_1,x_2\} > \bar{k}$.

For $k > 0$, there is a unique solution to the equation $e(k) = 0$, which is $\bar{k} = \pi^{\frac{1}{(a+b)c-1}}$. Now, $e'(k) = (a+b)ck^{(a+b)c-1} - \pi$ for all $k > 0$. Since $(a+b)c - 1 < 0$, we have that $e'(\bar{k}) = \pi\big((a+b)c-1\big) < 0$. Since $\bar{k}$ is the unique solution to $e(k) = 0$ for $k > 0$, e is continuous on its domain, and $e'(\overline{k}) < \underline{0}$, we have that $e(k) < 0$ for all $k > \overline{k}$. Therefore $f(x_1, x_2) < 0$ whenever $\max\{x_1, x_2\} > \overline{k}$.

Define the sets

$$\overline{C_Q} = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \,\middle|\, \max\{x_1, x_2\} \leq \overline{k} \right\}$$
$$\overline{C_R} = \left\{ (x_1, x_2) \in \mathbb{R}^2_+ \,\middle|\, \max\{x_1, x_2\} \leq \overline{k} \text{ and } g(x_1, x_2) \geq 0 \right\}$$

Since $(0,0) \in \overline{C_Q}$ and $(0,0) \in \overline{C_R}$, these sets are nonempty. Since $g(x_1, x_2)$ and $\max\{x_1, x_2\}$ are continuous functions on $\mathbb{R}^2_+$ and the sets are defined by weak inequalities, the sets are closed. And the restriction $\max\{x_1, x_2\} \leq \overline{k}$ means that the sets are bounded.

Now, for any $(x_1, x_2) \in C_Q \setminus \overline{C_Q}$, or for any $(x_1, x_2) \in C_R \setminus \overline{C_R}$, we have that $\max\{x_1, x_2\} > \overline{k}$, so it must be that $f(x_1, x_2) < 0 = f(0,0)$. So by the Extension of Weierstrass Theorem, there is some $\overline{x} \in C_Q$ that solves problem (Q) and some $\hat{x} \in C_R$ that solves problem (R).

(b) Seeking contradiction, suppose that $\overline{x}_1 = 0$ or $\overline{x}_2 = 0$. Then $f(\overline{x}_1, \overline{x}_2) \leq 0$. But for $0 < \varepsilon < (p+q)^{\frac{1}{(a+b)c-1}}$, we have that

$$\begin{aligned} f(\varepsilon, \varepsilon) &= \varepsilon^{ac}\varepsilon^{bc} - p\varepsilon - q\varepsilon \\ &= \varepsilon\left(\varepsilon^{(a+b)c-1} - p - q\right) \\ &> 0 \end{aligned}$$

Since $(\varepsilon, \varepsilon) \in C_Q$ and $f(\varepsilon, \varepsilon) > f(\overline{x}_1, \overline{x}_2)$, we have a contradiction of $\overline{x}$ solving problem (Q). So it must be that $\overline{x}_1 > 0$ and $\overline{x}_2 > 0$.

The idea for showing that $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$ is conceptually similar, but the proof is a bit trickier because we now have the constraint to worry about. First, define

$$\delta = (r+q)^{\frac{1}{(a+b)c-1}} > 0$$

This ensures that the constraint is satisfied: $g(\delta, \delta) = 0$. Seeking contradiction, suppose that $\hat{x}_1 = 0$ or $\hat{x}_2 = 0$. Then $f(\hat{x}_1, \hat{x}_2) \leq 0$. Now, since $r > p$ and $(a+b)c < 1$, we have that

$$\delta = (r+q)^{\frac{1}{(a+b)c-1}} < (p+q)^{\frac{1}{(a+b)c-1}}$$

Therefore $f(\delta, \delta) > 0$. Then $(\delta, \delta) \in C_R$ and $f(\delta, \delta) > f(\hat{x}_1, \hat{x}_2)$, which is a contradiction of $\hat{x}$ solving problem (R). So it must be that $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$.

(c) Define $X = \mathbb{R}^2_{++}$. Then X is open in $\mathbb{R}^2$ and the functions f and g are continuously differentiable on X. By parts (a) and (b), there is some $\bar{x} \in X$ that is a point of local maximum in problem (Q) and some $\hat{x} \in X$ that is a point of local maximum in problem (R).

We need to verify that the Arrow-Hurwicz-Uzawa Constraint Qualification is satisfied. We have that X is convex. For all $x \in X$, the Hessian of g at x is

$$H_g(x) = \begin{bmatrix} ac(1-ac)x_1^{ac-2}x_2^{bc} & abc^2 x_1^{ac-1}x_2^{bc-1} \\ abc^2 x_1^{ac-1}x_2^{bc-1} & bc(1-bc)x_1^{ac}x_2^{bc-2} \end{bmatrix}$$

Since $(a+b)c < 1$ with each parameter positive, it must be that $ac < 1$ and $bc < 1$. Therefore $ac(1-ac)x_1^{ac-2}x_2^{bc} > 0$ and $bc(1-bc)x_1^{ac}x_2^{bc-2} > 0$. Now, the determinant of the Hessian of g at any $x \in X$ is $abc^2(1-ac-bc)x_1^{2ac-2}x_2^{2bc-2} > 0$ since $(a+b)c < 1$. This means that the Hessian of g is positive semi-definite for all $x \in X$, which is equivalent to saying that g is a convex function on X. Therefore the Arrow-Hurwicz-Uzawa Constraint Qualification is satisfied.

By the Arrow-Hurwicz-Uzawa Necessity Theorem, then, there is some $\hat{\lambda} \in \mathbb{R}_+$ such that $\bar{x}$ satisfies the Kuhn-Tucker conditions for problem (Q) and $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions for problem (R).

Note that because of the result in part (b), we could apply the Arrow-Hurwicz-Uzawa Necessity Theorem to problem (R) only and use unconstrained optimization theory on problem (Q). This would require showing that f is concave on $\mathbb{R}^2_{++}$, so that the first order conditions are sufficient for a solution.

Also note that regarding problem (R), the result in part (b) allows us to leave the constraints $\hat{x}_1 > 0$ and $\hat{x}_2 > 0$ out of the Kuhn-Tucker conditions, so that we do not have to deal with two additional multipliers.

Now, the Kuhn-Tucker conditions for problem (Q) are

$$ac\bar{x}_1^{ac-1}\bar{x}_2^{bc} - p = 0 \tag{4.1}$$

$$bc\bar{x}_1^{ac}\bar{x}_2^{bc-1} - q = 0 \tag{4.2}$$

From (4.1) and (4.2), we can obtain

$$\frac{\bar{x}_1}{\bar{x}_2} = \frac{q}{p}\frac{a}{b} \tag{4.3}$$

The Kuhn-Tucker conditions for problem (R) are

$$ac\hat{x}_1^{ac-1}\hat{x}_2^{bc} - p + \hat{\lambda}\left(r - ac\hat{x}_1^{ac-1}\hat{x}_2^{bc}\right) = 0 \tag{4.4}$$

$$bc\hat{x}_1^{ac}\hat{x}_2^{bc-1} - q + \hat{\lambda}\left(q - bc\hat{x}_1^{ac}\hat{x}_2^{bc-1}\right) = 0 \tag{4.5}$$

$$r\hat{x}_1 + q\hat{x}_2 - \hat{x}_1^{ac}\hat{x}_2^{bc} \geq 0 \tag{4.6}$$

$$\hat{\lambda}\left(r\hat{x}_1 + q\hat{x}_2 - \hat{x}_1^{ac}\hat{x}_2^{bc}\right) = 0 \tag{4.7}$$

If $\hat{\lambda} = 0$ then the solution to problem (R) will be the same as the solution to problem (Q). But we want $\frac{\hat{x}_1}{\hat{x}_2} > \frac{\bar{x}_1}{\bar{x}_2}$, so we need to show that $\hat{\lambda} \neq 0$.

We can start by rewriting (4.5) as

$$(\hat{\lambda} - 1)\left(q - bc\hat{x}_1^{ac}\hat{x}_2^{bc-1}\right) = 0 \tag{4.8}$$

Now, if $\hat{\lambda} = 1$, then from (4.4) we have $r = p$, which is a contradiction of the given information that $r > p$. So it must be that $\hat{\lambda} \neq 1$. Then by (4.8), we have

$$q\hat{x}_2 = bc\hat{x}_1^{ac}\hat{x}_2^{bc} \tag{4.9}$$

Seeking contradiction, suppose $\hat{\lambda} = 0$. Then by (4.4), we have $p\hat{x}_1 = ac\hat{x}_1^{ac}\hat{x}_2^{bc}$. Using (4.9) in (4.6), we have $r\hat{x}_1 \geq (1 - bc)\hat{x}_1^{ac}\hat{x}_2^{bc}$. Now, divide the left side of this inequality by $p\hat{x}_1$ and the right side by $ac\hat{x}_1^{ac}\hat{x}_2^{bc}$, recalling that the divisors are equal. The result is $\frac{r}{p} \geq \frac{1-bc}{ac}$. But we are also given that $\frac{r}{p} \leq \frac{1}{(a+b)c}$, so we can form the inequality $\frac{1}{(a+b)c} \geq \frac{1-bc}{ac}$, which is equivalent to $a \geq (1 - bc)(a + b) = a + b - abc - b^2c$. Rearranging, we have that $b(1 - ac - bc) \leq 0$, which, since $b > 0$, implies that $(a + b)c \geq 1$. But this is a contradiction of the given information that $(a + b)c < 1$, so we can conclude that $\hat{\lambda} > 0$.

Using (4.7) and (4.9), then, we have

$$r\hat{x}_1 + (bc - 1)\hat{x}_1^{ac}\hat{x}_2^{bc} = 0 \tag{4.10}$$

Since $(a + b)c < 1$, we have $bc - 1 < -ac$, so (4.10) implies that $r\hat{x}_1 - ac\hat{x}_1^{ac}\hat{x}_2^{bc} > 0$. Multiplying (4.4) by $\hat{x}_1$, then, we have

$$ac\hat{x}_1^{ac}\hat{x}_2^{bc} < p\hat{x}_1 \tag{4.11}$$

We can divide the left side of (4.11) by $bc\hat{x}_1^{ac}\hat{x}_2^{bc}$ and the right side by $q\hat{x}_2$. Since the divisors are equal by (4.9), we have

$$\frac{\hat{x}_1}{\hat{x}_2} > \frac{q}{p}\frac{a}{b} \tag{4.12}$$

Combining (4.3) and (4.12), we have the result:

$$\frac{\hat{x}_1}{\hat{x}_2} > \frac{\overline{x_1}}{\overline{x_2}}$$

References:

This material is based on *Non-Linear Programming* by *O.L. Mangasarian* (Chapters 5, 7); *Mathematical Economics* by *A. Takayama* (Chapter 1); and *Convex Structures and Economic Theory* by *H. Nikaido* (Chapter 1).

The proof of Theorem 45 (the Kuhn-Tucker theorem), relying on Minkowski's separation theorem and Slater's condition is due to H. Uzawa: "The Kuhn-Tucker Theorem in Concave Programming" in *Studies in Linear and Non-Linear Programming* (1958), edited by K.J. Arrow, L. Hurwicz and H. Uzawa. For more on Theorem 48 (the Arrow-Hurwicz-Uzawa theorem), see K.J. Arrow, L. Hurwicz and H. Uzawa, "Constraint Qualifications in Maximization Problems", *Naval Research Logistic Quarterly*, 8 (1961), 175-191.

Chapter 11

Quasi-Concave Programming

11.1 Properties of Concave and Quasi-concave functions

For this section, it is convenient to denote $\mathbb{R}^n_+$ by Y and $\mathbb{R}^n_{++}$ by Z.

11.1.1 Concave Functions

Lemma 1. *If $h : Y \to \mathbb{R}$ is continuous on Y and concave on Z, then it is concave on Y.*

Proof. Let $x, \bar{x} \in Y$ and $0 < \theta < 1$ be given. Denote the vector $(1, \ldots, 1) \in Y$ by u and define:

$$x^s = x + (u/s), \bar{x}^s = \bar{x} + (u/s) \;\; for\; s = 1, 2, 3, \ldots \tag{11.1}$$

Then $x^s \in Z$ and $\bar{x}^s \in Z$ for each s. Since h is concave on Z, we have for each s:

$$h(\theta x^s + (1-\theta)\bar{x}^s) \geq \theta h(x^s) + (1-\theta) h(\bar{x}^s) \tag{11.2}$$

Now, let $s \to \infty$. Then, $x^s \to x, \bar{x}^s \to \bar{x}$ and $(\theta x^s + (1-\theta)\bar{x}^s) \to (\theta x + (1-\theta)\bar{x})$. Using (11.2) and the continuity of h on Y, we get:

$$h(\theta x + (1-\theta)\bar{x}) \geq \theta h(x) + (1-\theta) h(\bar{x}) \tag{11.3}$$

This establishes that h is concave on Y. ■

Lemma 2. *Let X be an open set containing Y. Let h be continuously differentiable on X and concave on Y. If $x, \bar{x} \in Y$, then:*

$$h(x) - h(\bar{x}) \leq (x - \bar{x}) \nabla h(\bar{x}) \tag{11.4}$$

Proof. Denote the vector $(1,\ldots,1) \in Y$ by u and define:

$$x^s = x + (u/s), \bar{x}^s = \bar{x} + (u/s) \quad for\ s = 1,2,3,\ldots \tag{11.5}$$

Then $x^s \in Z$ and $\bar{x}^s \in Z$ for each s. Since h is concave and continuously differentiable on the open convex set Z, we have for each s:

$$h(x^s) - h(\bar{x}^s) \leq (x^s - \bar{x}^s)\nabla h(\bar{x}^s) \tag{11.6}$$

Now, let $s \to \infty$. Then, $x^s \to x, \bar{x}^s \to \bar{x}$. Using (11.6) and the continuous differentiability of h on X, we get (11.4). ■

11.1.2 Quasi-concave functions

Lemma 3. *If $h: Y \to \mathbb{R}$ is continuous on Y and quasi-concave on Z, then it is quasi-concave on Y.*

Proof. Let $x, \bar{x} \in Y$ be given with $h(x) \geq h(\bar{x})$, and let $0 < \theta < 1$ be given. We have to show that:

$$h(\theta x + (1-\theta)\bar{x}) \geq h(\bar{x}) \tag{11.7}$$

Suppose, contrary to (11.7), we have:

$$h(\theta x + (1-\theta)\bar{x}) < h(\bar{x}) \tag{11.8}$$

Then, there is $\varepsilon > 0$ such that:

$$h(\theta x + (1-\theta)\bar{x}) < h(\bar{x}) - \varepsilon \tag{11.9}$$

Denote the vector $(1,\ldots,1) \in Y$ by u and define:

$$x^s = x + (u/s), \bar{x}^s = \bar{x} + (u/s) \quad for\ s = 1,2,3,\ldots \tag{11.10}$$

Then, since $x^s \to x$ and $\bar{x}^s \to \bar{x}$ as $s \to \infty$, and h is continuous on Y, we can find S such that for all $s \geq S$,

$$h(x^s) \geq h(x) - \varepsilon,\ h(\bar{x}^s) \geq h(\bar{x}) - \varepsilon \tag{11.11}$$

Since h is quasi-concave on Z, and $x^s \in Z, \bar{x}^s \in Z$ for $s \geq S$, we get:

$$\begin{aligned} h(\theta x^s + (1-\theta)\bar{x}^s) &\geq \min\{h(x^s), h(\bar{x}^s)\} \\ &\geq \min\{h(x) - \varepsilon, h(\bar{x}) - \varepsilon\} \\ &= h(\bar{x}) - \varepsilon \end{aligned} \tag{11.12}$$

Since $(\theta x^s + (1-\theta)\bar{x}^s) \to (\theta x + (1-\theta)\bar{x})$ as $s \to \infty$, and h is continuous on Y, (12) yields:

$$h(\theta x + (1-\theta)\bar{x}) \geq h(\bar{x}) - \varepsilon$$

which contradicts (11.9) and establishes (11.7). ■

Lemma 4. *Let X be an open set containing Y. Let h be continuously differentiable on X and quasi-concave on Y. If $x, \bar{x} \in Y$, and $h(x) \geq h(\bar{x})$, then:*

$$(x - \bar{x}) \nabla h(\bar{x}) \geq 0 \tag{11.13}$$

Proof. For $0 < \theta < 1$, define $x(\theta) = (\theta x + (1-\theta)\bar{x})$; then $x(\theta) \in Y$, since Y is a convex set. By quasi-concavity of h on Y,

$$h(x(\theta)) = h(\theta x + (1-\theta)\bar{x}) \geq h(\bar{x}) \tag{11.14}$$

Since h is continuously differentiable on X, we can apply the Mean-Value Theorem to obtain $z(\theta) \in Y$, such that:

$$h(x(\theta)) - h(\bar{x}) = [x(\theta) - \bar{x}] \nabla h(z(\theta)) = \theta [x - \bar{x}] \nabla h(z(\theta)) \tag{11.15}$$

where $z(\theta)$ is a convex combination of $x(\theta)$ and $\bar{x}$. Using (11.14) and (11.15), we obtain for each $0 < \theta < 1$:

$$[x - \bar{x}] \nabla h(z(\theta)) \geq 0 \tag{11.16}$$

Letting $\theta \to 0$, we see that $x(\theta) \to \bar{x}$ and so $z(\theta) \to \bar{x}$. Using the continuous differentiability of h on X,and (11.16), we get (11.13). ■

11.2 Definitions

Let X be an open set in $\mathbb{R}^n$, containing $\mathbb{R}^n_+$, and $f, G^j (j = 1, \ldots, r)$ be continuously differentiable on X. We are concerned with the optimization problem:

$$\left.\begin{array}{rll} Max & f(x) & \\ subject\ to & G^j(x) \geq 0 & for\ j = 1, \ldots, r \\ and & x \in \mathbb{R}^n_+ & \end{array}\right\} (P)$$

A pair $(\hat{x}, \hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$ satisfies the *Kuhn-Tucker conditions* if:

$$\left.\begin{array}{ll} (i)(a) & D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x}) \leq 0 \qquad i = 1, \ldots, n \\ (i)(b) & [D_i f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j D_i G^j(\hat{x})]\hat{x}_i = 0 \quad i = 1, \ldots, n \\ (ii)(a) & G^j(\hat{x}) \geq 0\ for\ j = 1, \ldots r \\ (ii)(b) & \sum_{j=1}^{r} \hat{\nu}_j G^j(\hat{x}) = 0 \end{array}\right\} (KT\ II) \tag{11.17}$$

For problem (P), these Kuhn-Tucker conditions $(KT\ II)$ are equivalent to the ones introduced in our discussion of concave programming in Chapter 10 (see handout on Kuhn-Tucker conditions).

The *constraint set* is defined as $C = \{x \in \mathbb{R}^n_+ : G^j(x) \geq 0 \text{ for } j = 1,\ldots,r\}$. We say that *Slater's Condition* is satisfied if there is $x^* \in C$, such that $G^j(x^*) > 0$ for $j = 1,\ldots,r$.

11.3 The Sufficiency Theorem of Arrow-Enthoven

Lemma 5. *Suppose $f, G^j(j = 1,\ldots,r)$ are continuously differentiable functions on X. Suppose there is a pair $(\hat{x},\hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$, such that $(\hat{x},\hat{\nu})$ satisfies the Kuhn-Tucker conditions. If each G^j is quasi-concave on $\mathbb{R}^n_+$, then*

$$x \in C \text{ implies } (x-\hat{x})\nabla f(\hat{x}) \leq 0 \tag{11.18}$$

Proof. Let $x \in C$. Then, we have

$$\begin{aligned}
(x-\hat{x})\nabla f(\hat{x}) &= (x-\hat{x})[\nabla f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x})] \\
&\quad -(x-\hat{x}) \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x}) \\
&= x[\nabla f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x})] \\
&\quad -(x-\hat{x}) \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x}) \\
&\leq -(x-\hat{x}) \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x})
\end{aligned}$$

If $\hat{\nu}_j = 0$ for some j, then $\hat{\nu}_j(x-\hat{x})\nabla G^j(\hat{x}) = 0$ for that j. If $\hat{\nu}_j > 0$ for some j, then $G^j(\hat{x}) = 0$ for that j, so $G^j(x) - G^j(\hat{x}) = G^j(x) \geq 0$. By quasi-concavity of G^j, we then have $\hat{\nu}_j(x-\hat{x})\nabla G^j(\hat{x}) \geq 0$ for that j, by Lemma 4. Thus, for each j, $\hat{\nu}_j(x-\hat{x})\nabla G^j(\hat{x}) \geq 0$. Consequently, we have $(x-\hat{x})\nabla f(\hat{x}) \leq 0$. ∎

To state the sufficiency result of Arrow and Enthoven, let us call an index $k \in \{1,\ldots,n\}$ a *relevant index* if there exists $x^* \in C$, such that $x^*_k > 0$. We define I as the set of relevant indices.

Theorem 50. *Suppose f, $G^j(j = 1,\ldots,r)$ are continuously differentiable on X, and quasi-concave on $\mathbb{R}^n_+$. Suppose there is a pair $(\hat{x},\hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$, such that $(\hat{x},\hat{\nu})$ satisfies the*

Kuhn-Tucker conditions. Suppose, further that at least one of the following conditions is satisfied:

(a) $D_i f(\hat{x}) < 0$ *for some* $i \in [1,\ldots,n]$.

(b) $D_i f(\hat{x}) > 0$ *for some* $i \in I$

(c) f *is concave on* X.

Then $\hat{x}$ *solves* (P).

Proof. If condition (a) holds, there is some index k, such that $D_k f(\hat{x}) < 0$. Let e^k be the k^{th} unit vector, and define $\tilde{x} = \hat{x} + e^k$. Then $\tilde{x} \in \mathbb{R}^n_+$, and

$$(\tilde{x} - \hat{x})\nabla f(\hat{x}) < 0 \tag{11.19}$$

Let $\bar{x}$ be an arbitrary vector in C. We have to show that $f(\bar{x}) \leq f(\hat{x})$. To this end, define for $0 < \theta < 1$, $x(\theta) = (1-\theta)\bar{x} + \theta\tilde{x}, y(\theta) = (1-\theta)\hat{x} + \theta\tilde{x}$. Then, using $\theta > 0$ and (11.19), we have

$$[y(\theta) - \hat{x}]\nabla f(\hat{x}) = \theta(\tilde{x} - \hat{x})\nabla f(\hat{x}) < 0 \tag{11.20}$$

Also, by Lemma 5,

$$[x(\theta) - y(\theta)]\nabla f(\hat{x}) = (1-\theta)(\bar{x} - \hat{x})\nabla f(\hat{x}) \leq 0 \tag{11.21}$$

Thus, adding (11.20) and (11.21), we get $[x(\theta) - \hat{x}]\nabla f(\hat{x}) < 0$. Since f is quasi-concave, we have $f(x(\theta)) < f(\hat{x})$ by Lemma 4. Letting $\theta \to 0$, we get $f(\bar{x}) \leq f(\hat{x})$.

Suppose condition (b) holds. If condition (a) still holds, we are already done. So, assume that (a) does not hold. That is $\nabla f(\hat{x}) \geq 0$, and $D_k f(\hat{x}) > 0$ for some index $k \in I$. Thus, there is $x^* \in C$ such that $x^*_k > 0$. Using Lemma 5,

$$\hat{x}\nabla f(\hat{x}) \geq x^* \nabla f(\hat{x}) > 0 \tag{11.22}$$

Let $\bar{x} \in C$. Then for $0 < \theta < 1$, by using Lemma 5 and (11.22), we get $(\theta\bar{x})\nabla f(\hat{x}) \leq \theta\hat{x}\nabla f(\hat{x}) < \hat{x}\nabla f(\hat{x})$. Using the quasi-concavity of f and Lemma 4, we have $f(\theta\bar{x}) < f(\hat{x})$. Letting $\theta \to 1$, we get $f(\bar{x}) \leq f(\hat{x})$.

Suppose condition (c) holds. If $\bar{x} \in C$, we have $f(\bar{x}) - f(\hat{x}) \leq (\bar{x} - \hat{x})\nabla f(\hat{x}) \leq 0$, by using Lemmas 2 and 5, and concavity of f. ■

Corollary 8. *Suppose* f, $G^j (j = 1,\ldots,r)$ *are continuously differentiable on* X, *and quasi-concave on* $\mathbb{R}^n_+$. *Suppose there is a pair* $(\hat{x},\hat{v}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$, *such that* $(\hat{x},\hat{v})$ *satisfies the Kuhn-Tucker conditions. Suppose there is* $x^* >> 0$, *such that* $G^j(x^*) \geq 0$ *for* $j = 1,\ldots,r$, *and* $\nabla f(\hat{x}) \neq 0$. *Then* $\hat{x}$ *solves* (P).

Proof. Since there is $x^* >> 0$, such that $G^j(x^*) \geq 0$ for $j = 1,\ldots,r$, all indices $i \in \{1,\ldots,n\}$ are relevant indices. Since $\nabla f(\hat{x}) \neq 0$, there is some index k for which $D_k f(\hat{x}) \neq 0$. If $D_k f(\hat{x}) < 0$ then condition (a) of Theorem 50 is satisfied. If $D_k f(\hat{x}) > 0$, then condition (b) of Theorem 50 is satisfied. Thus, in either case, the result follows from Theorem 50. ■

Corollary 9. *Suppose f, $G^j(j = 1,\ldots,r)$ are continuously differentiable on X, and quasi-concave on $\mathbb{R}^n_+$. Suppose there is a pair $(\hat{x},\hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$ such that $(\hat{x},\hat{\nu})$ satisfies the Kuhn-Tucker conditions. Suppose Slater's condition is satisfied, and $\nabla f(\hat{x}) \neq 0$. Then $\hat{x}$ solves (P).*

Proof. By Slater's condition, there is $\bar{x} \in C$ such that $G^j(\bar{x}) > 0$ for $j = 1,\ldots,r$. By continuity of $G^j(j = 1,\ldots,r)$, there is $x^* >> \bar{x}$, such that $G^j(x^*) > 0$ for $j = 1,\ldots,r$. Thus, $x^* >> 0$ and $x^* \in C$. So, the result follows directly from Corollary 8. ■

Remark 1. *To apply Theorem 50, Corollary 8 or Corollary 9, one has to check that $f, G^j(j = 1,\ldots,r)$ are continuously differentiable on X, and quasi-concave on $\mathbb{R}^n_+$. However, in view of Lemma 3, it is sufficient to check that $f, G^j(j = 1,\ldots,r)$ are continuously differentiable on X, and quasi-concave on $\mathbb{R}^n_{++}$.*

11.4 The Necessity Theorem of Arrow-Enthoven

Theorem 51. *Suppose $f, G^j(j = 1,\ldots,r)$ are continuously differentiable functions on X. Suppose $G^j(j = 1,\ldots,r)$ are quasi-concave on $\mathbb{R}^n_+$ and there is $x^* \in C$ such that $G^j(x^*) > 0$ for $j = 1,\ldots,r$. If $\hat{x} \in \mathbb{R}^n_+$ solves problem (P), and for each $j = 1,\ldots,r$, $\nabla G^j(\hat{x}) \neq 0$, then there is $\hat{\nu} \in \mathbb{R}^r_+$, such that $(\hat{x},\hat{\nu})$ satisfies the Kuhn-Tucker conditions $(KT\ II)$.*

Proof. Define $m = r + n$, and for $j \in \{1,\ldots,m\}$, define $g^j : X \to \mathbb{R}$ by $g^j(x) \equiv G^j(x)$ for $j = 1,\ldots,r, g^j(x) \equiv x_{j-r}$ for $j = r+1,\ldots,r+n$.

Let $E \equiv E(\hat{x})$ be the set of indices, denoted by k, for which $g^k(\hat{x}) = 0$. If E is the empty set, we can clearly apply the Arrow-Hurwicz-Uzawa theorem (from Chapter 10). If E is non-empty, we proceed as follows.

By continuity of G^j, there is $\bar{x} >> 0$, such that $G^j(\bar{x}) > 0$ for $j = 1,\ldots,r$. That is, there is $\bar{x} \in X$ such that $g^j(\bar{x}) > 0$ for all $j \in \{1,\ldots,m\}$.

Note that if $x \in C$, then for $k \in E$, $g^k(x) - g^k(\hat{x}) = g^k(x) \geq 0$. So, by quasi-concavity of g^k, we have:

$$(x - \hat{x})\nabla g^k(\hat{x}) \geq 0 \tag{11.23}$$

by Lemma 4.

We claim, now, that for each $k \in E$, $(\bar{x} - \hat{x})\nabla g^k(\hat{x}) > 0$. To see this, given any $k \in E$, define $x^k(\theta) = [\bar{x} - \theta\nabla g^k(\hat{x})]$ where $\theta > 0$ and sufficiently small to make $x^k(\theta) \geq 0$, and $G^j(x^k(\theta)) \geq 0$ for all $j \in \{1,\ldots,r\}$. [Since $\bar{x} >> 0$ and $G^j(\bar{x}) > 0$ for $j = 1,\ldots,r$, this is possible]. Thus $x^k(\theta) \in C$, and so by using (11.23), we have:

$$[x^k(\theta) - \hat{x}]\nabla g^k(\hat{x}) \geq 0 \tag{11.24}$$

But (11.24) implies that:

$$[\bar{x} - \hat{x}]\nabla g^k(\hat{x}) - \theta \left\|\nabla g^k(\hat{x})\right\|^2 \geq 0 \tag{11.25}$$

and so $(\bar{x} - \hat{x})\nabla g^k(\hat{x}) > 0$, since $\nabla g^k(\hat{x}) \neq 0$.

Define $h = (\bar{x} - \hat{x})$. Then $h\nabla g^k(\hat{x}) > 0$ for all $k \in E$, and we can again apply the Arrow-Hurwicz-Uzawa theorem (from Chapter 10). Thus, we have shown that whether E is empty or not, we can apply the Arrow-Hurwicz-Uzawa theorem to get $\hat{\lambda} = (\hat{\nu}, \hat{\mu}) \in \mathbb{R}^r_+ \times \mathbb{R}^n_+$, such that

(i) $\nabla f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x}) + \hat{\mu} = 0$

(ii) $\hat{\nu}G(\hat{x}) + \hat{\mu}\hat{x} = 0$.

From (i), and $\hat{\mu} \geq 0$, we get

$$\nabla f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x}) \leq 0$$

From (ii) and $[\hat{x}, G(\hat{x})] \geq 0, [\hat{\mu}, \hat{\nu}] \geq 0$, we get

(iii) $\hat{\mu}\hat{x} = 0$

(iv) $\hat{\nu}G(\hat{x}) = 0$

Multiplying (i) by $\hat{x}$ and using (iii), we get

(v) $\hat{x}[\nabla f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j \nabla G^j(\hat{x})] = 0$

Thus, the Kuhn-Tucker conditions $(KT\ II)$ are satisfied by $(\hat{x}, \hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$. ■

11.5 Worked Out Problems on Chapter 11

Problem 45 (Applying Arrow-Enthoven's theory of Quasi-Concave Programming)**.**

Let a, b be arbitrary positive real numbers. Consider the following constrained optimization problem:

$$\left.\begin{array}{ll} Maximize & x_1x_2 + [x_1/(1+x_1)] \\ subject\ to & ax_1 + bx_2 \le 1 \\ & (x_1, x_2) \in \mathbb{R}^2_+ \end{array}\right\} (P)$$

(a) Define $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > -1, x_2 > -1\}$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be defined by: $f(x_1, x_2) = x_1x_2 + [x_1/(1+x_1)]$, $g(x_1, x_2) = (1 - ax_1 - bx_2)$ for all $(x_1, x_2) \in X$. Verify that X is an open set in $\mathbb{R}^2$, and that f and g are continuously differentiable on X. Show that f and g are quasi-concave on $\mathbb{R}^2_+$, and that f is *not* concave on $\mathbb{R}^2_+$.

(b) Write down the Kuhn-Tucker conditions for problem (P) along the lines of Arrow-Enthoven for a pair $(\hat{x}, \hat{\lambda}) \in \mathbb{R}^2_+ \times \mathbb{R}_+$.

(c) Show that if $(\hat{x}, \hat{\lambda}) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ satisfies the Kuhn-Tucker conditions in (b) above, then (i) $(1 - a\hat{x}_1 - b\hat{x}_2) = 0$; (ii) $\hat{x}_1 > 0$; (iii) $\nabla f(\hat{x}) \neq 0$, and (iv) $\hat{\lambda} > 0$.

(d) Define a function $h : \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$h(z) = [b/(1+z)^2] - 2az + 1 \ for\ z \ge 0$$

Show that there is a unique positive solution to the equation: $h(z) = 0$. Call this solution c.

(e) Use the Arrow-Enthoven sufficiency theorem to show that (i) if $c > (1/a)$, then $\hat{x} = ((1/a), 0)$ solves problem (P); (ii) if $c \le (1/a)$, then $\hat{x} = (c, (1 - ac)/b)$ solves problem (P).

Solution.

(a) Define the set

$$X = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > -1 \text{ and } x_2 > -1 \right\}$$

For any $(\overline{x_1}, \overline{x_2}) \in X$, let $r = \min\{\overline{x_1} + 1, \overline{x_2} + 1\}$. Then we have $B(\overline{x}, r) \subset X$, so X is an open set in $\mathbb{R}^2$. Also, we have that $X \supset \mathbb{R}^2_+$.

Define f:$X \to \mathbb{R}$ and g:$X \to \mathbb{R}$ by

$$f(x_1, x_2) = x_1x_2 + \frac{x_1}{1+x_1} \quad \text{for all } (x_1, x_2) \in X$$
$$g(x_1, x_2) = 1 - ax_1 - bx_2 \quad \text{for all } (x_1, x_2) \in X$$

The functions f and g are continuously differentiable on X.

Note that we only have to establish quasi-concavity on $\mathbb{R}^2_+$. Since g is linear on $\mathbb{R}^2_+$, it is quasi-concave on $\mathbb{R}^2_+$. For all $x \in \mathbb{R}^2_{++}$, the bordered Hessian of f at x is

$$B_f(x) = \begin{bmatrix} 0 & x_2 + \frac{1}{(1+x_1)^2} & x_1 \\ x_2 + \frac{1}{(1+x_1)^2} & \frac{-2}{(1+x_1)^3} & 1 \\ x_1 & 1 & 0 \end{bmatrix}$$

The second leading principal minor of $B_f(x)$ is $-\left(x_2 + \frac{1}{(1+x_1)^2}\right)^2 < 0$ for all $x \in \mathbb{R}^2_{++}$. And for all $x \in \mathbb{R}^2_{++}$, the determinant of $B_f(x)$ is

$$\begin{aligned} \det B_f(x) &= -\left(x_2 + \frac{1}{(1+x_1)^2}\right)(-x_1) + x_1\left(x_2 + \frac{1}{(1+x_1)^2} + \frac{2x_1}{(1+x_1)^3}\right) \\ &= 2x_1x_2 + \frac{2x_1}{(1+x_1)^2} + \frac{2x_1^2}{(1+x_1)^3} \\ &> 0 \end{aligned}$$

Therefore f is quasi-concave on $\mathbb{R}^2_{++}$. Since f is continuous on $\mathbb{R}^2_+$ and quasi-concave on $\mathbb{R}^2_{++}$, by Lemma 3, Chapter 11, f is quasi-concave on $\mathbb{R}^2_+$. But for all $x \in \mathbb{R}^2_{++}$, the determinant of the Hessian of f at x is $-1 \ngeq 0$, so the Hessian of f at x is not negative semi-definite, which means that f is not concave on $\mathbb{R}^2_{++}$. Therefore f is not concave on $\mathbb{R}^2_+$.

Since f is quasi-concave but not concave on X, we need to verify two further conditions in order to apply the Arrow-Enthoven Sufficiency Theorem. In part (b) we will show that if a pair $(\hat{x}, \hat{\lambda}) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ satisfies the Kuhn-Tucker conditions, then $\nabla f(\hat{x}) \neq 0$. Slater's Conditions holds because $(0,0) \in \mathbb{R}^2_+$ and $g(0,0) = 1 > 0$.

Therefore all the conditions of the Arrow-Enthoven Sufficiency Theorem are met, so if some pair $(\hat{x}, \hat{\lambda}) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ satisfies the Kuhn-Tucker conditions, then $\hat{x}$ solves problem (P).

(b) A pair $(\hat{x}, \hat{\lambda}) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ satisfies the Kuhn-Tucker conditions for problem (P) if it satisfies the following:

$$\hat{x}_2 + \frac{1}{(1+\hat{x}_1)^2} - a\hat{\lambda} \le 0 \tag{5.1}$$

$$\hat{x}_1 - b\hat{\lambda} \le 0 \tag{5.2}$$

$$\hat{x}_1\left(\hat{x}_2 + \frac{1}{(1+\hat{x}_1)^2} - a\hat{\lambda}\right) = 0 \tag{5.3}$$

$$\hat{x}_2\big(\hat{x}_1 - b\hat{\lambda}\big) = 0 \tag{5.4}$$

$$1 - a\hat{x}_1 - b\hat{x}_2 \ge 0 \tag{5.5}$$

$$\hat{\lambda}\big(1 - a\hat{x}_1 - b\hat{x}_2\big) = 0 \tag{5.6}$$

(c) Seeking contradiction, suppose that $\hat{\lambda} = 0$. Then by (5.2), we have $\hat{x}_1 = 0$, so (5.1) gives $\hat{x}_2 \le -1$, which is a contradiction of $\hat{x}_2 \ge 0$. Therefore $\hat{\lambda} > 0$, so (5.6) yields

$$1 - a\hat{x}_1 - b\hat{x}_2 = 0 \tag{5.7}$$

Again seeking contradiction, suppose that $\hat{x}_1 = 0$. By (5.7), we have $\hat{x}_2 = \frac{1}{b} > 0$. Therefore by (5.4) we must have $\hat{x}_1 = b\hat{\lambda} > 0$ since $\hat{\lambda} > 0$, which is a contradiction. So it must be that $\hat{x}_1 > 0$. Since $D_2 f(x) = x_1$ and $\hat{x}_1 > 0$, we have that $\nabla f(\hat{x}) \ne 0$. Another consequence of $\hat{x}_1 > 0$ is that we can use (5.3) to write

$$\hat{x}_2 + \frac{1}{(1+\hat{x}_1)^2} - a\hat{\lambda} = 0 \tag{5.8}$$

(d) Define the function $h\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$h(z) = \frac{b}{(1+z)^2} - 2az + 1 \quad \text{for all } z \ge 0$$

Now, h is a continuous function on $\mathbb{R}_+$. Its derivative is

$$h'(z) = \frac{-2b}{(1+z)^3} - 2a < 0 \quad \text{for all } z > 0$$

With continuity of h on its domain, this shows that h is strictly decreasing on $\mathbb{R}_+$. In addition, we have $h(0) = b + 1 > 0$ and $\lim_{z\to\infty} h(z) = -\infty$. By the Intermediate Value Theorem, then, there is some $c > 0$ such that $h(c) = 0$.

(e) We can split the analysis into two cases, depending on whether (5.2) holds with equality or strict inequality.

Case 1: $\hat{x}_1 - b\hat{\lambda} < 0$. Then by (5.4), we have $\hat{x}_2 = 0$, so (5.7) gives $\hat{x}_1 = \frac{1}{a} > 0$. Using this and (5.8), we can rewrite the requirement for this case as

$$0 > \hat{x}_1 - b\hat{\lambda} = \frac{1}{a} - b\left(\frac{1}{a}\hat{x}_2 + \frac{1}{a(1+\hat{x}_2)^2}\right) = \frac{1}{a} - \frac{b}{a\left(1+\frac{1}{a}\right)^2}$$

This implies that

$$h\left(\frac{1}{a}\right) = \frac{b}{\left(1+\frac{1}{a}\right)^2} - 1 > 0 = h(c)$$

Since h is strictly decreasing on its domain, this requires $c > \frac{1}{a}$. Note that from (5.8), we have $\hat{\lambda} = \frac{a}{(a+1)^2} > 0$.

Case 2: $\hat{x}_1 - b\hat{\lambda} = 0$. Then, solving (5.7) for $\hat{x}_2$ and using this in (5.8), we have

$$\frac{1}{b} - \frac{a}{b}\hat{x}_1 + \frac{1}{(1+\hat{x}_1)^2} - \frac{a}{b}\hat{x}_1 = 0$$

Multiplying both sides by b, we have

$$\frac{b}{(1+\hat{x}_1)^2} - 2a\hat{x}_1 + 1 = 0$$

Since c is the unique point in $\mathbb{R}_+$ that gives $h(c) = 0$, it must be that $\hat{x}_1 = c$. Then by (5.7), we have that $\hat{x}_2 = \frac{1-ac}{b}$. Since we require $\hat{x}_2 \geq 0$, a requirement for this case is $c \leq \frac{1}{a}$. Note that $\hat{\lambda} = \frac{1}{b}\hat{x}_1 = \frac{c}{b} > 0$.

Recall that we showed in parts (a) and (b) that the Arrow-Enthoven Sufficiency Theorem applies. So we have the following conclusions:

- **(i)** If $c > \frac{1}{a}$ then $(\hat{x}, \hat{\lambda}) = \left(\frac{1}{a}, 0, \frac{a}{(a+1)^2}\right)$ is the unique solution to the Kuhn-Tucker conditions, so $\hat{x} = \left(\frac{1}{a}, 0\right)$ solves problem (P).
- **(ii)** If $c \leq \frac{1}{a}$ then $(\hat{x}, \hat{\lambda}) = \left(c, \frac{1-ac}{b}, \frac{c}{b}\right)$ is the unique solution to the Kuhn-Tucker conditions, so $\hat{x} = \left(c, \frac{1-ac}{b}\right)$ solves problem (P).

References:

This material is primarily based on K.J. Arrow and A.C. Enthoven,"Quasi-concave Programming", *Econometrica* (1961), 779-800. This paper also provides a further alternative condition (involving twice differentiability of the objective function, and a non-zero gradient vector of the objective function at $\hat{x}$) under which Theorem 1 is valid.

Theorem 2 is based on the Arrow-Hurwicz-Uzawa theorem discussed in Chapter 10. For this, see K.J. Arrow, L. Hurwicz and H. Uzawa, "Constraint Qualifications in Maximization Problems", *Naval Research Logistics Quarterly*, 8 (1961), 175-191.

The theory of quasi-concave programming can be further extended to the class of "pseudo-concave" objective functions [and quasi-concave constraint functions]. For this, see *Non-Linear Programming* by *O.L. Mangasarian* (Chapters 9, 10).

Chapter 12

Linear Programming

12.1 The Primal and Dual Problems

The theory of linear programming is concerned with the following problem:

$$(P)\begin{cases} & Max\, q'x & q'\, is\, 1\times n \\ Subject\, to & Ax \leq b & A\, is\, m\times n \\ & x \geq 0 & x\, is\, n\times 1 \\ & & b\, is\, m\times 1 \end{cases}$$

We will call (P) the *Primal* problem. Associated with (P) is the following problem:

$$(D)\begin{cases} & Min\, y'b & b\, is\, m\times 1 \\ Subject\, to & y'A \geq q' & A\, is\, m\times n \\ & y' \geq 0 & y'\, is\, 1\times m \\ & & q'\, is\, 1\times n \end{cases}$$

We will call (D) the *Dual* problem.

Define the set of *feasible solutions to the Primal* as

$$F(P) = \{x \geq 0\, such\, that\, Ax \leq b\}$$

Define the set of *feasible solutions to the Dual* as

$$F(D) = \{y' \geq 0\, such\, that\, y'A \geq q'\}$$

12.2 Optimality Criterion

In studying solutions to the Primal and Dual problems, we first establish the following optimality criterion: if x^o is a feasible solution to the primal and $y^{o\prime}$ a feasible solution to the dual, and their *values* are equal, [that is, $q'x^o = y^{o\prime}b$], then x^o is an optimal solution to the Primal [that is, x^o solves (P)] and $y^{o\prime}$ is an optimal solution to the Dual [that is, $y^{o\prime}$ solves (D)].

Lemma 6. *If $x^o \in F(P), y^{o\prime} \in F(D)$, then*

$$q'x^o \le y^{o\prime}Ax^o \le y^{o\prime}b \tag{12.1}$$

Proof. Since $y^{o\prime} \in F(D)$, and $x^o \ge 0$

$$q'x^o \le y^{o\prime}Ax^o \tag{12.2}$$

Since $x^o \in F(P)$, and $y^{o\prime} \ge 0$

$$y^{o\prime}b \ge y^{o\prime}Ax^o \tag{12.3}$$

Combining (12.2) and (12.3) yields (12.1). ∎

Theorem 52 (Optimality Criterion)**.** *If $x^o \in F(D)$, and*

$$q'x^o = y^{o\prime}b \tag{12.4}$$

then x^o solves (P) and $y^{o\prime}$ solves (D).

Proof. Let $x \in F(P)$. Then since $y^{o\prime} \in F(D)$, we can use Lemma 6 to get

$$q'x \le y^{o\prime}b$$

and we can use (12.4) to get $q'x \le q'x^o$. So x^o solves (P). Similarly $y^{o\prime}$ solves (D). ∎

12.3 The Basic Duality Theorems

The basic duality theorems of Linear Programming may be stated as follows. If both the primal and the dual have feasible solutions, then both have optimal solutions and the values of the optimal solutions are the same; if either the primal or the dual is infeasible, then neither has an optimal solution. We establish these results in Theorem 53 and 54 below.

Lemma 7 (Non-negative solutions of linear inequalities)**.** *Exactly one of the following alternative holds.*

(1) Either the inequality

$$Bu \leq a$$

has a non-negative solution for u

(2) Or the inequalities

$$v'B \geq 0,\ v'a < 0$$

have a non-negative solution for v'.

The proof, which is omitted, follows from the Farkas Lemma; see Chapter 7 on Convex Analysis

Lemma 8. *If there is x^o in $F(P)$, and $y^{o'}$ in $F(D)$ then there is $\tilde{x}$ in $F(P)$, and $\tilde{y}'$ in $F(D)$ such that*

$$q'\tilde{x} \geq \tilde{y}'b$$

Proof. Suppose not. Then

$$\left.\begin{array}{rr} Ax & \leq b \\ -A'y & \leq -q \\ -q'x + b'y & \leq 0 \end{array}\right\}$$

has no non-negative solution for (x,y). That is,

$$\begin{bmatrix} A & 0 \\ 0 & -A' \\ -q' & b' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} b \\ -q \\ 0 \end{bmatrix}$$

has no non-negative solution for (x,y). By Lemma 7, there is an m-vector $z' \geq 0$, an n-vector $w' \geq 0$, and a scalar $\theta \geq 0$, such that

$$[z', w', \theta] \begin{bmatrix} A & 0 \\ 0 & -A' \\ -q' & b' \end{bmatrix} \geq 0$$

and

$$[z' w', \theta] \begin{bmatrix} b \\ -q \\ 0 \end{bmatrix} < 0$$

That is,

$$z'A \geq \theta q' \tag{12.5}$$

$$w'A' \leq \theta b' \tag{12.6}$$

$$z'b < w'q \tag{12.7}$$

Now, we claim that $\theta \neq 0$. For if $\theta = 0$, then

$$\underbrace{z'b \geq z'Ax^o}_{(b \geq Ax^o)} \underbrace{\geq \theta q'x^0}_{\textit{Using (12.5)}} = 0$$

and

$$\underbrace{w'q \leq w'A'y^o}_{(q \leq A'y^o)} \underbrace{\leq \theta b'y^0}_{\textit{Using (12.6)}} = 0$$

Thus, $w'q \leq z'b$, which contradicts (12.7). This establishes our claim that $\theta \neq 0$. Since $\theta \geq 0$, we have $\theta > 0$. Using (12.5),

$$\frac{z'}{\theta}A \geq q', \qquad so \left(\frac{z'}{\theta}\right) \in F(D).$$

Using (12.6),

$$\frac{w'}{\theta}A' \leq b', \qquad so \left(\frac{w'}{\theta}\right) \in F(P).$$

So $q'\left(\frac{w}{\theta}\right) \leq \left(\frac{z'}{\theta}\right)b$ by Lemma 6, which contradicts (12.7) again. ■

Theorem 53 (Fundamental Theorem of Linear Programming)**.** *If there is x^o in $F(P)$ and $y^{o'}$ in $F(D)$, then there is $\tilde{x}$ in $F(P)$ and $\tilde{y}'$ in $F(D)$ such that*

(i) $q'\tilde{x} = \tilde{y}'b$

(ii) *$\tilde{x}$ solves (P), $\tilde{y}'$ solves (D).*

Proof. (i) By Lemma 8, there is $\tilde{x}$ in $F(P)$ and $\tilde{y}'$ in $F(D)$, such that

$$q'\tilde{x} \geq \tilde{y}'b$$

By Lemma 6,

$$q'\tilde{x} \leq \tilde{y}'b$$

So

$$q'\tilde{x} = \tilde{y}'b$$

(ii) By (i) and Theorem 52, $\tilde{x}$ solves (P) and $\tilde{y}'$ solves (D). ■

Theorem 54. *If either $F(P) = \phi$, or $F(D) = \phi$, then (P) has no solution and (D) has no solution.*

Proof. Suppose $F(P) = \phi$. Then (P) clearly has no solution. To show that (D) has no solution, suppose, on the contrary, that $y^{o\prime}$ solves (D).

Since $Ax \leq b$ has no non-negative solution there is $y' \geq 0$ such that

$$y'A \geqq 0, y'b < 0$$

by Lemma 7. Then $[y^{o\prime} + y'] \in F(D)$ since

$$y^{o\prime} + y' \geq 0$$

$$[y^{o\prime} + y']A \geq q'$$

Also,

$$[y^{o\prime} + y']b = y^{o\prime}b + y'b < y^{o\prime}b$$

So $y^{o\prime}$ does not solve (D), a contradiction. A similar proof can be worked out by supposing $F(D) = \phi$. ∎

12.4 Complementary Slackness

We finally state a result which has been very useful in applications of linear programming. It says that if x^o and $y^{o\prime}$ are optimal solutions to the primal and dual problems, then (i) whenever a constraint (say the j^{th}) of the primal problem is not "binding" (at x^o), the optimal solution of the corresponding dual variable (that is, y^o_j) must be zero; (ii) whenever a constraint (say the i^{th}) of the dual problem is not "binding" (at $y^{o\prime}$), the optimal solution of the corresponding primal variable (that is x^o_i) must be zero. We establish this in Theorem 55 below.

Corollary 10. *If x^o solves (P) and $y^{o\prime}$ solves (D), then $q'x^o = y^{o\prime}Ax^o = y^{o\prime}b$.*

Proof. Using Lemma 2, $y^{o\prime}b \geq y^{o\prime}Ax^o \geq q'x^o$. Since x^o solves (P) and $y^{o\prime}$ solves (D), we have $y^{o\prime}b = q'x^o$, by Theorem 53. Hence $y^{o\prime}b = y^{o\prime}Ax^o = q'x^o$. ∎

Corollary 11. *If x^o solves (P) and $y^{o\prime}$ solves (D), then*

(a) $y^o_j[Ax^o - b]_j = 0$ $\quad$ *for each j*

(b) $[y^{o\prime}A - q']_i x^o_i = 0$ $\quad$ *for each i*

Proof. By Corollary 10,

$$y^{o\prime}[Ax^o - b] = 0 \tag{12.8}$$

Since $y^{o\prime} \geq 0, Ax^o - b \leq 0$, we obtain $y_j^{o\prime}[Ax^o - b]_j \leq 0$ for each j. This, together with (12.8) implies $y_j^{o\prime}[Ax^o - b]_j = 0$ for $j = 1, \ldots, m$. This proves (a). The proof of (b) is similar. ∎

Theorem 55 (Goldman-Tucker)**.** *If $x^o \in F(P), y^{o\prime} \in F(D)$, then the following two statements are equivalent:*

(i) x^0 *solves* (P) *and* $y^{o'}$ *solves* (D)

(ii) *(a)* $y_j^o = 0$ *whenever* $[Ax^o]_j < b_j$, *and*

(b) $x_i^0 = 0$ *whenever* $[y^{o'}A]_i > q_i$

Proof. **[(ii) implies (i)]**. If $[Ax^o]_j = b_j$, then $y_j^o[Ax^o]_j = y_j^o b_j$. If $[Ax^o]_j < b_j$, then $y_j^o[Ax^o]_j = 0 = y_j^o b_j$, by (ii) (a). Thus $y^{o\prime}Ax^o = y^{o\prime}b$. Similarly $y^{o\prime}Ax^o = q'x^o$. So $y^{o\prime}b = q'x^o$. Hence x^o solves (P), $y^{o\prime}$ solves (D), by Theorem 52.

[(i) implies (ii)]. Since x^o solves (P) and $y^{o\prime}$ solves (D), we have by Corollary 11,

$$y_j^o[Ax^o - b]_j = 0 \qquad \text{for each } j$$

and

$$[y^{o\prime}A - q']_i x_i^o = 0 \qquad \text{for each } i$$

Thus, whenever $[Ax^o - b]_j < 0$, we must have $y_j^o = 0$. And, whenever $[y^{o\prime}A - q']_i > 0$, we must have $x_i^o = 0$. ∎

12.5 Worked Out Problems on Chapter 12

Problem 46. *The theory of linear programming is concerned with the following problem:*

$$(P)\quad \begin{cases} \text{Maximize} & q'x \\ \text{subject to} & Ax \leq b \\ \text{and} & x \geq 0, \end{cases}$$

where q and x are is $1 \times n$*, A is* $m \times n$ *and b is* $m \times 1$*. We will call (P) the Primal problem. Associated with (P) is the following problem:*

$$(D)\quad \begin{cases} \text{Minimize} & y'b \\ \text{subject to} & y'A \geq q' \\ \text{and} & y \geq 0, \end{cases}$$

where y is $m \times 1$*. We will call (D) the Dual problem. Define the set of feasible solutions to the Primal as*

$$F(P) = \{x \in \mathbb{R}^n_+ : Ax \leq b\}.$$

Define the set of feasible solutions to the Dual as

$$F(D) = \{y \in \mathbb{R}^m_+ : y'A \geq q'\}.$$

(a) Suppose $\bar{x} \in F(P)$*,* $\bar{y} \in F(D)$ *and* $(\bar{x}, \bar{y})$ *satisfies*

$$\bar{y}'(b - A\bar{x}) = (\bar{y}'A - q')\bar{x} = 0.$$

Show that $(\bar{x}, \bar{y})$ *satisfies the Kuhn-Tucker conditions (KT II) for problem (P). Use the Arrow-Enthoven sufficiency theorem to conclude that* $\bar{x}$ *solves problem (P). Show that* $\bar{y}$ *solves problem (D) by using an analogous argument.*
(b) Suppose $\bar{x}$ *solves (P) and* $\bar{y}$ *solves (D). Using the fact that* $\bar{x} \in F(P)$ *and* $\bar{y} \in F(D)$*, show that*

$$q'\bar{x} \leq \bar{y}'A\bar{x} \leq \bar{y}'b. \tag{1}$$

Using the fact that $\bar{x}$ *solves (P) and applying the Arrow-Hurwicz-Uzawa necessity theorem, show that there exists* $\hat{y} \in F(D)$ *such that*

$$\hat{y}'(b - A\bar{x}) = (\hat{y}'A - q')\bar{x} = 0. \tag{2}$$

Using the fact that $\bar{y}$ *solves (D), and (2), show that* $q'\bar{x} \geq \bar{y}'b$*. Using this inequality and (1), show that*

$$\bar{y}'(b - A\bar{x}) = (\bar{y}'A - q')\bar{x} = 0. \tag{3}$$

Solution (a) Let $X = \mathbb{R}^n$ be an open set containing $\mathbb{R}^n_+$. Let us define $f(x) = q'x$ and $G^j(x) = (b - Ax)_j$ on X, $j = 1, \ldots, m$. A pair $(\bar{x}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ satisfies the KT II conditions for the problem (P) if and only if

$$\begin{array}{lrl}
(i)(a) & D_i f(\bar{x}) + \sum_{j=1}^m \bar{y}_j D_i G^j(\bar{x}) \le 0 & \text{for } i = 1, \ldots, n, \\
(i)(b) & \bar{x}_i(D_i f(\bar{x}) + \sum_{j=1}^m \bar{y}_j D_i G^j(\bar{x})) \le 0 & \text{for } i = 1, \ldots, n, \\
(ii)(a) & G^j(\bar{x}) \ge 0 & \text{for } j = 1, \ldots, m, \\
(ii)(b) & \sum_{j=1}^m \bar{y}_j G^j(\bar{x}) = 0. &
\end{array}$$

For the linear problem (P) these conditions look like

$$\begin{array}{lrl}
(i)(a) & q_i + \sum_{j=1}^m \bar{y}_j(-a_{ji}) \le 0 & \text{for } i = 1, \ldots, n, \\
(i)(b) & \bar{x}_i(q_i + \sum_{j=1}^m \bar{y}_j(-a_{ji})) = 0 & \text{for } i = 1, \ldots, n, \\
(ii)(a) & (b - A\bar{x})_j \ge 0 & \text{for } j = 1, \ldots, m, \\
(ii)(b) & \sum_{j=1}^m \bar{y}_j(b - A\bar{x})_j = 0. &
\end{array}$$

Conditions (i)(a) can be rewritten as $(q' - \bar{y}'A)_i \le 0$ and hence are satisfied because $\bar{y} \in F(D)$. Conditions (ii)(a) follows from $\bar{x} \in F(P)$. Conditions (i)(b) are implied by (i)(a), $\bar{x} \in \mathbb{R}^n_+$ and $(\bar{y}'A - q')\bar{x} = 0$. Condition (ii)(b) follows from (ii)(a), $\bar{y} \in \mathbb{R}^m_+$ and $\bar{y}'(b - A\bar{x}) = 0$.

Since f and g^j, $j = 1, \ldots, m$, are linear on the open set $X = \mathbb{R}^n$ containing $\mathbb{R}^n_+$, these functions are continuously differentiable on $\mathbb{R}^n$ and quasi-concave on $\mathbb{R}^n_+$. Since f is concave on X, it follows that condition (c) of the Arrow-Enthoven sufficiency theorem (Theorem 50, p. 79 in the Lecture Notes) is satisfied. Therefore if a pair $(\bar{x}, \bar{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+$ satisfies the KT II conditions for the problem (P), then the Arrow-Enthoven sufficiency theorem implies that $\bar{x}$ solves problem (P). By the analogous argument it follows that $\bar{y}$ solves problem (D).

(b) If $\bar{x} \in F(P)$ and $\bar{y} \in F(D)$, we have $A\bar{x} \le b$ and $\bar{y}'A \ge q'$. Since $\bar{x} \ge 0$ and $\bar{y} \ge 0$, premultiplying by $\bar{y}'$ and postmultiplying by $\bar{x}$ yield $\bar{y}'A\bar{x} \le \bar{y}'b$ and $\bar{y}'A\bar{x} \ge q'\bar{x}$.

Let us define $g^j(x) = (b - Ax)_j$ and $g^{m+i}(x) = x_i$ on $X = \mathbb{R}^n$, $j = 1, \ldots, m$, $i = 1, \ldots, n$. Then $X = \mathbb{R}^n$ is open and convex set, functions f and g^k, $k = 1, \ldots, n+m$, are continuously differentiable on X, and function g^k is convex on X for $k = 1, \ldots, n+m$. Hence condition (b) of the AHU constraint qualification is satisfied. Therefore, if $\bar{x}$ solves (P), then $\bar{x}$ is a point of constrained local maximum for f, and by the Arrow-Hurwicz-Uzawa necessity theorem (Theorem 48, p.73 in the Lecture Notes) there exists $\hat{\lambda} \in \mathbb{R}^{n+m}_+$ such that $(\bar{x}, \hat{y})$ satisfy the Kuhn-Tucker (KT I) conditions:

$$\begin{array}{lll}
\text{(i)} & D_i f(\bar{x}) + \sum_{k=1}^{m+n} \hat{\lambda}_k D_i g^k(\bar{x}) = 0, & i = 1, \ldots, n, \\
\text{(ii)} & g^k(\bar{x}) \ge 0, & k = 1, \ldots, n+m, \\
\text{(iii)} & \hat{\lambda}_k g^k(\bar{x}) = 0, & k = 1, \ldots, n+m.
\end{array}$$

Hence

$$\begin{array}{lll}
\text{(i)} & q_i + \sum_{j=1}^{m} \hat{\lambda}_j(-a_{ji}) + \hat{\lambda}_{m+i} = 0, & i = 1,\ldots,n, \\
\text{(ii)} & (b - A\bar{x})_j \ge 0, & j = 1,\ldots m, \\
\text{(ii}') & \bar{x}_k \ge 0, & k = m+1,\ldots,m+n, \\
\text{(iii)} & \hat{\lambda}_j(b - A\bar{x})_j = 0, & j = 1,\ldots,m, \\
\text{(iii}') & \hat{\lambda}_{m+i}\bar{x}_i = 0, & i = 1,\ldots,n.
\end{array}$$

If we define $\hat{y} = (\hat{\lambda}_1,\ldots,\hat{\lambda}_m)'$, then (ii) implies $\hat{y} \in F(D)$ and (iii) implies $\hat{y}'(b - A\bar{x}) = 0$. Also it follows from (i) that

$$(\hat{y}'A - q')\bar{x} = \sum_{i=1}^{n} \bar{x}_i(\hat{y}'A - q')_i = \sum_{i=1}^{n} \bar{x}_i \left(-q_i + \sum_{j=1}^{m} \hat{\lambda}_j a_{ji} \right) = \sum_{i=1}^{n} \bar{x}_i \hat{\lambda}_{m+i} = 0.$$

From (2) we immediately have $q'\bar{x} = \hat{y}'A\bar{x} = \hat{y}'b$. Now, since $\bar{y}$ solves (D) and $\hat{y} \in F(D)$, we have $\bar{y}'b \le \hat{y}'b$, therefore $q'\bar{x} = \hat{y}'b \ge \bar{y}'b$. From this inequality and from (1) it follows that $q'\bar{x} = \bar{y}'A\bar{x} = \bar{y}'b$, hence (3) is satisfied.

Problem 47. *Consider the following linear programming problem:*

$$\left.\begin{array}{rl}
\textit{Maximize} & 3x_1 + 4x_2 + 3x_3 \\
\textit{subject to} & x_1 + x_2 + 3x_3 \le 12, \\
 & 2x_1 + 4x_2 + x_3 \le 42 \\
\textit{and} & (x_1, x_2, x_3) \in \mathbb{R}^3_+
\end{array}\right\} \quad \textit{(Q)}$$

Find a solution to problem (Q) showing your procedure clearly. [Hint: you might want to use your solution to problem 1 of PS11, and the duality theory for linear programming developed in problem 1 of PS12].
(a) Write down the Kuhn-Tucker conditions for problem (P).
(b) Use the Kuhn-Tucker (sufficiency) theorem to show that $\hat{y} = (2, 1/2)$ solves problem (P)
(c) Draw an appropriate diagram to illustrate your solution.

Solution (a) If we define

$$q = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 42 \end{pmatrix},$$

then problem (Q) and its dual problem (QD) can be written as

$$\text{(Q)} \quad \left\{ \begin{array}{rl} \text{Maximize} & q'x \\ \text{subject to} & Ax \le b \\ \text{and} & x \in \mathbb{R}^3_+, \end{array} \right. \qquad \text{(QD)} \quad \left\{ \begin{array}{rl} \text{Minimize} & y'b \\ \text{subject to} & y'A \ge q' \\ \text{and} & y \in \mathbb{R}^2_+. \end{array} \right. ,$$

where problem (QD) reads as

$$
(\text{QD}) \quad \begin{cases} \text{Minimize} & 12y_1 + 42y_2 \\ \text{subject to} & y_1 + 2y_2 \geq 3, \\ & y_1 + 4y_2 \geq 4, \\ & 3y_1 + y_2 \geq 3 \\ \text{and} & y \in \mathbb{R}^2_+. \end{cases}
$$

Let us recall that $\hat{y} = (2, 1/2)$ solves problem (QD′) from problem 1 of PS11 given by

$$
(\text{QD}') \quad \begin{cases} \text{Minimize} & 12y_1 + 42y_2 \\ \text{subject to} & y_1 + 2y_2 \geq 3, \\ & y_1 + 4y_2 \geq 4 \\ \text{and} & y \in \mathbb{R}^2_+. \end{cases}
$$

Since $\hat{y} = (2, 1/2)'$ solves (QD′) and the third constraint $3y_1 + y_2 \geq 3$ from problem (QD) is satisfied at $\hat{y}$, we can conclude that $\hat{y} = (2, 1/2)$ also solves the dual problem (QD). Assume $\hat{x}$ solves the primal problem (Q). Then complementary slackness conditions (Corollary 11, p. 87 in the Lecture Notes) are satisfied:

$$
\begin{array}{lll} (a) & \hat{y}_j\,(A\hat{x} - b)_j = 0 & \text{for } j = 1, 2, \\ (b) & (\hat{y}'A - q')_i\,\hat{x}_i = 0 & \text{for } i = 1, 2, 3. \end{array}
$$

Since

$$
\hat{y}'A - q' = \begin{pmatrix} 2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 7/2 \end{pmatrix},
$$

and hence $(\hat{y}'A - q')_3 = 7/2 > 0$, it follows from (b) that $\hat{x}_3 = 0$. Since $\hat{y}_1 > 0$ and $\hat{y}_2 > 0$, we can conclude from (a) that $(A\hat{x} - b)_1 = (A\hat{x} - b)_2 = 0$, i.e.

$$
\begin{cases} \hat{x}_1 + \hat{x}_2 & = & 12, \\ 2\hat{x}_1 + 4\hat{x}_2 & = & 42. \end{cases}
$$

It immediately follows that $\hat{x}_1 = 3$ and $\hat{x}_2 = 9$. Therefore $\hat{x} = (3, 9, 0)'$ is the only candidate for the solution to problem (Q). Let us prove now that $\hat{x}$ actually solves (Q) by checking condition (ii) of the Goldman-Tucker theorem (Theorem 55, p. 87 in the Lecture Notes). Since

$$
A\hat{x} - b = 0 \quad \text{and} \quad \hat{y}'A - q' = (0, 0, 7/2),
$$

it follows that

$$
\begin{array}{lll} (ii, a) & \hat{y}_j = 0 & \text{whenever } (A\hat{x})_j < b_j \text{ (never)}, \\ (ii, b) & \hat{x}_i = 0 & \text{whenever } (\hat{y}'A)_i > q_i \text{ (for } i = 3), \end{array}
$$

Therefore Goldman-Tucker theorem implies that $\hat{x} = (3, 9, 0)'$ solves problem (Q). [Please note that since in this problem we have $a_{ij} > 0$ for all i and j, it immediately follows that constraint set for problem (Q) is bounded, and hence the existence of the solution to problem (Q) follows form the Weierstrass theorem. In general, some elements of A may be non-positive, and applicability of the Weierstrass theorem might be more difficult to justify.]

References: This material is based almost entirely on D. Gale, *The Theory of Linear Economic Models*, McGraw-Hill, New York, 1960, chapter 3.