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Notes on Kuhn-Tucker's Sufficiency Theorem

1 Introduction

Kuhn-Tucker's sufficiency theorem is one of the most useful results for solving constrained optimization problems. Given a point $\hat{x}$ in the constraint set, and a vector $\hat{\lambda}$ of Kuhn-Tucker multipliers, which jointly satisfy the Kuhn-Tucker conditions, the theorem provides conditions on the objective and constraint functions *sufficient* to ensure that the point $\hat{x}$ is a point of constrained global maximum. [In fact, it does more: it ensures that $(\hat{x}, \hat{\lambda})$ is a saddle-point, but we are mainly interested in the constrained global maximum property, when we are trying to solve a constrained optimization problem].

The Kuhn-Tucker conditions involve derivatives, so one needs *differentiability* of the objective and constraint functions. The sufficient conditions involve *concavity* of the objective and constraint functions. The version of the Kuhn-Tucker theorem given in Theorem 46 of the *Lecture Notes* assumes both the differentiability property and the concavity property on an open set X. However, it is possible to separate the domains on which these two properties hold for the class of problems of interest in economics. These notes indicate how this separation can be carried out, by following the approach of Arrow and Enthoven. The resulting theorem (Theorem 1 below) assumes concavity of the objective and constraint functions on $\mathbb{R}^n_+$, while assuming continuous differentiability of these functions on an open set X containing $\mathbb{R}^n_+$.

The differentiability assumption of Theorem 1 allows one to differentiate the objective and constraint functions not just in the interior of $\mathbb{R}^n_+$, but also at boundary points of $\mathbb{R}^n_+$. In applying the result to solve constrained optimization problems, the differentiability assumption will be typically met in those cases where the derivatives of the functions approach well-defined (finite) limits as one approches points of the boundary from points in the interior of $\mathbb{R}^n_+$.

If one of the partial derivatives of the objective function or one of the constraint functions becomes unbounded as one approches points of the boundary from points in the interior of $\mathbb{R}^n_+$, then the differentiability assumption of Theorem 1 will not hold. However, in such cases, a result with a milder differentiability assumption will typically be applicable. We present such a result in Theorem 2. Examples are given to illustrate the typical applications of Theorems 1 and 2.

2 Basic Inequality for Differentiable Concave Functions

We start by reviewing the result in Theorem 27 of the *Lecture Notes* on differentiable concave functions (which was proved in Problem Set 7). Notice the separation of the domains of differentiability and concavity of the function in Lemma 1 below, in contrast to Theorem 27 of the *Lecture Notes*. [Lemma 1 below appears as Lemma 2 in Chapter 11 of the *Lecture Notes*].

Lemma 1 *Let X be an open set in $\mathbb{R}^n$, with $\mathbb{R}^n_+ \subset X$, and let $h : X \to \mathbb{R}$ be continuously differentiable on X. If h is concave on $\mathbb{R}^n_+$, then for all $x, x' \in \mathbb{R}^n_+$,*

$$h(x) - h(x') \leq \nabla h(x')(x - x') \tag{1}$$

Proof. Define $x(n) = x + (u/n), x'(n) = x' + (u/n)$, where $u = (1, 1, ..., 1)$ in $\mathbb{R}^n$, and $n \in \mathbb{N}$. Then, $x(n)$ and $x'(n)$ belong to $\mathbb{R}^n_{++}$ for all $n \in \mathbb{N}$, and h is concave on the open set $\mathbb{R}^n_{++}$, so we can apply Theorem 27 of the *Lecture Notes* to get:

$$h(x(n)) - h(x'(n)) \leq \nabla h(x'(n))(x(n) - x'(n)) \tag{2}$$

Now, we let $n \to \infty$. Then $x(n) \to \bar{x}, x'(n) \to x'$; by continuity of h on X (which contains $\mathbb{R}^n_+$), $h(x(n)) \to h(x)$ and $h(x'(n)) \to h(x')$; by continuity of $D_i h$ on X (which contains $\mathbb{R}^n_+$), we have $D_i h(x'(n)) \to D_i h(x')$. Using these facts in (2) and noting that weak inequalities are preserved in the limit, we obtain (1). // ■

3 The Kuhn-Tucker Sufficiency Theorem

We can now use Lemma 1 above to provide the following version of the Kuhn-Tucker Sufficiency Theorem. As in Lemma 1, notice the separation of the domains of differentiability and concavity of the objective and constraint functions in Theorem 1 below, in contrast to Theorem 46 of the *Lecture Notes.*

Theorem 1 *Let X be an open set in $\mathbb{R}^n$, with $\mathbb{R}^n_+ \subset X$, and let f, G^j $(j = 1, ..., r)$ be continuously differentiable functions from X to $\mathbb{R}$. Suppose $(\hat{x}, \hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$ satisfies the Kuhn-Tucker conditions:*

$$\left.\begin{array}{l} (i)\ D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x}) \leq 0 \ \text{ for } i = 1, ..., n \\ (ii)\ \hat{x}_i[D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] = 0 \ \text{ for } i = 1, ..., n \\ (iii)\ G^j(\hat{x}) \geq 0 \ \text{ for } j = 1, ..., r \\ (iv)\ \hat{\nu}_j G^j(\hat{x}) = 0 \ \text{ for } j = 1, ..., r \end{array}\right\} \tag{3}$$

If f, G^j $(j = 1, ..., m)$ are concave on $\mathbb{R}^n_+$, then $\hat{x}$ solves the problem:

$$\left.\begin{array}{ll} \text{Max} & f(x) \\ \text{subject to} & G^j(x) \geq 0 \ \text{ for } j = 1, ..., r \\ \text{and} & x \in \mathbb{R}^n_+ \end{array}\right\} \tag{4}$$

Proof. Define the constraint set, C, of problem (4) by:

$$C = \{x \in \mathbb{R}^n_+ : G^j(x) \geq 0 \ \text{ for } j = 1, ..., r\} \tag{5}$$

Note that since $\hat{x} \in \mathbb{R}^n_+$, and (3)(iii) holds, we have $\hat{x}$ in the constraint set C.

Define:

$$\phi(x, \hat{\nu}) = f(x) + \sum_{j=1}^r \hat{\nu}_j G^j(x) \ \text{ for all } x \in X \tag{6}$$

Then, given $\hat{\nu}$, we know that ϕ is continuously differentiable on X, and concave on $\mathbb{R}^n_+$ (since f, G^j $(j = 1, ..., r)$ are concave functions on $\mathbb{R}^n_+$, and $\hat{\nu} \in \mathbb{R}^r_+$). Applying Lemma 1 to the function ϕ (given $\hat{\nu}$), we have for all $x \in \mathbb{R}^n_+$,

$$\phi(x, \hat{\nu}) - \phi(\hat{x}, \hat{\nu}) \leq \nabla\phi(\hat{x}, \hat{\nu})(x - \hat{x}) = \nabla\phi(\hat{x}, \hat{\nu})x - \nabla\phi(\hat{x}, \hat{\nu})\hat{x} \tag{7}$$

Since $\hat{x}_i D_i \phi(\hat{x}, \hat{\nu}) = \hat{x}_i[D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] = 0$ for $i = 1, ..., n$ by (3)(ii), we obtain $\nabla\phi(\hat{x}, \hat{\nu})\hat{x} = 0$. Since $D_i \phi(\hat{x}, \hat{\nu}) = D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] \leq 0$ for $i = 1, ..., n$ by (3)(i), and $x \in \mathbb{R}^n_+$, we have $\nabla\phi(\hat{x}, \hat{\nu})x \leq 0$. Using these observations in (7), we obtain:

$$f(x) + \sum_{j=1}^m \hat{\nu}_j G^j(x) = \phi(x, \hat{\nu}) \leq \phi(\hat{x}, \hat{\nu}) = f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j G^j(\hat{x}) \tag{8}$$

using the definition of ϕ in (6). Using (3)(iv) in (8), we obtain:

$$f(x) + \sum_{j=1}^{r} \hat{\nu}_j G^j(x) \leq f(\hat{x}) \tag{9}$$

Now consider any $x \in C$. Then, $G^j(x) \geq 0$ for $j = 1, ..., r$, and so:

$$\sum_{j=1}^{r} \hat{\nu}_j G^j(x) \geq 0 \tag{10}$$

since $\hat{\nu} \in \mathbb{R}^r_+$. Using (10) in (9), we get $f(x) \leq f(\hat{x})$, which shows that $\hat{x}$ solves problem (4). // ■

Remark 1:

The Kuhn-Tucker conditions appearing in (3) look somewhat different from the Kuhn-Tucker conditions appearing in Section 10.3 of the *Lecture Notes*. Actually, they are equivalent, as discussed in detail in Section 10.6 of the *Lecture Notes*. The difference is that in (3), we are only keeping track of the Kuhn-Tucker multipliers associated with the constraints $G^j(x) \geq 0$ for all $j = 1, ..., r$, but we are not explicitly mentioning the multipliers associated with the constraints $x \geq 0$. A conversion from the form used in (3) to the form used in Section 10.3 can be accomplished as follows.

Define $g^j(x) = G^j(x)$ for all $x \in X$, and $j = 1, ..., r$; further define $g^j(x) = x_{j-r}$ for all $x \in X$, and for $j = r + 1, ..., r + n$. Then, defining $m = n + r$, we have m constraint functions (which are defined and continuously differentiable on the open set X in $\mathbb{R}^n$).

Given (3), define:

$$\hat{\lambda}_j = \hat{\nu}_j \text{ for } j = 1, ..., r \tag{11}$$

and:

$$\hat{\lambda}_j = -[D_{j-r} f(\hat{x}) + \sum_{k=1}^{r} \hat{\nu}_k D_{j-r} G^k(\hat{x})] \text{ for } j = r + 1, ..., r + n \tag{12}$$

Then $\hat{\lambda} \in \mathbb{R}^m_+$ by (3)(i), and:

$$D_i f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j D_i g^j(\hat{x}) = 0 \text{ for } i = 1, ..., n \tag{13}$$

Further, since $\hat{x} \in \mathbb{R}^n_+$ and (3)(iii) holds, we have:

$$g^j(\hat{x}) \geq 0 \text{ for } j = 1, ..., m \tag{14}$$

Finally, by using (3)(ii) and (3)(iv), we have:

$$\hat{\lambda}_j g^j(\hat{x}) = 0 \text{ for } j = 1, ..., m \tag{15}$$

Thus, we have $(\hat{x}, \hat{\lambda}) \in X \times \mathbb{R}^m_+$, satisfying the Kuhn-Tucker conditions (13), (14), (15) as defined in Section 10.3.

Remark 2:

The Kuhn-Tucker sufficiency theorem is most useful in optimization problems in which the solution to the problem can be at the boundary or in the interior of $\mathbb{R}^n_+$, depending on the values of the parameters. To handle both cases, one needs to be assured of being able to differentiate the objective and constraint functions not just in the interior of $\mathbb{R}^n_+$, but also at boundary points of $\mathbb{R}^n_+$. This is the principal role of the assumption of continuous differentiability of these functions on an open set X, which contains

$\mathbb{R}^n_+$. Concavity of these functions, however, is only needed on $\mathbb{R}^n_+$ because that suffices to establish the crucial inequality (7) above, by applying Lemma 1.

As an illustration, consider the problem of obtaining a demand function in the case of a quasi-linear utility function. Let p_1, p_2 and w be positive parameters. The relevant optimization problem is:

$$\left.\begin{array}{ll} \text{Max} & v(x_1) + x_2 \\ \text{subject to} & p_1 x_1 + p_2 x_2 \leq w \\ \text{and} & (x_1, x_2) \geq 0 \end{array}\right\} (P)$$

Because $x \in \mathbb{R}^2_+$ is a constraint on the problem, the objective and constraint functions are naturally going to be defined for all x in $\mathbb{R}^2_+$ (otherwise the problem would not be well-defined). Other properties of these functions (if any) are also going to be specified on the domain $\mathbb{R}^2_+$.

However, in order that one can take derivatives of these functions at the boundary of $\mathbb{R}^2_+$, we want an open set X which contains $\mathbb{R}^2_+$, such that the objective and constraint functions will be continuously differentiable on X. This set X will obviously not be given, but is rather a set that you choose appropriately to apply the Kuhn-Tucker sufficiency theorem.

Example 1:

Suppose, as a concrete example,

$$v(x_1) = A \ln(1 + x_1) \text{ for all } x_1 \geq 0 \tag{16}$$

where A is a positive parameter. Then, we can define $f : \mathbb{R}^2_+ \to \mathbb{R}$ by $f(x_1, x_2) = A \ln(1 + x_1) + x_2$ for all $x \in \mathbb{R}^2_+$, and $G : \mathbb{R}^2_+ \to \mathbb{R}$ by $G(x_1, x_2) = w - (p_1 x_1 + p_2 x_2)$ for all $x \in \mathbb{R}^2_+$. This would convert problem (P) to the form (4), with $n = 2$ and $r = 1$. It would be easy to check that f and G are concave on $\mathbb{R}^2_+$.

Now, one can choose, for instance, $X = \{x \in \mathbb{R}^2 : x_1 > -1 \text{ and } x_2 > -1\}$. This is an open set in $\mathbb{R}^2$, containing $\mathbb{R}^2_+$, and the f and the G functions defined in the previous paragraph on $\mathbb{R}^2_+$ are in fact well-defined on this larger set X. Further, as can be checked, on this larger set X, both f and G are continuously differentiable. We are now all set to apply Theorem 1 to problem (P).

Example 2:

Consider an alternative example in which

$$v(x_1) = A(x_1)^{1/2} \text{ for all } x_1 \geq 0 \tag{17}$$

where A is a positive parameter. Then, we can define $f : \mathbb{R}^2_+ \to \mathbb{R}$ by $f(x_1, x_2) = A(x_1)^{1/2} + x_2$ for all $x \in \mathbb{R}^2_+$, and $G : \mathbb{R}^2_+ \to \mathbb{R}$ by $G(x_1, x_2) = w - (p_1 x_1 + p_2 x_2)$ for all $x \in \mathbb{R}^2_+$. This would convert problem (P) to the form (4), with $n = 2$ and $r = 1$. It would be easy to check that f and G are concave on $\mathbb{R}^2_+$.

But, now, you will not be able to define an open set X, containing $\mathbb{R}^2_+$, on which f is continuously differentiable. This is because the derivative of v goes to infinity as x_1 approaches 0 from the right. This makes it inappropriate to try to apply Theorem 1 above to this optimization problem. Of course, we have other methods at our disposal to solve this problem.

4 Weakening the Differentiability Conditions

Keeping Example 2 in mind, one might try to establish a version of the Kuhn-Tucker sufficiency theorem, where the domain on which one requires continuous differentiability of the objective and constraint functions is more restricted.

To this end, a variation of Lemma 1 can be stated and proved as follows. It replaces the continuous differentiability of h on an open set containing $\mathbb{R}^n_+$ with the assumption that h is continuously differentiable in a neighborhood of the point x' in $\mathbb{R}^n_+$, where the gradient is evaluated in the statement of Lemma 1. Concavity of h on $\mathbb{R}^n_+$ is still enough to ensure the basic inequality obtained in Lemma 1.

Lemma 2 *Let $x' \in \mathbb{R}^n_+$ and assume there is $\delta > 0$ such that h is a function from $\mathbb{R}^n_+ \cup B(x', \delta)$ to $\mathbb{R}$, which is continuously differentiable on $B(x', \delta)$, the open ball in $\mathbb{R}^n$ with center x' and radius δ. If h is concave on $\mathbb{R}^n_+$, then for all $x \in \mathbb{R}^n_+$,*

$$h(x) - h(x') \leq \nabla h(x')(x - x') \tag{18}$$

Proof. Define $X = B(x', \delta)$. We can find $0 < \lambda < 1$ such that $\bar{x} \equiv \lambda x' + (1 - \lambda)x$ is in X; clearly, $\bar{x}$ also belongs to $\mathbb{R}^n_+$. Define $\bar{x}(n) = \bar{x} + (u/n), x'(n) = x' + (u/n)$, where $u = (1, 1, ..., 1)$ in $\mathbb{R}^n$, and $n \in \mathbb{N}$. We can find $N \in \mathbb{N}$, such that for all $n \geq N$, we have $\bar{x}(n)$ and $x'(n)$ in X. Then, $\bar{x}(n)$ and $x'(n)$ belong to $\mathbb{R}^n_{++} \cap X$ for all $n \geq N$, and h is concave on the open set $\mathbb{R}^n_{++}$, so we can apply Theorem 27 of the *Lecture Notes* to get:

$$h(\bar{x}(n)) - h(x'(n)) \leq \nabla h(x'(n))(\bar{x}(n) - x'(n)) \tag{19}$$

Now, we let $n \to \infty$. Then $\bar{x}(n) \to \bar{x}, x'(n) \to x'$; by continuity of h on X, $h(\bar{x}(n)) \to h(\bar{x})$ and $h(x'(n)) \to h(x')$; by continuity of $D_i h$ on X, we have $D_i h(x'(n)) \to D_i h(x')$. Using these facts in (19) and noting that weak inequalities are preserved in the limit, we get:

$$h(\bar{x}) - h(x') \leq \nabla h(x')(\bar{x} - x') \tag{20}$$

Using the definition of $\bar{x}$, and the concavity of h on $\mathbb{R}^n_+$, we obtain $(\bar{x} - x') = (1 - \lambda)(x - x')$, while $h(\bar{x}) - h(x') \geq \lambda h(x') + (1 - \lambda)h(x) - h(x') = (1 - \lambda)(h(x) - h(x'))$. Substituting these expressions in (20) implies:

$$(1 - \lambda)(h(x) - h(x')) \leq \nabla h(x')(1 - \lambda)(x - x')$$

which proves (18), since $(1 - \lambda) > 0$. ■

Using Lemma 2 in place of Lemma 1, and imitating the steps in the proof of Theorem 1, one can state and prove the following Theorem.

Theorem 2 *Suppose $(\hat{x}, \hat{\nu}) \in \mathbb{R}^n_+ \times \mathbb{R}^r_+$, and there is $\delta > 0$ such that f, G^j $(j = 1, ..., m)$ are functions from $\mathbb{R}^n_+ \cup B(x', \delta)$ to $\mathbb{R}$, which are continuously differentiable on $B(\hat{x}, \delta)$, the open ball in $\mathbb{R}^n$ with center $\hat{x}$ and radius δ. Suppose f, G^j $(j = 1, ..., m)$ are concave functions on $\mathbb{R}^n_+$. If $(\hat{x}, \hat{\nu})$ satisfies the Kuhn-Tucker conditions:*

$$\left.\begin{array}{l} (i)\ D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x}) \leq 0 \ \ for\ i = 1, ..., n \\ (ii)\ \hat{x}_i[D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] = 0 \ \ for\ i = 1, ..., n \\ (iii)\ G^j(\hat{x}) \geq 0 \ \ for\ j = 1, ..., r \\ (iv)\ \hat{\nu}_j G^j(\hat{x}) = 0 \ \ for\ j = 1, ..., r \end{array}\right\} \tag{21}$$

then $\hat{x}$ solves the problem:

$$\left.\begin{array}{ll} Max & f(x) \\ subject\ to & G^j(x) \geq 0 \ \ for\ j = 1, ..., r \\ and & x \in \mathbb{R}^n_+ \end{array}\right\} \tag{22}$$

Proof. Define the constraint set, C, of problem (22) by:

$$C = \{x \in \mathbb{R}^n_+ : G^j(x) \geq 0 \ \text{ for } j = 1, ..., r\} \tag{23}$$

Note that since $\hat{x} \in \mathbb{R}^n_+$, and (21)(iii) holds, we have $\hat{x}$ in the constraint set C.

Define $X = B(\hat{x}, \delta)$, and:

$$\phi(x, \hat{\nu}) = f(x) + \sum_{j=1}^r \hat{\nu}_j G^j(x) \ \text{ for all } x \in X \cup \mathbb{R}^n_+ \tag{24}$$

Then, given $\hat{\nu}$, we know that ϕ is continuously differentiable on X, and concave on $\mathbb{R}^n_+$ (since f, G^j $(j = 1, ..., r)$ are concave functions on $\mathbb{R}^n_+$, and $\hat{\nu} \in \mathbb{R}^r_+$). Applying Lemma 2 to the function ϕ (given $\hat{\nu}$), we have for all $x \in \mathbb{R}^n_+$,

$$\phi(x, \hat{\nu}) - \phi(\hat{x}, \hat{\nu}) \leq \nabla\phi(\hat{x}, \hat{\nu})(x - \hat{x}) = \nabla\phi(\hat{x}, \hat{\nu})x - \nabla\phi(\hat{x}, \hat{\nu})\hat{x} \tag{25}$$

Since $\hat{x}_i D_i\phi(\hat{x}, \hat{\nu}) = \hat{x}_i[D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] = 0$ for $i = 1, ..., n$ by (21)(ii), we obtain $\nabla\phi(\hat{x}, \hat{\nu})\hat{x} = 0$. Since $D_i\phi(\hat{x}, \hat{\nu}) = [D_i f(\hat{x}) + \sum_{j=1}^r \hat{\nu}_j D_i G^j(\hat{x})] \leq 0$ for $i = 1, ..., n$ by (21)(i), and $x \in \mathbb{R}^n_+$, we have $\nabla\phi(\hat{x}, \hat{\nu})x \leq 0$. Using these observations in (25), we obtain:

$$f(x) + \sum_{j=1}^{m} \hat{\nu}_j G^j(x) = \phi(x, \hat{\nu}) \leq \phi(\hat{x}, \hat{\nu}) = f(\hat{x}) + \sum_{j=1}^{r} \hat{\nu}_j G^j(\hat{x}) \tag{26}$$

using the definition of ϕ in (24). Using (21)(iii) in (26), we obtain:

$$f(x) + \sum_{j=1}^{r} \hat{\nu}_j G^j(x) \leq f(\hat{x}) \tag{27}$$

Now consider any $x \in C$. Then, $G^j(x) \geq 0$ for $j = 1, ..., r$, and so:

$$\sum_{j=1}^{r} \hat{\nu}_j G^j(x) \geq 0 \tag{28}$$

since $\hat{\nu} \in \mathbb{R}^r_+$. Using (28) in (27), we get $f(x) \leq f(\hat{x})$, which shows that $\hat{x}$ solves problem (22). // ■

Remarks:

We return now to a discussion of Example 2 above. Here, it is easy to check that $(\hat{x}, \hat{\nu})$ given by:

$$\hat{x}_1 = \frac{A^2 p_2^2}{4p_1^2},\ \hat{x}_2 = w - \frac{A^2 p_2^2}{4p_1},\ \hat{\nu} = \frac{1}{p_2} \tag{29}$$

satisfies the Kuhn-Tucker conditions, when the parameters satisfy:

$$\frac{A^2 p_2^2}{4p_1} < w \tag{30}$$

Since $\hat{x}_1 > 0$ in (29), there is $\delta > 0$ such that f and G are continuously differentiable on $B(\hat{x}, \delta)$, and Theorem 2 is applicable. Thus, $\hat{x}$ given in (29) solves problem (P), given (17), when the condition on the parameters satisfies the inequality in (30).

Similarly, it can be checked that$(\hat{x}, \hat{\nu})$ given by:

$$\hat{x}_1 = \frac{w}{p_1},\ \hat{x}_2 = 0,\ \hat{\nu} = \frac{A}{2w^{1/2}p_1^{1/2}} \tag{31}$$

satisfies the Kuhn-Tucker conditions, when the parameters satisfy:

$$\frac{A^2 p_2^2}{4p_1} \geq w \tag{32}$$

Again, since $\hat{x}_1 > 0$ in (31), there is $\delta > 0$ such that f and G are continuously differentiable on $B(\hat{x}, \delta)$, and Theorem 2 is applicable. Thus, $\hat{x}$ given in (31) solves problem (P), given (17), when the condition on the parameters satisfies the inequality in (32).