Cowles Foundation
for Research in Economics
at Yale University

Cowles Foundation Discussion Paper No. 1896RR

THE LIMITS OF PRICE DISCRIMINATION

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May 2013
Revised April 2014

The Limits of Price Discrimination[*]

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April 14, 2014

Abstract

We analyze the welfare consequences of a monopolist having additional information about consumers' tastes, beyond the prior distribution; the additional information can be used to charge different prices to different segments of the market, i.e., carry out "third degree price discrimination".

We show that the segmentation and pricing induced by the additional information can achieve *every* combination of consumer and producer surplus such that: (i) consumer surplus is non-negative, (ii) producer surplus is at least as high as profits under the uniform monopoly price, and (iii) total surplus does not exceed the surplus generated by efficient trade.

KEYWORDS: First Degree Price Discrimination, Second Degree Price Discrimination, Third Degree Price Discrimination, Private Information, Privacy, Bayes Correlated Equilibrium, Concavification.

JEL CLASSIFICATION: C72, D82, D83.

1 Introduction

A classic issue in the economic analysis of monopoly is the impact of discriminatory pricing on consumer and producer surplus. A monopolist engages in *third degree price discrimination* if he uses additional information about consumer characteristics to offer different prices to different segments of the aggregate market. A large and important literature (reviewed below) examines the impact of particular segmentations on consumer and producer surplus, as well as on output and prices.

In this paper, we characterize what could happen to consumer and producer surplus for all possible segmentations of the market. We know that at least two points will be attained. If the monopolist has no information beyond the prior distribution of valuations, there will be no segmentation. The producer charges the uniform monopoly price and gets the associated monopoly profit, which is always a lower bound on producer surplus; consumers receive a positive surplus, the standard information rent. This is marked by point A in Figure 1. On the other hand, if the monopolist has complete information about the valuations of the buyers, then he can charge each buyer their true valuation, i.e., engage in perfect or *first degree price discrimination*. The resulting allocation is efficient, but consumer surplus is zero and the producer captures all of the gains from efficient trade. This is marked by point B in Figure 1.
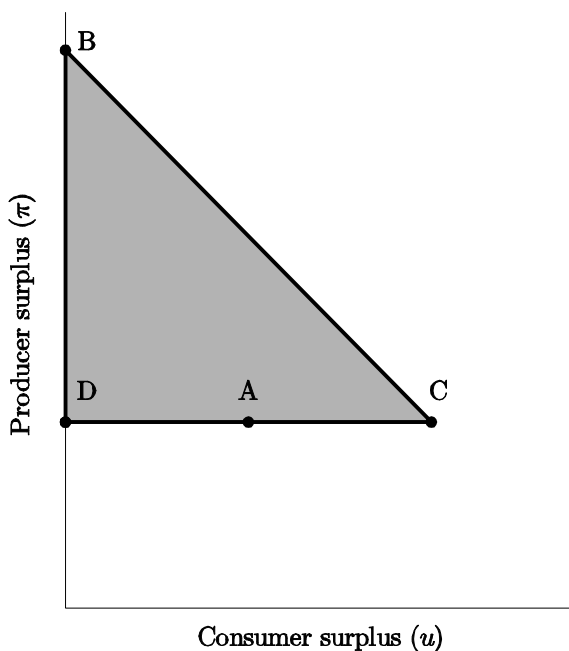

Figure 1: The surplus triangle

We are concerned with the welfare consequences of all possible segmentations, in addition to the

two mentioned above. To begin with, we can identify some elementary bounds on consumer and producer surplus in any market segmentation. First, consumer surplus must be non-negative as a consequence of the participation constraint; a consumer will not buy the good at a price above his valuation. Second, the producer must get at least the surplus that he could get if there was no segmentation and he charged the uniform monopoly price. Third, the sum of consumer and producer surplus cannot exceed the total value that consumers receive from the good, when that value exceeds the marginal cost of production. The shaded right angled triangle in Figure 1 illustrates these three bounds.

Our main result is that *every* welfare outcome satisfying these constraints is attainable by some market segmentation. This is the entire shaded triangle in Figure 1. The point marked C is where consumer surplus is maximized; in particular, the producer is held down to his uniform monopoly profits, but at the same time the outcome is efficient and consumers receive all of the gains in efficiency relative to no discrimination. At the point marked D, social surplus is minimized by holding producer surplus down to uniform monopoly profits and holding consumer surplus down to zero.

We can explain these results most easily in the case where there is a finite set of possible consumer valuations and the cost of production is zero. The latter is a normalization of constant marginal cost we will maintain in the paper. We will first explain one intuitive way to maximize consumer surplus, i.e., realize point C. The set of market prices will consist of every valuation less than or equal to the uniform monopoly price. Suppose that we can divide the market into segments corresponding to each of these prices in such a way that (i) in each segment, the consumers' valuations are always greater than or equal to the price for that segment; and (ii) in each segment, the producer is indifferent between charging the price for that segment and charging the uniform monopoly price. Then the producer is indifferent to charging the uniform monopoly price on *all* segments, so producer surplus must equal uniform monopoly profit. The allocation is also efficient, so consumers must obtain the rest of the efficient surplus. Thus, (i) and (ii) are sufficient conditions for a segmentation to maximize consumer surplus.

We now describe a way of constructing such a market segmentation iteratively. Start with a "lowest price segment" where a price equal to the lowest valuation will be charged. All consumers with the lowest valuation go into this segment. For each higher valuation, a share of consumers with that valuation also enters into the lowest price segment. While the *relative* share of each higher valuation (with respect to each other) is the same as in the prior distribution, the proportion of all of the higher valuations is lower than in the prior distribution. We can choose that proportion

between zero and one such that the producer is indifferent between charging the segment price and the uniform monopoly price. We know this must be possible because if the proportion were equal to one, the uniform monopoly price would be profit maximizing for the producer (by definition); if the proportion were equal to zero—so only lowest valuation consumers were in the market—the lowest price would be profit maximizing; and, by keeping the relative proportions above the lowest valuation constant, there is no price other than these two that could be optimal. Now we have created one market segment satisfying properties (i) and (ii) above. But notice that the consumers not put in the lowest price segment are in the same relative proportions as they were in the original population, so the original uniform monopoly price will be optimal on this "residual segment". We can apply the same procedure to construct a segment in which the market price is the second lowest valuation: put all the remaining consumers with the second lowest valuation into this market; for higher valuations, put a fixed proportion of remaining consumers into that segment; choose the proportion so that the producer is indifferent between charging the second highest valuation and the uniform monopoly price. This construction iterates until the segment price reaches the uniform monopoly price, at which point we have recovered the entire population and point C is attained.

For the formal proof of our results, we make use of a deeper geometric argument. This establishes an even stronger conclusion: any point where the monopolist is held down to his uniform monopoly profits—including outcomes A, C, and D in Figure 1—can be achieved *with the same segmentation*! In this segmentation, consumer surplus varies because the monopolist is indifferent between charging different prices. The existence of such a segmentation is a consequence of the following key property: Consider the set of all markets where a given monopoly price is optimal. This set is convex, so any aggregate market with the given monopoly price can be decomposed as a weighted sum of markets which are extreme points of this set, which in turn defines a segmentation. We show that these extreme points, or extremal markets, must take a special form: In any extremal market, the monopolist will be indifferent to setting any price in the support of consumers' valuations. Thus, each subset of valuations that includes the given monopoly price generates an extremal market. If the monopolist charges the uniform monopoly price on each extreme segment, we get point A. If he charges the lowest value in the support of each segment (which is also an optimal price, by construction), we get point C; and if he charges the highest value in the support, we get point D. Beyond its welfare implications, this argument also highlights the multiplicity of segmentations that can achieve extreme outcomes, since there are many sets of extremal markets which can be used to decompose a given market.

Thus, we are able to demonstrate that points B, C, and D can be attained. Every point in their convex hull, i.e., the shaded triangle in Figure 1, can also be attained by segmenting a share of the market using extremal markets as in the previous paragraph and segmenting the rest of the market to facilitate perfect price discrimination. Such a segmentation always gives a fixed level of producer surplus between the uniform monopoly profits and perfect price discrimination, and the monopolist is indifferent between prices that yield a consumer surplus of zero and prices that maximize social surplus. This gives us a complete characterization of all possible welfare outcomes.

While we focus on welfare implications, we can also completely characterize possible output levels and derive implications for prices. An upper bound on output is the efficient quantity, and this is realized by any segmentation along the efficient frontier. In particular, it is attained in any consumer surplus maximizing segmentation. In such segmentations, prices are always weakly below the uniform monopoly price. We are also able to obtain a tight lower bound on output. Note that in any segmentation the monopolist must receive at least his uniform monopoly profits, so this profit is a lower bound on social surplus. We say that an allocation is conditionally efficient if conditional on the good being sold, it sold to those with the highest valuations. Such allocations minimize output for a given level of social surplus. In fact, we construct a social surplus minimizing segmentation that results in a conditionally efficient allocation and therefore attains a lower bound on output. In this segmentation, and indeed in any social surplus minimizing segmentation, prices are always weakly higher than the uniform monopoly price.

Though the results described above are for discrete distributions, we are able to prove similar results for any market that can be described by a Borel measure over valuations. For such distributions, we use a limit argument to establish the existence of segmentations that attain points C and D.

We contribute to a large literature on third degree price discrimination, starting with the classic work of Pigou (1920). This literature examines what happens to prices, quantity, consumer surplus, producer surplus, and social welfare as a market is segmented. Pigou (1920) considered the case of two segments with linear demand. In the special case where both segments are served when there is a uniform price, he showed that output does not change under price discrimination. Since different prices are charged in the two segments, this means that some high valuation consumers are replaced by low valuation consumers, and thus social welfare decreases. We can visualize the results of Pigou (1920) and other authors in Figure 1. Pigou (1920) showed that this particular segmentation resulted in a west-northwest move (i.e., a move from point A to a point below the negative 45° line going

through A). A literature since then has focused on identifying sufficient conditions on the shape of demand for social welfare to increase or decrease with price discrimination. A recent paper of Aguirre, Cowan, and Vickers (2010) unifies and extends this literature and, in particular, identifies sufficient conditions for price discrimination to either increase or decrease social welfare (i.e., move above or below the negative 45° line through A). Restricting attention to market segments that have concave profit functions and an additional property ("increasing ratio condition") that they argue is commonly met, they show that welfare decreases if the direct demand in the higher priced market is at least as convex as that in the lower priced market; welfare is higher if prices are not too far apart and the inverse demand function in the lower priced market is locally more convex than that in the higher priced market. They note how their result ties in with an intuition of Robinson (1933): concave demand means that price changes have a small impact on quantity, while convex demand means that prices have a large impact on quantity. If the price rises in a market with concave demand and falls in a market with convex demand, the increase in output in the low-price market will outweigh the decrease in the high price market, and welfare will go up. A recent paper of Cowan (2013) gives sufficient conditions for consumer surplus (and thus total surplus) to increase under third degree price discrimination.

Our paper also gives sufficient conditions for particular welfare impacts of segmentation. However, unlike most of the literature, we allow for segments with non-concave profit functions. Indeed, the segmentations giving rise to extreme points in welfare space (i.e., consumer surplus maximization at point C and social surplus minimization at point D) generally rely on non-concave profit functions within segments. This ensures that the type of local conditions highlighted in the existing literature will not obtain. Our non-local results suggest some very different intuitions. Of course, consumer surplus always increases if prices drop in all markets. We show that for any demand curve, low valuation consumers can be pooled with high valuation consumers in such a way that the producer has an incentive to offer prices below the monopoly price; but if this incentive is made arbitrarily weak, the consumers capture the efficiency gains.

The literature also has results on the impact of segmentation on output and prices. On output, the focus is on identifying when an increase in output is necessary for an increase in welfare, as in Schmalensee (1981) and Varian (1985). Although we do not analyze the question in detail in this paper, a given output level is associated with many different levels of producer, consumer and social surplus. We do provide a sharp characterization of the highest and lowest possible output over all market segmentations. On prices, Nahata, Ostaszewski, and Sahoo (1990) offer examples with

non-concave profit functions where third degree price discrimination may lead prices in all market segments to move in the same direction; it may be that all prices increase or all prices decrease. We show that one can create such segmentations *for any* demand curve. In other words, in constructing our critical market segmentations, we show that it is always possible to have all prices rise or all prices fall (although profit functions in the segments cannot all be strictly concave, as shown by Nahata, Ostaszewski, and Sahoo (1990)).

If market segmentation is exogenous, one might argue that the segmentations that deliver extremal surpluses are special and might be seen as atypical. But given the amount of information presently being collected on the internet about consumer valuations, it might be argued that there is increasing endogeneity in the market segmentations that arise. To the extent that producers control how information is disseminated, they will have an incentive to gather as much information as possible, ideally engaging in perfect price discrimination. Suppose, however, that an internet intermediary wanted to release its information about consumers to producers for free in order to maximize consumer welfare, say, because of regulatory pressure or the relation to a broader business model. Our results describe how such a consumer-minded internet company would choose to structure this information.[1]

Third degree price discrimination is a special case of the classic screening problem, in which a principal is designing a contract for an agent who has private information about the environment. If the principal has no information about the agent's type, then he must offer the same menu to all agents, which yields a uniform menu profit (or producer surplus) for the principal with a corresponding information rent (or consumer surplus) for the agent, leading to a point analogous to A in Figure 1. If the principal was perfectly informed about the agent's type, he could extract all the potential surplus from the relationship, leading to point analogous to B in Figure 1. And, as in the third degree price discrimination problem, there are bounds on surplus pairs for any intermediate segmentation given by a triangle BCD. However, it is not possible in general to find a segmentation that attains every point in BCD, and we do not have a characterization of what happens in general screening problems. Nonetheless, we do examine the robustness of our main result by seeing what happens as we move from third degree price discrimination to more general settings.

To understand our robustness exercise, observe that our main result would be unchanged if instead of restricting consumers to either not getting the good or getting one unit, we allowed consumption

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[1]An important subtlety of this story, however, is that this could only be done by randomly allocating consumers with the same valuation to different segments with different prices. Thus it could be done by a benevolent intermediary who already knew consumers' valuations, but not by one who needed consumers to truthfully report their values.

of quantities between 0 and 1 and linearly interpolated the consumer's valuation of intermediate quantities. This follows from the classical observation that as long as consumers' valuations are linear in quantities or probabilities of getting the object, a posted price is an optimal mechanism. This would no longer be true if valuations are nonlinear in quantity. In Section 5, we examine the robustness of our result by adding a small amount of concavity to consumers' valuations. In this case, the monopolist will wish to engage in "second degree price discrimination", since for each consumer the marginal value of the good varies with quantity, as in Maskin and Riley (1984). As long as only a finite set of quantities are possible, our geometric analysis can be extended to characterize the set of distributions over consumer valuations where a given menu is optimal. At extreme points of that set, the producer will be indifferent between different menus. Thus, there will be a range of consumer surpluses that are consistent with the producer being held down to his uniform menu profits. However, it will not generally be the case—as it was with linear preferences—that the aggregate market can be segmented in ways such that the producer is held down to his uniform menu profit, but the allocation is either efficient—giving point C—or results in zero consumer surplus—leading to point D. The reason is that with linear preferences it is possible to make the monopolist indifferent to any subset of prices, whereas with general concave utility, the monopolist can only be indifferent among menus that differ by small amounts. Yet, we do show that as we approach the linear case, the equilibrium surplus pairs converge to the triangle and, in this sense, our main result is robust to small deviations from linearity.

Our work has a methodological connection to two strands of literature. Kamenica and Gentzkow (2011)'s study of "Bayesian persuasion" considers how a sender would choose to transmit information to a receiver, if he could commit to an information revelation strategy before observing his private information. They provide a characterization of such optimal communication strategies as well as applications. If we let the receiver be the producer choosing prices, and let the sender be a planner maximizing some weighted sum of consumer and producer surplus, our problem belongs to the class of problems analyzed by Kamenica and Gentzkow (2011). They show that if one plots the utility of the "sender" as a function of the distribution of the sender's types, his highest attainable utility can be read off from the "concavification" of that function.[2] The concavification arguments are especially

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[2]Aumann and Maschler (1995), show that the concavification of the (stage) payoff function represents the limit payoff that an informed player can achieve in a repeated zero sum game with incomplete information. In particular, their Lemma 5.3, the "splitting lemma", derives a partial disclosure strategy on the basis of a concavified payoff function.

powerful in the case of two types. While we do not use concavification arguments in the proof of our main result, we illustrate their use in our two type analysis of second degree price discrimination.

Bergemann and Morris (2013a) examine the general question, in strategic many-player settings, of what behavior could arise in an incomplete information game if players observe additional information not known to the analyst. They show that behavior that might arise is equivalent to an incomplete information version of correlated equilibrium termed "Bayes correlated equilibrium", which reduces to the problem Kamenica and Gentzkow (2011) in the case of one player.[3] Using the language of Bergemann and Morris (2013a), the present paper considers the game of a producer making take-it-or-leave-it offers to consumers. Here, consumers have a dominant strategy to accept all offers strictly less than their valuation and reject all offers strictly greater than their valuation, and we select for equilibria in which consumers accept offers that make them indifferent. We characterize what could happen for any information structure that players might observe, as long as consumers know their own valuations. Thus, we identify possible payoffs of the producer and consumers in all Bayes correlated equilibria of the price setting game. Thus, our results are a striking application of the methodologies of Bergemann and Morris (2013a), (2013b) and Kamenica and Gentzkow (2011) to the problem of price discrimination. We also make use of these methodologies as well as results from the present paper in our analysis of what can happen for all information structures in a first-price auction, in Bergemann, Brooks, and Morris (2013a).

We present our model of monopoly price discrimination with discrete valuations in Section 2, with our main results in Section 3. We first give a characterization of the equilibrium surplus pairs using the extremal segmentations described above. Though the argument for this characterization is non-constructive, we also exhibit some constructive approaches to achieve extreme welfare outcomes. In this Section, we also characterize a tight lower bound on output that can arise under price discrimination. In Section 4, we briefly extend our results to general settings with a continuum of values, so that the demand curve consists of a combination of mass points and densities. The basic economic insights extend to this setting unchanged. In Section 5, we describe how our results do change as we move to more general screening environments, where the utility of the buyer and/or the cost of the seller are not linear in quantity, and thus give rise to second degree price discrimination. We conclude in Section 6. All omitted proofs are contained in the Appendix.

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[3]In Bergemann and Morris (2013b), these insights were developed in detail in games with a continuum of players, linear-quadratic payoffs and normally distributed uncertainty.

2   Model

A monopolist sells a good to a continuum of consumers, each of whom demands one unit. We normalize the total mass of consumers to one and the constant marginal cost of the good to zero. There are K possible values $v_k \in V \subseteq \mathbb{R}_+$ that the consumers might have:

$$0 < v_1 < \cdots < v_k < \cdots < v_K.$$

We will extend the analysis to a continuum of valuations in Section 4. A *market* x is a distribution over the K valuations, with the set of all markets being:

$$X \triangleq \left\{ x \in \mathbb{R}_+^V \,\middle|\, \sum_{k=1}^{K} x\left(v_k\right) = 1 \right\}.$$

This set can be identified with the $(K-1)$-dimensional simplex, and to simplify notation we will write x_k for $x\left(v_k\right)$, which is the proportion of consumers who have valuation v_k. Thus, a market x corresponds to a step demand function, where $\sum_{j \geq k} x_j$ is the demand for the good at any price in the interval $(v_{k-1}, v_k]$ (with the convention that $v_0 = 0$).

While we will focus on this interpretation of the model throughout the paper, there is a well-known alternative interpretation that there is a single consumer with unit demand whose valuation is distributed according to probability distribution x. The analysis is unchanged and all results can be translated into this alternative interpretation.

Throughout the analysis, we hold a given *aggregate market* as fixed and identify it by:

$$x^* \in X. \tag{1}$$

We say that the price v_k is *optimal* for market x if the expected revenue from price v_k satisfies:

$$v_k \sum_{j \geq k} x_j \;\geq\; v_i \sum_{j \geq i} x_j, \;\; \text{for all } i = 1, ..., K. \tag{2}$$

X_k denotes the set of markets where price v_k is optimal:

$$X_k \triangleq \left\{ x \in X \,\middle|\, v_k \sum_{j=k}^{K} x_j \geq v_i \sum_{j=i}^{K} x_j, \text{ for all } i = 1, ..., K \right\}.$$

Now write $v^* \triangleq v_{i^*}$ for the optimal uniform price for the aggregate market x^*. Thus $x^* \in X^* \triangleq X_{i^*}$. The maximum feasible surplus is:

$$w^* \triangleq \sum_{k=1}^{K} v_k x_k^*, \tag{3}$$

corresponding to all consumers buying the good. The uniform price producer surplus is then:

$$\pi^* \triangleq v^* \sum_{k=i^*}^{K} x_k^* = \max_{i \in \{1,\dots,K\}} v_i \sum_{k=i}^{K} x_k^*, \qquad (4)$$

and the uniform price consumer surplus is

$$u^* \triangleq \sum_{k=i^*}^{K} (v_k - v^*) x_k^*.$$

We will use a simple example to illustrate many of the results to follow.

Example 1 (Three Values with Uniform Probability)

There are three valuations $\{1, 2, 3\}$ which arise in equal proportions. Thus, $K = 3$, $v_k = k$, and $x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. The feasible social surplus is $w^* = \frac{1}{3}(1 + 2 + 3) = 2$. The uniform monopoly price is $v^* = 2 = i^*$. Under the uniform monopoly price, profit is $\pi^* = \frac{2}{3} \times 2 = \frac{4}{3}$ and consumer surplus is $u^* = \frac{1}{3}(3 - 2) + \frac{1}{3}(2 - 2) = \frac{1}{3}$.

We can visualize the markets consisting of three possible valuations as being points in the two-dimensional probability simplex, as depicted in Figure 2. Each point in the triangle corresponds to the weighted sum of the three vertices with weights corresponding to the respective proportions. We have divided the simplex into three regions corresponding to X_1, X_2, and X_3 where prices 1, 2 and 3 respectively are optimal. Note that the restriction that revenue from price v_k is greater than revenue from price v_i is a linear restriction, and thus the region X_1, for example, is the intersection of the region in which price 1 is better than 2 and the region where price 1 is better than 3. The aggregate market x^* is the centroid $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which lies in the interior of the set X_2, since price 2 is strictly optimal.

A *segmentation* is a division of the aggregate market into different markets. Thus, a segmentation σ is a simple probability distribution on X, with the interpretation that $\sigma(x)$ is the proportion of the population in market x. A segmentation can be viewed as a two stage lottery on outcomes $\{1, \dots, K\}$ whose reduced lottery is x^*. Writing **supp** for the support of a distribution, the set of possible segmentations is:

$$\Sigma = \left\{ \sigma \in \Delta(X) \ \middle| \ \sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \cdot x = x^*, \ |\mathbf{supp}\,\sigma| < \infty \right\}.$$

We restrict attention to finitely many segments so that $|\mathbf{supp}\,\sigma| < \infty$. This is without loss of generality in the present environment with finitely many valuations, in that finite segmentations will suffice to prove tightness of our bounds on welfare outcomes.
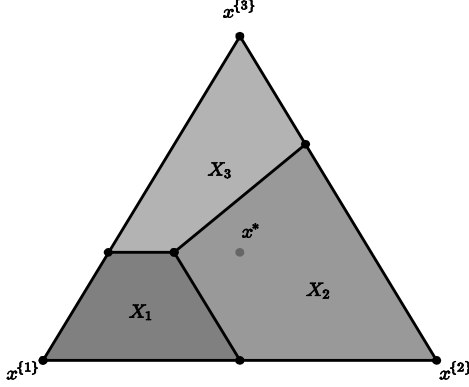
Figure 2: The simplex of markets with $v_k \in \{1, 2, 3\}$.

A *pricing rule* for a segmentation σ specifies a distribution over prices for each market in the support of σ:

$$\phi : \mathbf{supp}\,\sigma \to \Delta\left\{v_1, ..., v_K\right\}. \tag{5}$$

We will write $\phi_k(x)$ for the probability of charging price v_k in market x. A pricing strategy is *optimal* if, for each x, $v_k \in \mathbf{supp}\,\phi(x)$ implies $x \in X_k$, i.e., all prices charged with positive probability on market x must be profit maximizing for market x.

An example of a segmentation and an associated optimal pricing rule is given by the case of perfect (or first degree) price discrimination. In this case, there are at least as many segments as possible valuations, and each segment contains consumers of a single valuation. The optimal pricing rule charges the unique valuation that appears in the segment. For Example 1, perfect price discrimination consists of three market segments with three associated prices as illustrated in the table below:

	value 1	value 2	value 3	price	weight
market 1	1	0	0	1	$\frac{1}{3}$
market 2	0	1	0	2	$\frac{1}{3}$
market 3	0	0	1	3	$\frac{1}{3}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		

(6)

This segmentation can be visualized as simply saying that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ can be decomposed as the convex combination of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Given segmentation σ and pricing rule ϕ, consumer surplus is:

$$\sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) \sum_{j=k}^{K} (v_j - v_k) x_j;$$

producer surplus is:

$$\sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) v_k \sum_{j=k}^{K} x_j;$$

and the total surplus is:

$$\sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) \sum_{j=k}^{K} v_j x_j.$$

3   The Limits of Discrimination

We now turn to the characterization of the set of welfare outcomes which can arise under third degree price discrimination. We will demonstrate that the welfare bounds described in the Introduction are tight, using the special geometry that comes from grouping the markets by corresponding optimal prices. This geometry is the subject of Section 3.1, and we use it to prove our main result in Section 3.2. Though the argument we present is non-constructive, there are in fact many ways of constructing segmentations that achieve the bounds, and we give examples of constructions in Section 3.3. We will also provide a tight characterization of limits on output in Section 3.4.

3.1   Extremal Markets

Our first result is a linear algebraic characterization of the set X_k of markets where price v_k is optimal. Write $\mathcal{V}$ for the set of non-empty subsets of $V = \{v_1, ..., v_K\}$. For every support set $S \in \mathcal{V}$, we define a market $x^S \in X$, with the properties that: (i) no consumer has a valuation outside the set S; and (ii) the monopolist is indifferent between charging any price inside the set S. Writing $\mathcal{V}_k$ for the set of subsets of V containing v_k and letting $S \in \mathcal{V}_k$, we define the market x^S by the *indifference conditions* that if $v_i \in S$, then:

$$v_i \sum_{j=i}^{K} x_j^S = v_k \sum_{j=k}^{K} x_j^S; \tag{7}$$

and by the *inclusiveness condition* that:

$$\sum_{\{j \mid v_j \in S\}} x_j^S = 1. \tag{8}$$

Thus, we have K equations in K unknowns with a unique solution in X. Writing $\min S$ for the smallest element of S, (7) implies that profits from any price in the support must be $\min S$ and thus

we must have:

$$v_i \sum_{j=i}^{K} x_j^S = \min S \tag{9}$$

for all $v_i \in S$. Writing $\mu(v_i, S)$ for the smallest element of S which is strictly greater than v_i, we must have:

$$x_i^S \triangleq \begin{cases} 0, & \text{if} & v_i \notin S; \\ \min S \left(\frac{1}{v_i} - \frac{1}{\mu(v_i,S)} \right), & \text{if} & v_i \in S \text{ and } v_i \neq \max S; \\ \frac{\min S}{\max S}, & \text{if} & v_i = \max S. \end{cases} \tag{10}$$

An implication is that in every market x^S, the discrete version of the virtual utility is zero for every element $v_i \in S$ except $\max S$:

$$v_i \in S \setminus \{\max S\} \Leftrightarrow v_i - (\mu(v_i, S) - v_i) \frac{1 - \sum_{k \leq i} x_k^S}{x_i^S} = 0.$$

A remarkable and useful property of the set X_k is that every $x \in X_k$ can be represented as a convex combinations of the markets $\{x^S\}_{S \in \mathcal{V}_k}$. For this reason, we will refer to any market of the form x^S for some non-empty S as an *extremal market*.

Lemma 1 (Extremal Markets)

X_k *is equal to the convex hull of* $\{x^S\}_{S \in \mathcal{V}_k}$.

Proof. The inclusion of the convex hull of $\{x^S\}_{S \in \mathcal{V}_k}$ in X_k is immediate, since by definition $x^S \in X_k$ for any $S \in \mathcal{V}_k$, and X_k is convex, being the intersection of the convex simplex and the half spaces in which price v_k is better than price v_i for all $i \neq k$.

Moreover, X_k is finite-dimensional and compact, as it is the intersection of closed sets with the compact simplex. Thus, by the Minkowski-Caratheodory Theorem (see Simon (2011), Theorem 8.11) X_k is equal to the convex hull of its extreme points. We will show that every extreme point of X_k is equal to x^S for some $S \in \mathcal{V}_k$. First observe that if v_i is an optimal price for market x, then $x_i > 0$. Otherwise the monopolist would want to deviate to a higher price if $\sum_{j=i+1}^{K} x_j > 0$ or a lower price if this quantity is zero, either of which contradicts the optimality of v_i.

Now, the set X_k is characterized by the linear constraints that for any $x \in X_k$:

$$\sum_{i=1}^{K} x_i = 1,$$

the non-negativity constraints:

$$x_i \geq 0, \text{ for all } i,$$

and the optimality (of price v_k) constraints:

$$v_k \sum_{j=k}^{K} x_j \geq v_i \sum_{j=i}^{K} x_j \text{ for } i \neq k.$$

Any extreme point of X_k must lie at the intersection of at least K of these constraints (see Simon (2011), Proposition 15.2). One binding constraint is always $\sum_{i=1}^{K} x_i = 1$, and since v_k is an optimal price, the non-negativity constraint $x_k \geq 0$ is always slack. Thus, there must be at least $K - 1$ binding optimality and non-negativity constraints for $i \neq k$.

But as we have argued, we cannot have both the optimality and non-negativity constraints bind for a given i, so for each $i \neq k$ precisely one of these is binding. This profile of constraints defines x^S, where S is the set valuations for which the optimality constraint binds. ∎

The following is an alternative and intuitive explanation as to why any $\widehat{x} \in X_k \setminus \left\{ x^S \,|\, S \in \mathcal{V}_k \right\}$ cannot be an extreme point of X_k. Let $\widehat{S}$ be the support of $\widehat{x}$ and consider moving from $\widehat{x}$ either towards $x^{\widehat{S}}$ or in the opposite direction. Price v_k will continue to be optimal, because complete indifference at $x^{\widehat{S}}$ to all prices in the support means moving in either direction will not change optimal prices. Also, for small perturbations, we will remain in the simplex, since $\widehat{x}$ and $x^{\widehat{S}}$ have the same support by construction. Since we can move in opposite directions and remain within X_k, it follows that $\widehat{x}$ is not an extreme point of X_k.

We illustrate in Figure 3 the extremal markets for X_2 in the probability simplex of Example 1. Since the uniform monopoly price was $v^* = 2$, the extremal markets x^S corresponding to $S \in \mathcal{V}^*$ are $x^{\{2\}}, x^{\{1,2\}}, x^{\{2,3\}}$, and $x^{\{1,2,3\}}$. We will refer to a segmentation consisting only of extremal markets as an *extremal segmentation* and a segmentation consisting only of extremal markets in X^* as a *uniform profit preserving extremal segmentation*. It is a direct consequence of Lemma 1 that a uniform profit preserving extremal segmentation exists.

An example of a uniform profit preserving extremal segmentation for Example 1 is given below.[4]

	value 1	value 2	value 3	weight
market $\{1,2,3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
market $\{2,3\}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$
market $\{2\}$	0	1	0	$\frac{1}{6}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$\tag{11}$$

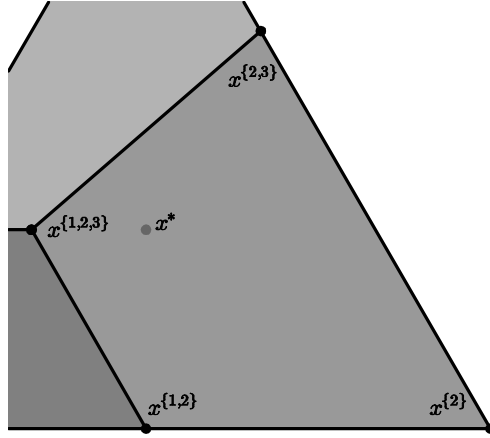[4]This segmentation is the output of the "greedy" algorithm we will describe in Section 3.3.

Figure 3: Uniform Profit Preserving Extremal Markets for $x^* = (1/3, 1/3, 1/3)$.

An example of an extremal segmentation that is not uniform profit preserving is the perfect price discriminating segmentation described in table (6), since this necessarily uses markets $x^{\{v_k\}}$ with $v_k \neq v^*$, which are not in X^*.

3.2 Limits of Welfare

For a given market x, we define the minimum pricing rule $\underline{\phi}(x)$ to deterministically charge $\min \mathbf{supp}\, x$ and, similarly, we define the maximum pricing rule $\overline{\phi}(x)$ to deterministically charge $\max \mathbf{supp}\, x$. The minimum pricing rule always implies an efficient allocation in the market x and the maximum pricing rule always implies an allocation in the market x where there is zero consumer surplus. When combined with extremal segmentations, the minimum and maximum pricing rules are especially powerful:

Proposition 1 (Extremal Segmentations)

In every extremal segmentation, minimum and maximum pricing rules are optimal. Total surplus is w^* under the minimum pricing rule, and consumer surplus is zero under the maximum pricing rule. If the extremal segmentation is uniform profit preserving, then producer surplus is π^* under every optimal pricing rule, and consumer surplus is $w^* - \pi^*$ under the minimum pricing rule.

Proof. By construction of the extremal markets, any price in S is an optimal price in market x^S. This implies that minimum and maximum pricing rules are both optimal. Under the minimum pricing rule, all consumers purchase the good, so the efficient total surplus is attained. Consumer

surplus is always zero under the maximum pricing strategy because consumers who purchase pay exactly their value. If the extremal segmentation is uniform profit preserving, setting the price equal to v^* in every segment is optimal, so producer surplus must be exactly π^* under any optimal pricing rule. Combining this with the fact that total surplus is w^* under minimum pricing, we conclude that consumer surplus is $w^* - \pi^*$. ∎

This Proposition implies that with a uniform profit preserving extremal segmentation, aggregate consumer surplus must be weakly greater under the minimum pricing rule and weakly lower (in particular zero) under the maximum pricing rule. In fact, the same predictions hold conditional on each possible valuation of the consumer. With the minimum pricing rule $\underline{\phi}(x)$, we observe that all efficient trades are realized (as opposed to only those with a value equal or above the uniform price $v_i \geq v^*$), and by construction of the minimum pricing rule $\underline{\phi}(x)$, all sales are realized at prices below or equal to v^*. Thus, conditional on each valuation, consumer surplus must increase. As for the maximum pricing rule $\overline{\phi}(x)$, only the buyer with the highest value in the segment x purchases the product but has to pay exactly his valuation. Hence, the expected net utility conditional on a purchase is zero, and so is the expected net utility without a purchase. Thus, all valuation types are weakly worse off relative to the uniform price in the aggregate market.

Combining the previous analysis, we have our main result on the welfare limits of price discrimination:

Theorem 1 (Surplus Triangle)

There exists a segmentation and optimal pricing rule with consumer surplus u and producer surplus π if and only if (u, π) satisfy $u \geq 0$, $\pi \geq \pi^*$ and $\pi + u \leq w^*$.

Proof. First we argue necessity. That consumer surplus must be non-negative and that total surplus is bounded above by w^* follows directly from the definitions. For producer surplus, a price offered under an optimal pricing rule must generate weakly greater revenue than would v^*. Summing this inequality over all markets in the segmentation and over all prices induced by the rule yields the desired result:

$$
\sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) v_k \sum_{j=k}^{K} x_j \geq \sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) v^* \sum_{j=i^*}^{K} x_j
$$
$$
= v^* \sum_{j=i^*}^{K} x_j^*
$$
$$
= \pi^*.
$$

For sufficiency, a direct consequence of Lemma 1 is that a uniform profit preserving extremal segmentation σ always exists. By Proposition 1, the minimum and maximum pricing rules under this segmentation achieve the surplus pairs $(w^* - \pi^*, \pi^*)$ and $(0, \pi^*)$ respectively. The segmentation:

$$\sigma'(x) = \begin{cases} x_k^*, & \text{if} \quad x = x^{\{v_k\}}; \\ 0, & \text{if} \quad \text{otherwise}; \end{cases}$$

together with any optimal pricing rule attains the surplus pair $(0, w^*)$, in which the seller receives the entire surplus. It follows that the three vertices of the surplus triangle can be attained. But now any point in the surplus triangle can be written as a convex combination:

$$\alpha(0, w^*) + (1 - \alpha)[\beta(w^* - \pi^*, \pi^*) + (1 - \beta)(0, \pi^*)],$$

with $\alpha, \beta \in [0, 1]$. The extremal segmentation:

$$\sigma''(x) = \alpha\sigma(x) + (1 - \alpha)\sigma'(x)$$

together with the optimal pricing rule:

$$\phi_k(x) = \beta\underline{\phi}_k(x) + (1 - \beta)\overline{\phi}_k(x).$$

achieves the desired welfare outcome. ∎

Note that the property that the producer cannot be driven below his uniform monopoly profits reflects the general result that information has positive value in single-person decision problems. In oligopolistic settings, however, partial information can drive sellers below their profits with no information (see, e.g., Bergemann and Morris (2013b) for a different setting where this occurs).

3.3   Constructive Approaches and Direct Segmentations

Our results thus far establish the existence of segmentations that achieve extreme welfare outcomes, based on the fact that any market can be decomposed as a convex combination of extremal markets in X^*. In general, there will be many such segmentations. One reason is that there may be many subsets of extremal markets in X^* whose convex hulls contain x^*, and therefore many uniform profit preserving extremal segmentations with different supports. A second reason is that extremal segmentations are just one kind of segmentation; welfare bounds can also be attained with segments that are not extremal. We will briefly describe two constructive algorithms in order to gives a sense

of this multiplicity, to give some intuition for what critical segmentations will end up looking like, and to make some additional observations about the number of segments required.

We start with a construction of a uniform profit preserving extremal segmentation, through the following "greedy" procedure. First, pack as many consumers as possible into the market $x^{\mathbf{supp}\, x^*}$, i.e., the extremal market in which the monopolist is indifferent between charging all prices in the support of x^*. At some point, we will run out of mass of some valuation in $\mathbf{supp}\, x^*$, and define the residual market to be the distribution of all remaining consumers. We then proceed inductively with a segment that puts as much mass as possible on the extremal market corresponding to all remaining valuations in the residual market; and so on. At each step, we eliminate one valuation from the residual market, so the process will necessarily terminate after at most K rounds.

More formally, let $Y^S = \{x \in X \,|\, \mathbf{supp}\, x \subseteq S\}$, which is the subset of the simplex with support in S. It is a fact that the extreme points of $Y^S \cap X^*$ are simply those $x^{S'}$ such that $i^* \in S'$ and $S' \subseteq S$. Also, the only extreme point of $Y^S \cap X^*$ that is in the relative interior of Y^S is x^S,[5] since any point on the relative boundary of Y^S has $x_k = 0$ for some $k \in S$. With these observations in hand, we can define the greedy algorithm as the following iterative procedure. At the end of iteration $l \geq 0$, the "residual" market of valuations not yet assigned to a segment is x^l, with $x^0 = x^*$, and the support of this residual is defined to be $S_l = \mathbf{supp}\, x^l$.

We now describe what happens at iteration l, taking as inputs the residual and support from the previous iteration $\left(x^{l-1}, S_{l-1}\right)$. If $x^{l-1} = x^{S_{l-1}}$, then we define $\alpha^l = 1$. Otherwise, we find the unique $\widehat{t}$ for which the market $z\left(t\right) = x^{S_{l-1}} + t\left(x^{l-1} - x^{S_{l-1}}\right)$ is on the relative boundary of $Y^{S_{l-1}}$, and define $x^l = z\left(\widehat{t}\right)$ and α^l by $x^{l-1} = \alpha^l x^{S_{l-1}} + \left(1 - \alpha^l\right) x^l$. Note that moving away from $x^{S_{l-1}}$ will never take us out of X^*, since this transformation preserves the set of optimal prices. In particular, for any $v_i \in S_l$, the loss in revenue from pricing at v_i instead of v^* is:

$$v^* \sum_{j=i^*}^{K} z_j\left(t\right) - v_i \sum_{j=i}^{K} z_j\left(t\right) = t\left(v^* \sum_{j=i^*}^{K} x_j^{l-1} - v_i \sum_{j=i}^{K} x_j^{l-1}\right),$$

which is non-negative as long as $t \geq 0$. Also observe that this transformation preserves the fact that $\sum_{j=1}^{K} z_j\left(t\right) = 1$. Finally, since $x^{l-1} \neq x^{S_{l-1}}$, having $x_i^{l-1} \geq x_i^{S_{l-1}}$ for all i would violate probabilities in x^{l-1} summing to one. Therefore there is at least one i for which $x_i^{l-1} < x_i^{S_{l-1}}$, so that $z_i\left(t\right)$ eventually hits zero. The desired $\widehat{t}$ is $\inf\{t|t \geq 0, z_i\left(t\right) < 0 \text{ for some } i\}$.

---

[5]The relative interior of Y^S is the set of $y \in Y^S$ such that for all $z_1, z_2 \in Y^S$, $\{y + \varepsilon z, y - \varepsilon z\} \subseteq Y^S$ for ε sufficiently small, where $z = z_1 - z_2$.
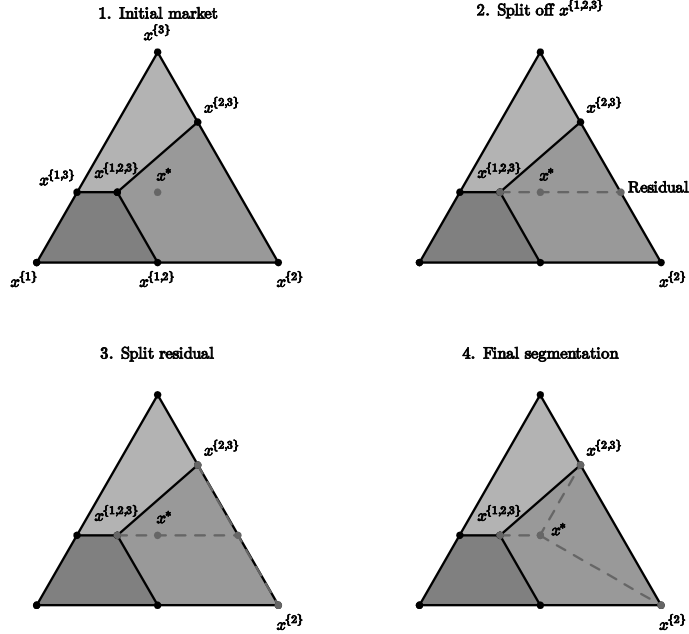
Figure 4: The Greedy Segmentation

Now inductively define $S^l = \mathbf{supp}\, x^l$, which is a strict subset of S^{l-1} since x^l is on the relative boundary of $Y^{S_{l-1}}$. The inductive hypothesis is that:

$$x^* = \sum_{j=0}^{l} \alpha^j x^{S_j} \prod_{i=0}^{j-1} \left(1 - \alpha^i\right) + x^l \prod_{i=0}^{l} \left(1 - \alpha^i\right),$$

which by is trivially satisfied for the base case $l = 0$ with the convention that $\alpha^0 = 0$. Our choice of α^l guarantees that if the inductive hypothesis holds at $l - 1$, it continues to hold at l as well. The algorithm terminates at iteration $L + 1$ when $x^L = x^{S_L}$, which certainly has to be the case when $|S_L| = 1$, and we define the segmentation to have support equal to $\left\{x^{S_l}\right\}_{l=0}^{L}$ with $\sigma\left(x^{S_l}\right) = \alpha^{l+1} \prod_{j=0}^{l} \left(1 - \alpha^j\right)$.

For Example 1, this decomposition is visually depicted in Figure 4, and results in the following segmentation which we previously used to illustrate uniform profit preserving extreme segmentations in (11):

	value 1	value 2	value 3	weight	$\underline{\phi}(x)$	$\overline{\phi}(x)$
market $\{1, 2, 3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	1	3
market $\{2, 3\}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	2	3
market $\{2\}$	0	1	0	$\frac{1}{6}$	2	2
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1		

This algorithm incidentally establishes constructively that at most K segments are required to attain points on the bottom of the welfare triangle, and thus at most $2K$ segments to attain all points in the welfare triangle.

In the greedy segmentation of Example 1, under either the minimum or the maximum pricing rule, there are multiple segments in which the same price is charged. For example, if we focus on maximizing consumer surplus, then the monopolist is to charge price 2 when the segment is $x^{\{2,3\}}$ or when the segment is $x^{\{2\}}$. Note that from the monopolist's point of view, he would also be happy to charge price 2 if we just told him that the market was one of $x^{\{2,3\}}$ and $x^{\{2\}}$, but we did not specify which one. The reason is that price 2 is also optimal in the "merged" market:

$$\frac{\sigma(x^{\{2,3\}})}{\sigma(x^{\{2,3\}}) + \sigma(x^{\{2\}})} x^{\{2,3\}} + \frac{\sigma\left(x^{\{2\}}\right)}{\sigma(x^{\{2,3\}}) + \sigma(x^{\{2\}})} x^{\{2\}}.$$

Given this observation, a natural class of segmentations (and associated pricing rules) are those in which any given price is charged in at most one segment. Formally, we define a *direct segmentation* σ to be one that has support on at most K markets, indexed by $k \in \{1, .., K\}$ such that $x^k \in X_k$. In other words, price v_k is optimal on its corresponding segment x^k. The *direct pricing rule* is the rule that puts probability one on price v_k being charged in market x^k, i.e., $\phi_k(x^k) = 1$. This notation is in contrast to the extremal markets where the upper case superscript S in x^S referred to the support of the distribution. Here, the lower case superscript k in x^k refers to price v_k charged in the direct segment x^k. By construction, the direct pricing rule is optimal for direct segmentations, and whenever we refer to a direct segmentation in the subsequent discussion, it is assumed that the monopolist will use direct pricing.

Extremal segmentations and direct segmentations are both rich enough classes to achieve any welfare outcome. The reason is that the welfare outcome is completely determined by the joint distribution over prices and valuations that is induced by the segmentation and the pricing rule, and both classes of segmentations can achieve any such joint distribution. This result is formalized in the following proposition:

Proposition 2 (Extremal and Direct Segmentations)
For any segmentation and optimal pricing rule (σ, ϕ), there exist: (i) an extremal segmentation and an optimal pricing strategy (σ', ϕ') and (ii) a direct segmentation σ'' (and associated direct pricing strategy ϕ'') that achieve the same joint distribution over valuations and prices. As such, they achieve the same producer surplus, consumer surplus, total surplus, and output.

Proof. To find an extremal segmentation, each market $x \in \mathbf{supp}\,\sigma$ can itself be decomposed using extremal markets with a segmentation σ_x, using only those indifference sets S which contain $\mathbf{supp}\,\phi(x)$. The extremal segmentation of (σ, ϕ) is then defined by:

$$\sigma'(x^S) \triangleq \sum_{x \in \mathbf{supp}\,\sigma} \sigma(x)\sigma_x(x^S),$$

and the corresponding pricing rule is:

$$\phi'_k(x^S) \triangleq \frac{1}{\sigma'(x^S)} \sum_{x \in \mathbf{supp}\,\sigma} \sigma(x)\sigma_x(x^S)\phi_k(x).$$

Similarly, the direct segmentation σ'' can be defined by:

$$\sigma''(x^k) \triangleq \sum_{x \in \mathbf{supp}\,\sigma} \sigma(x)\phi_k(x),$$

and therefore:

$$x^k \triangleq \frac{1}{\sigma''(x^k)} \sum_{x \in \mathbf{supp}\,\sigma} \sigma(x)\phi_k(x) \cdot x$$

yields the corresponding composition of each direct segment x^k. ∎

As an example, the direct segmentation corresponding to the consumer surplus maximizing greedy extremal segmentation of Example 1 is:

	value 1	value 2	value 3	price	weight
Market 1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	1	$\frac{2}{3}$
Market 2	0	$\frac{2}{3}$	$\frac{1}{3}$	2	$\frac{1}{3}$
total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		1

(12)

where the market for price 3 is degenerate. Note that Market 1 is extremal but Market 2 is not. This direct segmentation is visually represented in Figure 4.2 and is realized after the first step of the greedy algorithm, namely after splitting $x^{\{1,2,3\}}$ from the aggregate market. This example illustrates the observation that, while any surplus pair can be achieved by either a direct segmentation or an extremal segmentation, it is generally not possible to attain any surplus pair with a segmentation that is *both* extremal *and* direct. To see why there is not another segmentation which is both extremal and direct corresponding this outcome, observe that in Example 1, any extremal segmentation must use at least three segments, while any direct segmentation that attains maximum consumer surplus must use at most two segments.

Direct segmentations are a convenient tool for constructing some alternative and intuitive segmentations that attain the welfare bounds. Let us give a formal description of the first segmentation described in the Introduction that attains maximum consumer surplus. For each $k \leq i^*$, let market x^k have the features that: (i) the lowest valuation in the support is v_k; (ii) all values of v_{k+1} and above appear in the same relative proportion as in the aggregate population:

$$x_i^k \triangleq \begin{cases} 0, & \text{if } i < k, \\ 1 - \gamma_k \displaystyle\sum_{i=k+1}^{K} x_i^*, & \text{if } i = k, \\ \gamma_k x_i^*, & \text{if } i > k, \end{cases} \tag{13}$$

where $\gamma_k \in [0,1]$ uniquely solves:

$$v_k \left(x_k^* + \gamma_k \left(\sum_{i=k+1}^{K} x_i^* \right) \right) = \gamma_k v^* \left(\sum_{i=i^*}^{K} x_i^* \right).$$

By construction of the above equality, both v_k and v^* are optimal prices for segment x^k. We can always construct a segmentation of the aggregate market x^* that uses only $\left(x^k \right)_{k=1}^{i^*}$. We establish the construction inductively, letting:

$$\sigma \left(x^1 \right) \triangleq \frac{x_1^*}{x_1^1}, \tag{14}$$

and:

$$\sigma \left(x^k \right) \triangleq \frac{x_k^* - \displaystyle\sum_{i=1}^{k-1} \sigma \left(x^i \right) x_k^i}{x_k^k}. \tag{15}$$

We can verify that this segmentation generates maximum consumer surplus by charging in segment x^k the price v_k. The direct pricing rule is optimal and gives rise to an efficient allocation, and because the monopolist is always indifferent to charging v^*, producer surplus is π^*.

Direct segmentations correspond to direct mechanisms in mechanism design. They are minimally informative in the sense that among all information structures under which a given joint distribution over prices and values can arise, the information structure in the direct mechanism is the least informative according to the ranking of Blackwell (1951). While extremal segmentations have been a key tool in this setting, in many related applications (such as those in Kamenica and Gentzkow (2011) for the single player case and in Bergemann and Morris (2013b) and Bergemann, Brooks, and Morris (2013a) for the many player case), it is more convenient to work with direct segmentations.

3.4   Limits of Output

While our focus so far has been on welfare outcomes, we can also report tight bounds on how output can vary across segmentations and optimal price rules. For a segmentation σ and pricing rule ϕ, output is given by:

$$\sum_{x \in \mathbf{supp}(\sigma)} \sigma(x) \sum_{k=1}^{K} \phi_k(x) \sum_{j=k}^{K} x_j.$$

An upper bound on output among all segmentations and optimal pricing rules is selling to all consumers, and this bound is achieved by any efficient segmentation.[6] Characterizing the lowest possible output is more subtle. We will first establish a lower bound and then show that it can be attained.

To establish a lower bound on output, recall that the producer must get at least the uniform monopoly profits π^*, and this requires some positive output. The smallest output delivering π^* will arise in a *conditionally efficient* allocation where the good is always sold to those with the highest valuation. In our discrete valuations model, there must be a critical valuation $v_{\underline{i}}$ such that the good is always sold to all consumers with valuations above $v_{\underline{i}}$ and never sold to consumers with valuations below $v_{\underline{i}}$. Thus letting $\underline{i}$ and $\underline{\beta} \in (0,1]$ uniquely solve:

$$v_{\underline{i}} \underline{\beta} x_{\underline{i}}^* + \sum_{j=\underline{i}+1}^{K} v_j x_j^* = \pi^*, \tag{16}$$

we obtain a lower bound $\underline{q}$ given by:

$$\underline{q} \triangleq \underline{\beta} x_{\underline{i}}^* + \sum_{j=\underline{i}+1}^{K} x_j^*. \tag{17}$$

The additional variable $\underline{\beta} \in (0,1]$ describes the proportion of buyers at the threshold value $v_{\underline{i}}$ who must purchase the good to achieve equality (16) in this discrete setting.

With respect to the earlier Example 1, we have $\underline{q} = \underline{\beta} = \frac{1}{2}$ and $\underline{i} = i^* = 2$. In fact, the greedy segmentation for this Example displayed in (11) in combination with maximum pricing strategy supports the conditionally efficient allocation with $\underline{q} = \underline{\beta} = \frac{1}{2}$; all of the consumers with valuation 3 and exactly half of the consumers with valuation 2 purchase the good. However, it need not be the case more generally that every uniform profit preserving extremal segmentation delivers a conditionally efficient outcome under the maximum pricing strategy. This is illustrated in the following:

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[6]If we had a positive constant marginal cost and some consumers had valuations below the marginal cost, then the producer could never be induced to sell to those consumers, so it would still be the case that the efficient output would be an upper bound that was attained.

Example 2 (Three Values without Uniform Probability)

The setup is the same as in Example 1, except that now the proportion of valuation 1 consumers is $x_1^* = \frac{3}{5}$, and the proportions of valuations 2 and 3 are $x_2^* = x_3^* = \frac{1}{5}$. The monopoly price is 1, $\pi^* = 1$, $u^* = \frac{3}{5}$, and $w^* = \frac{8}{5}$. The minimum output is $\underline{q} = \frac{2}{5}$, $\underline{i} = 2$, and $\underline{\beta} = 1$.

Two alternative uniform profit preserving extremal segmentations are displayed below in (18) and graphically represented in Figure 5. However, only Segmentation 2 leads to a conditionally efficient allocation with the maximum pricing rule, in which case only prices 2 and 3 are used. In Segmentation 1, the price 1 will sometimes be charged under maximum pricing, thus creating conditional inefficiency.

	value 1	value 2	value 3	price	weight
Segmentation 1					
market $\{1,2,3\}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	3	$\frac{3}{5}$
market $\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{1}{5}$
market $\{1\}$	1	0	0	1	$\frac{1}{5}$
Segmentation 2					
market $\{1,3\}$	$\frac{2}{3}$	0	$\frac{1}{3}$	3	$\frac{3}{5}$
market $\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{2}{5}$
total	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$		1

(18)

In spite of this apparent complication, it turns out that for any aggregate market, it is always possible to find a uniform profit preserving extremal segmentation that, together with the maximum pricing rule, results in a conditionally efficient outcome. Our approach is analogous to that employed for Lemma 1 and Proposition 1. We will show that when looking for a uniform profit preserving extremal segmentation, it is without loss of generality to look at a particular subset of extremal markets in X^*. The use of these markets will then imply that the outcome under the maximum pricing rule is conditionally efficient. In particular, we divide the simplex into regions $X_{k,l}$ for $l = k, \ldots, K$, which is the set of markets in X_k for which v_l is the lowest valuation receiving the good in the minimum quantity conditionally efficient outcome. In other words:

$$X_{k,l} \triangleq \left\{ x \in X_k \,\middle|\, \sum_{j=l+1}^{K} v_j x_j \leq v_k \sum_{j=k}^{K} x_j \leq \sum_{j=l}^{K} v_j x_j \right\}.$$

We will refer to these two additional constraints in the definition of $X_{k,l}$ as the lower and upper output constraints (LC and UC, respectively). Note that with $l = K$, the left-hand side of the lower

constraint is zero, and with $l = k$, the right-hand side of the upper constraint is necessarily at least π^*, so

$$\bigcup_{l=1}^{K} X_{k,l} = \left\{ x \in X_k \,\middle|\, 0 \le v_k \sum_{j=k}^{K} x_j \le \sum_{j=k}^{K} v_j x_j \right\} = X_k.$$

Let $\mathcal{V}_{k,l}$ be the subsets of values:

$$\mathcal{V}_{k,l} \triangleq \left\{ S \in \mathcal{V}_k \,\middle|\, \begin{array}{l} S \cap \{v_l, \dots, v_K\} \neq \emptyset, \\ |S \cap \{v_{l+1}, \dots, v_K\}| \le 1 \end{array} \right\}.$$

We have the following linear algebraic characterization of the sets $X_{k,l}$ in terms of extremal markets with supports in $\mathcal{V}_{k,l}$, which mirrors Lemma 1:

Lemma 2 (Extremal Markets with Output Constraints)

$X_{k,l}$ *is the convex hull of* $\left\{ x^S \right\}_{S \in \mathcal{V}_{k,l}}$.

Now, by the definition of lowest valuation $v_{\underline{i}}$ in the support of the conditionally efficient allocation, it must be that $x^* \in X_{i^*, \underline{i}}$, and taking extremal segmentations with support in $X_{i^*, \underline{i}}$, we can verify that the outcome under the maximum pricing rule is conditionally efficient:

Proposition 3 (Quantity Minimizing Segmentation)

For every market, there exists a uniform profit preserving extremal segmentation such that the allocation under the maximum pricing rule is conditionally efficient. As a result, producer surplus is π^*, *consumer surplus is* 0, *and output is* $\underline{q}$.

Proof. Since the $X_{i^*, l}$ cover X_{i^*}, $x^* \in X_{i^*, l}$ for some l. As such, x^* can be written as a convex combination of extreme points of $X_{i^*, l}$. For every such market, $S \cap \{v_l, \dots, v_K\} \neq \emptyset$ implies only consumers with valuations weakly greater than v_l receive the good, and $|S \cap \{v_{l+1}, \dots, v_K\}| \le 1$ implies that all consumers with valuations strictly greater than v_l purchase the good. For if not, there must be some market in which consumers with valuation $v_i > v_l$ have strictly positive mass but the price charged is $v_{i'} > v_i$. But this can only happen if $\{v_i, v_{i'}\} \subseteq S$, a contradiction. As a result, the allocation is conditionally efficient, but because the segmentation is uniform profit preserving extremal, producer surplus is π^*, and under the maximum pricing rule consumer surplus is 0. Hence, output must be $\underline{q}$. ∎

For any aggregate market with three possible valuations, the geometry is quite simple when the optimal uniform price is $v^* = 2$ or $v^* = 3$. In those cases, every extremal segmentation results in a
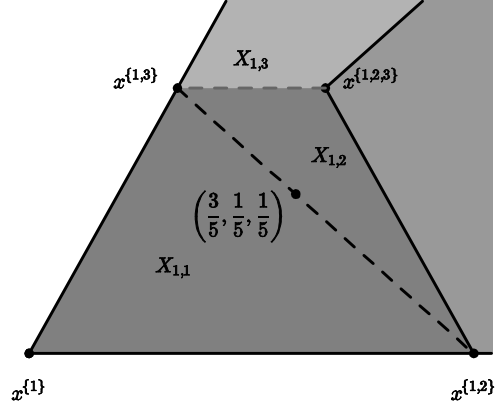
Figure 5: Multiple Extremal Segmentations

conditionally efficient outcome whenever the maximum pricing rule is used: For $v^* = 3$, the maximum pricing rule always induces a price of 3, and when $v^* = 2$, one can verify from the definitions that $X_{2,2} = X_2$. However, when $v^* = 1$, as in Example 2, then the sets $X_{1,1}$ and $X_{1,2}$ have disjoint and non-empty interiors. $X_{1,3}$ is just the line segment connecting $x^{\{1,3\}}$ and $x^{\{1,2,3\}}$. This is illustrated in detail in Figure 5. Here, the market $x^* = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$ lies on the boundary of $X_{1,1}$ and $X_{1,2}$, and it also lies in the convex hull of $\left\{x^{\{1\}}, x^{\{1,2\}}, x^{\{1,2,3\}}\right\}$. The latter sets of extremal markets appear in the segmentation generated by the greedy algorithm.

4   A Continuum of Valuations

Until now, we have considered markets that have a finite and fixed support of valuations. The finite structure has allowed us to make simple geometric arguments to characterize the limits of price discrimination. Nonetheless, our results generalize in a straightforward manner to environments with infinitely many valuations. In this Section, we give a simple convergence argument showing why this is the case and report some examples of critical segmentations for continuous demand curves.

For the present analysis, we redefine a market to be an element $x \in X = \Delta\left([0, \overline{v}]\right)$, which is the set of Borel probability measures on the interval $[0, \overline{v}]$, and we endow the set X with the weak-* topology. We will write $x(Y)$ for the measure of a set $Y \in \mathcal{B}\left([0, \overline{v}]\right)$, which is the collection of Borel subsets of $[0, \overline{v}]$. As before, we fix an aggregate market $x^* \in X$, and let v^* denote a uniform monopoly

price that solves:

$$v^* \in \arg\max_{v \in [0,\overline{v}]} v\, x^* \left([v,\overline{v}]\right).$$

Note that for any Borel measure, such a maximizer exists due to the fact that $x^*\left([v,\overline{v}]\right)$ is monotonically decreasing and continuous from the left.

The set X_v is defined to be the set of markets in which v^* is a maximizer of $v\,x\left([v,\overline{v}]\right)$. We let $\overline{X}_v$ denote the set of markets in X_v such that the seller is indifferent between setting any price in the support, i.e.:

$$\overline{X}_v = \left\{ x \in X_v \,|\, v'\, x\left([v',\overline{v}]\right) = \min\mathbf{supp}\,x, \quad \forall v' \in \mathbf{supp}\,x \right\}.$$

We write $\widehat{X}_v$ for the subset of $\overline{X}_v$ with finite support. A preliminary result asserts that a convergent sequence of measures in $\widehat{X}_v$ must converge to an element of $\overline{X}_v$.

Lemma 3 (Closure)

$cl\,\widehat{X}_v \subseteq \overline{X}_v$.

In fact, the closure of $\widehat{X}_v$ is equal to $\overline{X}_v$, but the weaker property is sufficient for our goals. Lemma 3 should not be viewed as a continuous analogue of Lemma 1, but rather as the "glue" that binds the discrete characterization of extremal markets to the continuous characterization. Extremal markets in the continuous case are precisely the elements of $\overline{X}_v$ for some v. But in order to find an extremal segmentation in the proof of Theorem **??**, which appears in the Appendix, we will take limits of extremal segmentations of finite approximations to x^*, and convergence is guaranteed by Lemma 3.

To that end, we redefine a *segmentation* of the market x^* to be an element $\sigma \in \Sigma$, where:

$$\Sigma = \left\{ \sigma \in \Delta\left(X\right) \,\middle|\, \int_{x' \in X} x'\left(Y\right) \sigma\left(dx'\right) = x^*\left(Y\right) \ \forall Y \in \mathcal{B}\left([0,\overline{v}]\right) \right\}.$$

A segmentation σ is *uniform profit preserving extremal* if its support is contained in $\overline{X}_{v^*}$. A *pricing rule* is a mapping $\phi: \mathbf{supp}\,\sigma \to X$, and the pricing rule is *optimal* if for all $x \in \mathbf{supp}\,\sigma$, $\mathbf{supp}\,\phi\left(x\right) \subseteq \arg\max_{v \in [0,\overline{v}]} v\, x\left([v,\overline{v}]\right)$. The *minimum pricing rule* and *maximum pricing rule* put probability one on the minimum and maximum of $\mathbf{supp}\,x$ for all $x \in \mathbf{supp}\,\sigma$, respectively. Consumer surplus is:

$$\int_{x' \in X} \int_{v' \in [0,\overline{v}]} \int_{v'' \in [v',\overline{v}]} \left(v'' - v'\right) x'\left(dv''\right) \phi\left(dv'\right) \sigma\left(dx'\right);$$

producer surplus is:

$$\int_{x' \in X} \int_{v' \in [0,\overline{v}]} v'\, x'\left([v',\overline{v}]\right) \phi\left(dv'\right) \sigma\left(dx'\right);$$

and total surplus is:

$$\int_{x' \in X} \int_{v' \in [0, \bar{v}]} \int_{v'' \in [v', \bar{v}]} v'' \, x' \, (dv'') \, \phi \, (dv') \, \sigma \, (dx') \, .$$

We can then re-establish the earlier Proposition 1 and Theorem 1 for the environment with a continuum of **B**orel measurable values.

Proposition 1B (Extremal Segmentations (with a Continuum of Values))

In every extremal segmentation, minimum and maximum pricing rules are optimal. Total surplus is w^* under the minimum pricing rule, and consumer surplus is zero under the maximum pricing rule. If the extremal segmentation is uniform profit preserving, then producer surplus is π^* under every optimal pricing rule, and consumer surplus is $w^* - \pi^*$ under the minimum pricing rule.

Combining results, we have the following:

Theorem 1B (Surplus Triangle (with a Continuum of Values))

There exists a segmentation and optimal pricing rule with consumer surplus u and producer surplus π if and only if (u, π) satisfy $u \geq 0$, $\pi \geq \pi^*$ and $\pi + u \leq w^*$.

We conclude this Section with an Example of an aggregate market with continuously distributed values for which we can derive explicit segmentations with a convex support for all segmentations.

Example 3 (Unit Interval with Uniform Density)

The valuations of the consumers are uniformly distributed between 0 and 1, so that $x^* \, ([v, 1]) = 1 - v$ for all $v \in [0, 1]$. The uniform monopoly price is $v^* = \frac{1}{2}$, uniform monopoly profits are $\pi^* = \frac{1}{4}$, and the efficient surplus is $w^* = \frac{1}{2}$.

We will construct a uniform profit preserving extremal segmentation σ of this x^*, in which there is a uniform distribution of market segments x_p for $p \in \left[0, \frac{1}{2}\right]$. Each segment x_p has support of the form $[p, z \, (p)]$, and is defined by:

$$x_p \, ([v, 1]) = \begin{cases} 1, & \text{if} \quad v \leq p; \\ \frac{p}{v}, & \text{if} \quad p < v \leq z \, (p) \, ; \\ 0, & \text{if} \quad v > z \, (p) \, ; \end{cases}$$

and the upper boundary point is given by:

$$z \, (p) = \frac{1 + \sqrt{1 - 4p^2}}{2},$$

which is monotonically decreasing and has range $\left[\frac{1}{2}, 1\right]$. Thus, the support sets of the segments can be ordered by the strong set order. By construction, all of the segments are in $X_{\frac{1}{2}}$, and in fact the segmentation arises as the solution to the continuous version of the greedy algorithm constructed in Section 3.3.

Let us briefly verify that σ is in fact a segmentation of the aggregate market. It is sufficient to check that the density of a valuation v integrates to one. For $v \in \left[0, \frac{1}{2}\right]$, the density in market x_p is $\frac{p}{v^2}$ when $p \leq v$ and zero otherwise, so the aggregate density is:

$$\int_{p=0}^{v} 2\frac{p}{v^2} dp = \frac{p^2}{v^2}\bigg|_{p=0}^{v} = 1.$$

If $v \in \left[\frac{1}{2}, 1\right]$, the density in market x_p is $\frac{p}{v^2}$ when $z(p) > v$, and there is a conditional mass point of size $\frac{p}{v}$ in the market x_p such that $z(p) = v$, which is when $p = \sqrt{v(1-v)}$. Note that the probability that the maximum of the support of x_p is less than w is $1 - 2\sqrt{w(1-w)}$, so the density at v is $\frac{2v-1}{\sqrt{v(1-v)}}$. Therefore the aggregate density for every valuation v is equal to:

$$\frac{2v-1}{\sqrt{v(1-v)}}\frac{\sqrt{v(1-v)}}{v} + \int_{p=0}^{\sqrt{v(1-v)}} 2\frac{p}{v^2} dp = \frac{2v-1}{v} + \frac{1-v}{v} = 1.$$

We conclude that σ does in fact segment the aggregate market, as it preserves the aggregate density.

The proof of Theorem 1B establishes the existence of extremal segmentations for general Borel measurable distributions. In Bergemann, Brooks, and Morris (2013b), we establish a related existence result for direct segmentations, stated as Theorem 2 in that paper. In addition, there we show that when we narrow the analysis to aggregate markets with differentiable distribution functions, we can explicitly construct direct segmentations that achieve the extreme welfare outcomes as solutions of differential equations. The resulting segmentations, given in Bergemann, Brooks, and Morris (2013b) as Proposition 3 and 4, mirror those in the finite environment (see our current Proposition 2).

We will illustrate these results with examples of direct segmentations for the uniform environment of Example 3. The consumer surplus maximizing segmentation, as derived there in Proposition 3, leads to an associated distribution function of prices $\overline{H}(p)$ given by:

$$\overline{H}(p) = 1 - \frac{1-p}{1-2p}e^{-\frac{2p}{1-2p}}, \quad \text{for } p \in \left[0, \frac{1}{2}\right]. \tag{19}$$

By contrast, the segmentation of the consumers in the total surplus and output minimizing allocation as described there by Proposition 4 leads to a distribution function of prices given by:

$$\underline{H}(p) = 2p^2 - 1, \quad \text{for } p \in \left[\frac{1}{\sqrt{2}}, 1\right].$$

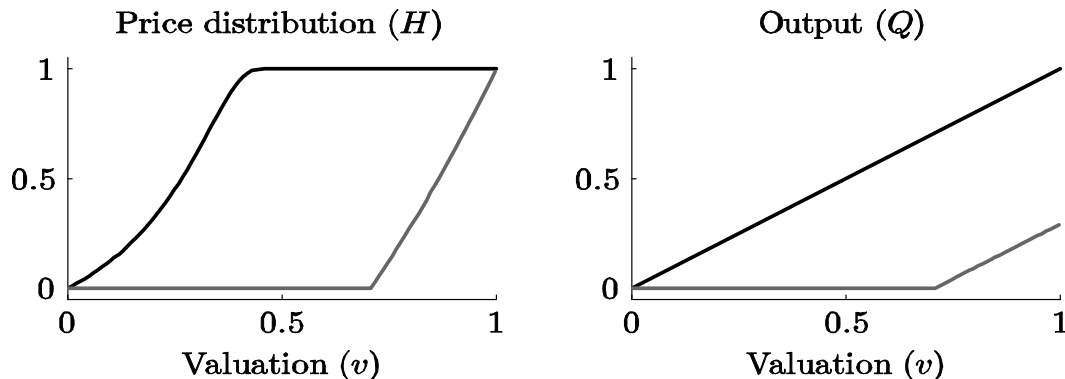Price distribution (H)   Output (Q)

Figure 6: Price and Output Distribution in Direct Surplus Minimizing and Surplus Maximizing Segmentations

The distributions of prices induced by these distinct direct segmentations are displayed in the left panel of Figure 6, where the upper curve represents the consumer surplus maximizing, the lower curve the total surplus minimizing distribution of prices.

The surplus minimizing and maximizing distributions represent optimal pricing policies for distinct segmentations of the *same* aggregate market. Even though they share the same aggregate market, the support sets of prices do not overlap. These distinct price distributions also lead to very different allocations. The surplus maximizing pricing policy generates all efficient sales, and hence the cumulative distribution of sales, $\overline{Q}(v) = v$, exactly replicates the aggregate distribution of consumers' valuations. By contrast, the surplus minimizing distribution truncates sales for values v below $1/\sqrt{2}$. As we described in the current Proposition 2, the allocation is conditionally efficient, and hence $\underline{Q}(v) = v - 1/\sqrt{2}$ for $v \in [1/\sqrt{2}, 1]$, and zero elsewhere. These different patterns of sales are displayed in the right panel of Figure 6, where the upper curve represents the surplus maximizing, the lower curve the surplus minimizing distribution of output.

5 Beyond the Linear Case

We have thus far established the limits of price discrimination in the canonical model of monopoly pricing. The monopoly problem with unit demand may be viewed as special case of a more general class of screening problems, as considered in the seminal papers by Mussa and Rosen (1978) and Maskin and Riley (1984) and often referred to as second degree price discrimination. In these problems, the utility of the consumer, or the cost of the producer, or both, can be non-linear in

the quantity (or quality) of the object. In contrast, our benchmark results only apply to settings in which utility is linear or the monopolist is restricted to selling a single unit to each consumer, so that posted price mechanisms are optimal. Nonetheless, the same welfare question can be posed in the general screening environment: what are the feasible pairs of consumer and producer surplus that can be induced through optimal behavior by the monopolist under some segmentation of the market? While we do not provide a complete answer to this question, we can report general features of how our results change as we move towards more general screening environments.

The limits of price discrimination are characterized by the surplus triangle, which is defined by the participation constraint of the consumer, the uniform price profit lower bound of the producer, and efficient surplus upper bound. In the non-linear case, there are analogous restrictions on consumer surplus and total surplus, and the monopolist must get at least the profit he would obtain with the uniform monopoly menu (rather than just a posted price). As we introduce non-linearity, these bounds can no longer be attained exactly, but the central features of the limits survive as follows:

(i) With a finite set of allocations, the geometric approach, in particular the characterization of the feasible surplus pairs by means of the critical markets, remains valid. A continuum of consumer surplus values can still be supported while maintaining the producer surplus associated with the uniform monopoly menu. Thus, we maintain the "bottom flat" of the surplus set;

(ii) With a continuum of allocations, the "bottom flat" typically disappears. Critical markets where several distinct allocations are simultaneously optimal do not exist anymore. Nonetheless, the entire surplus set remains "fat" in the sense that many levels of consumer surplus are consistent with profit levels strictly above the "uniform menu profit," where the producer offers the same menu to all consumers;

(iii) As the nonlinear environment approaches the linear one, the surplus set of the nonlinear environment continuously approaches the surplus triangle of the linear environment.

We will illustrate these points with simple examples that add a small amount of concavity to the utility function of the consumer, namely the linear quadratic utility function:

$$u_k(q) = v_k q + \varepsilon q(1-q), \quad \varepsilon > 0. \tag{20}$$

As before, we maintain zero marginal cost and let $q \in [0,1]$. The concave model can be interpreted as one of quantity discrimination with a constant marginal cost of production, as in Maskin and Riley (1984). Alternatively, we could have considered a convex cost function, and then relate the subsequent results to quality discrimination as in Mussa and Rosen (1978). As ε converges to zero,

the model converges to the linear model in which a uniform price for the entire object, $q = 1$, is always an optimal policy. We observe that the concavity in the utility function is independent of the type v, and so is the socially efficient allocation (as long as ε is sufficiently small). That is, provided that $v > 0$, the socially efficient allocation is to assign each type the entire object, $q = 1$. In consequence, the efficient boundary of the surplus triangle is independent of the quadratic term and of the size of ε.

We choose to present results in this Section for this one parameter example and report calculations in an Supplemental Online Appendix. For the cases considered in this Section, with a small number of outcomes or types, the arguments extend in a straightforward manner to more general models with single crossing payoff functions. However, we will not characterize this general case.[7]

Finite Set of Allocations   First consider the case of finitely many allocations and finitely many types. With second degree price discrimination, the seller offers a menu consisting of price-quantity pairs from which the consumer can choose. In every optimal menu, prices are uniquely determined given the quantities through binding incentive compatibility constraints. Thus, any optimal menu is described by a vector of quantities rather than a single price p, as in the preceding analysis of third degree price discrimination. A menu is then a $K-$dimensional vector, $q = (q_1, ..., q_K)$, where $q_k \in Q \subseteq [0,1]$ and $|Q| < \infty$. The menu prescribes for every type v_k the quantity q_k that that type will choose. The only restriction on the feasible set of allocation vectors q is that the entries of the vector q are weakly increasing. This monotonicity requirement comes from the incentive compatibility constraints: $v_{k+1} \geq v_k \Rightarrow q_{k+1} \geq q_k$.[8] The binary allocation space of Sections 3 and 4 with $q_k \in \{0,1\}$ is simply a special case of the more general finite allocation space. With this generalization of the space of optimal policies, from a single price to a menu of quantities, the earlier results of Lemma 1 and Proposition 1 generalize, with the modification that now it is only possible to make the monopolist indifferent between particular sets of menus. This is in contrast to the linear case, where the monopolist can simultaneously be indifferent among charging any set of prices in the

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[7]In Bergemann, Brooks, and Morris (2013b), we performed another robustness check, seeing how the results changed if only "partial segmentation" was possible, so that segmentations were required to be convex combinations of some fixed set of markets. As in the case of second degree price discrimination, we no longer attain the entire welfare triangle but attain fat regions of possible welfare outcomes.

[8]Equivalently, we could represent the allocation vector by a vector of prices rather than quantities. However, as usual, the monotonicity condition of the incentive compatibility is more immediate to express in the quantity than in the price dimension.

support of x^*. Nonetheless, we can identify a uniform profit preserving extremal segmentations and identify the corresponding minimum and maximum pricing strategies, as in Proposition 1.

We illustrate this by returning to Example 1 with three valuations given by $v_k \in \{1, 2, 3\}$, but now allow for an intermediate quantity between 0 and 1, say $\frac{1}{2}$, and thus:

$$q_k \in \left\{0, \frac{1}{2}, 1\right\}.$$

In Figure 7 we display the resulting optimal policies given the concave utility of the consumers (for the case of $\varepsilon = 0.6$). There are now six distinct allocation vectors q, each of which is optimal for a subset of markets. The number of optimal allocation vectors is much smaller than the combinatorial upper bound, 3^3, would suggest as each vector has to be monotone by incentive compatibility, and by the familiar "no distortion at the top" result, allocation for the highest type must be the efficient quantity, $q_3 = 1$. It is informative to contrast the revenue maximizing allocations in Figure 7 with
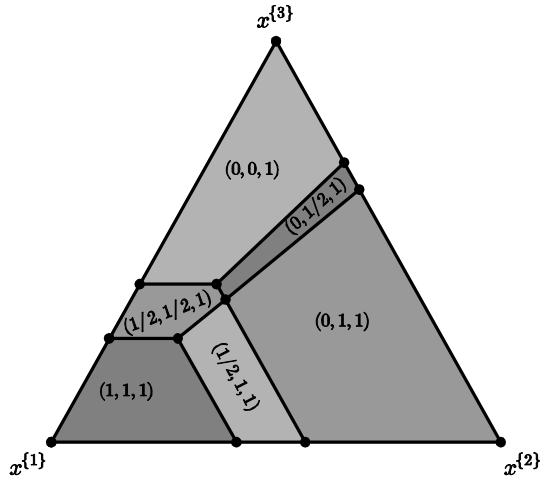

Figure 7: Extremal Markets with Menus

those in Figure 2. In the preceding analysis with unit demand, the choice of a single price p uniquely determined the allocation vector, so for example $p = 1$ represented the associated allocation vector $q = (1, 1, 1)$, in which every type receives the object, or $p = 2$ represented the allocation $q = (0, 1, 1)$ in which the low type does not receive the good. Between the previously used allocations, namely $(1, 1, 1)$, $(0, 1, 1)$ and $(0, 0, 1)$, there now appears a band of markets where the use of the intermediate quantity $\frac{1}{2}$ appears as an element of an optimal menu. In other words, with concave utility, the revenue maximizing policy sometimes uses the intermediate quantity to screen consumers within a segment.

In Figure 7, we observe at the base of the simplex, where only the low and intermediate valuation have positive probability, that the intermediate quantity $\frac{1}{2}$ already appears as part of the optimal allocation. Since we can analyze this case insightfully and completely, we therefore restrict attention further to the binary type model.

Binary Types With the binary type model, $v_k \in \{1, 2\}$, the aggregate market is described by the prior probability of the low type:

$$x^* \triangleq \Pr(v_k = 1).$$

In the binary type space, the probability simplex becomes the unit interval and we can determine the extremal markets at which multiple, here exactly two, menus are simultaneously optimal. The extremal market $\underline{x}$ identifies the prior probability at which the seller is indifferent between offering the menus $(0, 1)$ and $\left(\frac{1}{2}, 1\right)$; and the extremal market $\overline{x}$ identifies the prior probability at which the seller is indifferent between the menus $\left(\frac{1}{2}, 1\right)$ and $(1, 1)$. These extremal segments are illustrated in Figure 8.
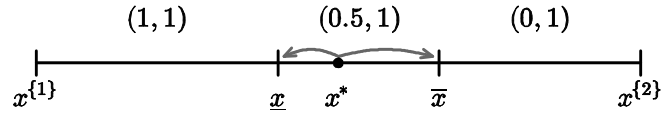

Figure 8: Probability Simplex with Binary Types

We can ask how the nonlinearity affects the limits of discrimination. The benchmark is now provided by the *uniform menu profit* rather than the uniform price profit, namely the profit that the seller could achieve with a single menu offered to the aggregate market x^*. We continue to refer to the uniform menu profit as π^*. The earlier Proposition 1 still applies and we can construct minimum and maximum pricing strategies to construct the horizontal boundary at the *uniform menu profit* π^*, but the resulting bounds on the consumer surplus are weaker in the non-linear screening environment. In the linear environment, the uniform profit preserving extremal segmentation had the property that there existed one pricing strategy that generates the socially efficient allocation and the consumer surplus is $w^* - \pi^*$, and another pricing strategy at which consumers receives zero surplus. By contrast, in the present case of second degree price discrimination, one or even both of the optimal pricing strategies involve intermediate quantities for any given market. As intermediate quantities are neither socially efficient nor do they suppress the information rents completely, the lower and upper bounds on consumer surplus, 0 and $w^* - \pi^*$, can no longer be achieved.

We can divide the binary type simplex into three regions in which each of the menus $(1,1)$, $\left(\frac{1}{2},1\right)$, and $(0,1)$ are respectively optimal. The corresponding set of extremal markets is given by $\{0,\underline{x},\overline{x},1\}$. Now, for every aggregate market $x^* \in [0,1] \setminus \{0,\underline{x},\overline{x},1\}$, a uniform profit preserving extremal segmentation involves the two markets adjacent to x^*, and at least one of them is necessarily $\underline{x}$ or $\overline{x}$. In consequence, the intermediate menu $\left(\frac{1}{2},1\right)$ is used under either the maximum or the minimum pricing policy. Either way, the intermediate menu does not generate the entire social surplus w^* nor does it reduce the consumer surplus to 0.

In Figure 9, we illustrate the attainable surplus set and the optimal allocation at an aggregate market $x^* = \frac{1}{2}$, at which point the menu $\left(\frac{1}{2},1\right)$ is strictly optimal since $\underline{x} < \frac{1}{2} < \overline{x}$. Thus, the uniform profit preserving extremal segmentation uses the markets $\underline{x}$ and $\overline{x}$, as indicated by the arrows in Figure 8. The minimum pricing strategy asks the seller to offer the most efficient among the optimal menus at each segment, which is $(1,1)$ at the segment $\overline{x}$ and $\left(\frac{1}{2},1\right)$ at the segment $\underline{x}$. This is analogous to the minimum pricing strategy, in that the allocation of the low type is as efficient as possible subject to optimal behavior on the part of the monopolist. However, these menus do not lead to the efficient allocation with probability one, and hence the realized surplus pair is below the efficient frontier. By a similar consideration, the maximum pricing strategy involves at every extremal market the least efficient among the optimal allocations, and hence offers $(0,1)$ at $\underline{x}$ and $\left(\frac{1}{2},1\right)$ at $\overline{x}$. Here again, we do not succeed in lowering the information rent of consumers to zero. It then follows that the set of attainable surplus pairs is strictly smaller than the bounds that defined the surplus triangle. But importantly, the characterization in terms of the extremal segmentations and the resulting large or "fat" set of possible surplus pairs remain valid.Of course, uniform profit preserving extremal segmentations give us only the lower and upper boundary of the set of consumer surpluses that are consistent with the uniform menu profit π^*. Thus, while we now have the boundary of the surplus set at π^*, the question arises how to identify the entire surplus set. With the linear environment, this question could be answered immediately as we directly attained the vertices of the surplus triangle. By contrast, in the present nonlinear environment, additional work remains to be done. Clearly, we can still form convex combinations of the extremal markets at π^* and the complete information outcome, the case of perfect discrimination. However, there are other surplus points that can be achieved using extremal but non-uniform profit preserving segmentations, as depicted in Figure 9 (for $\varepsilon = 0.6$). In particular, it is possible to achieve zero consumer surplus if the aggregate market is segmented using $\underline{x}$ and $x^{\{2\}}$, and the monopolist offers $(1,1)$ at $\underline{x}$ and $(0,1)$ (which is strictly optimal) at $x^{\{2\}}$. Similarly, zero consumer surplus can be reached by segmenting into $x^{\{1\}}$ and $\overline{x}$ and
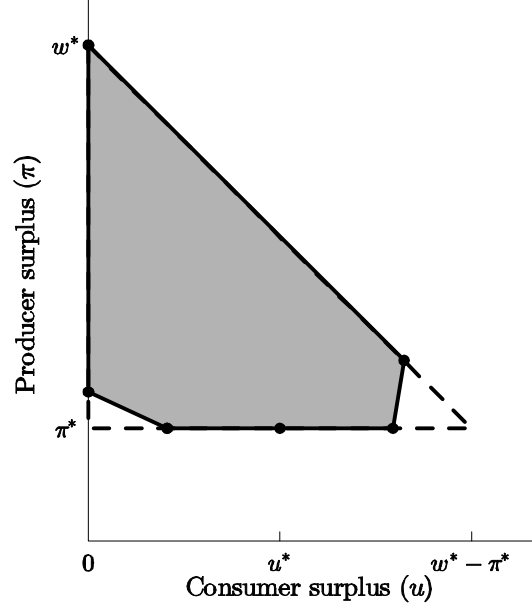
Figure 9: Surplus Set with Second Degree Price Discrimination

offering $(1, 1)$ at $x^{\{1\}}$ (again strictly optimal) and $(0, 1)$ at $\overline{x}$. We will see that with a continuum of allocations, the frontier becomes significantly more complicated, as in Figure 11.

A Continuum of Allocations   We conclude our discussion by staying with the binary type space $\{1, 2\}$ and the quadratic utility function given above by (20), but we now allow for a continuous choice of allocations with $q_k \in [0, 1]$. In other words, we are giving up the finiteness in the allocation space that we assumed so far. With this amount of flexibility in the allocation, the revenue maximizing menu will continuously respond to the composition of the market. In particular, for every market $x \in [0, 1]$, there exists a single menu that is revenue optimal. Thus, there are no longer any extremal markets at which multiple distinct menus are simultaneously optimal. In consequence, we cannot use the multiplicity of optimal menus to identify an interval of possible surplus pairs (u, π^*) varying in u that can be achieved by uniform profit preserving extremal segmentations. In fact, the set of attainable surplus pairs, while still large, now has a unique pair (u^*, π^*) at which the monopolist is held down to lower bound profits. Nonetheless, we can use segmentation across different profit levels to vary both the consumer and the producer surplus.

We will solve for the entire set of feasible surplus pairs in this example using the "concavification" methodology of Aumann and Maschler (1995) and Kamenica and Gentzkow (2011), discussed in the Introduction. This technique is especially powerful and transparent in the two type case considered

here. In the language of Bayesian persuasion (Kamenica and Gentzkow (2011)), the problem is as follows: Suppose a sender could commit, before observing his type, to a noisy signal that he will transmit to a receiver conditional on each type. The receiver in turn takes an action with payoff consequences for both sender and receiver, where this action maximizes the receiver's payoff given his posterior beliefs about the type conditional on the signal he received. The sender's payoff from the optimal signal structure can be identified by the concavification of the sender's payoff as a function of the receiver's posterior beliefs, where this payoff is induced by the receiver's optimal action given those beliefs. In particular, the maximum payoff of the sender over all signal structures is the concavification evaluated at the prior distribution of the type. In the present problem, the "sender" is a social planner maximizing a weighted sum of consumer and producer surplus with the type being the consumer's valuation, and the "receiver" is the monopolist choosing his privately optimal menu as a function of the posterior belief about the consumer's valuation. In effect, both the signal and the posterior beliefs are a market segment, and Bayesian updating requires that the average segment be the aggregate distribution of types. We can use this methodology to calculate the surplus frontier by finding the concavification of every possible objective of the social planner.

With this perspective, the profit function of the seller as a function of the proportion of low types x is given by:

$$\pi\left(x\right) \triangleq \begin{cases} \left(1-x\right)u_2\left(1\right), & \text{if } 0 \leq x \leq \underline{x}; \\ xu_1\left(q\left(x\right)\right)+\left(1-x\right)\left(u_2\left(1\right)-u_1\left(q\left(x\right)\right)+u_2\left(q\left(x\right)\right)\right), & \text{if } \underline{x} \leq x \leq \overline{x}; \\ u_1\left(1\right), & \text{if } \overline{x} \leq x \leq 1; \end{cases}$$ (21)

and likewise the consumer surplus is:

$$u\left(x\right) \triangleq \begin{cases} 0, & \text{if } 0 \leq x \leq \underline{x}; \\ \left(1-x\right)\left(u_2\left(q\left(x\right)\right)-u_1\left(q\left(x\right)\right)\right), & \text{if } \underline{x} \leq x \leq \overline{x}; \\ \left(1-x\right)\left(u_2\left(1\right)-u_1\left(1\right)\right), & \text{if } \overline{x} \leq x \leq 1. \end{cases}$$ (22)

The profit and consumer surplus expressions follow directly from the optimal and incentive compatible prices charged by the seller.

In Figure 10, we illustrate the shape of producer and consumer surplus for the quadratic utility function and $q \in [0,1]$ and for the case of $\varepsilon = 0.6$, as well as their respective concavified versions. The concavified version has a graph equal to the convex hull of the graph of the original function. Now, in segments where the concavification is strictly greater than the original function, we find that a convex combination of the critical markets that form the concavification maximizes producer and

consumer surplus, respectively. These illustrations immediately indicate some elementary properties of the profit maximizing or consumer surplus maximizing segmentations, which hold true for all concave utility functions $u(v, q)$. The concavified profit function strictly dominates the convex profit function $\pi(x)$ and hence the seller always prefers perfect segmentation, i.e., segments which contain either only low or only high valuations customers. By contrast, it is indicated by the concavified consumer surplus function that the maximal consumer surplus is attained without any segmentation with a large share x of low valuation buyers, whereas a small share x of low valuation buyers requires market segmentation to achieve maximal consumer surplus.
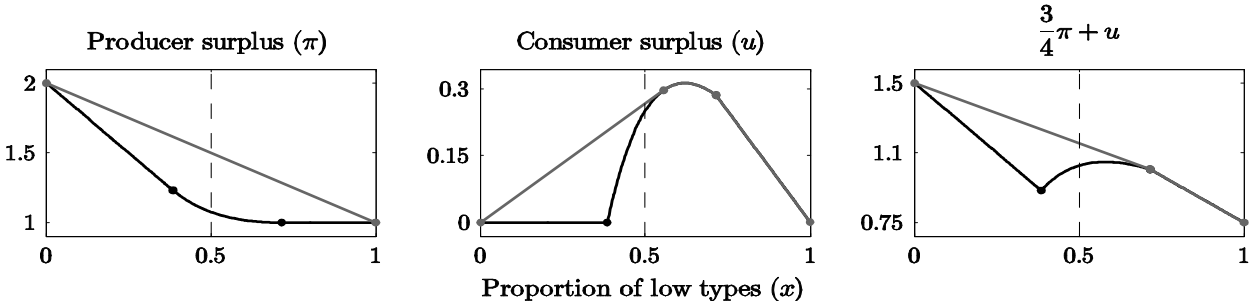

Figure 10: Consumer and producer surplus, and their concavified versions.

The entire boundary of the set of feasible pairs of consumer and producer surplus can be constructed by concavifying the weighted sum of these two expressions. We would have to allow for negative weights on either term to able to reach the lower bounds of the surplus set as well.

We illustrate the shape of the surplus triangle in Figure 11 for four different values of ε. As ε decreases, the concave utility comes closer to the linear utility. At the same time, the surplus set increases in the strong set order and eventually approximates the boundaries that identify the surplus triangle. Importantly, even though for every ε there is a unique pair of (u^*, π^*) at the Bayes Nash equilibrium profit level π^*, the overall surplus set is still "fat" in the sense that it forms a large open set inside the boundaries identified earlier. Ultimately, we find that the dramatic and concise characterization of feasible surplus pairs in our benchmark setting does depend on the linearity of payoffs. Nonetheless, the qualitative features of the main results remain approximately true for small deviations from linearity. The complete characterization of the limits of price discrimination in non-linear settings seems to be an open direction for future research.
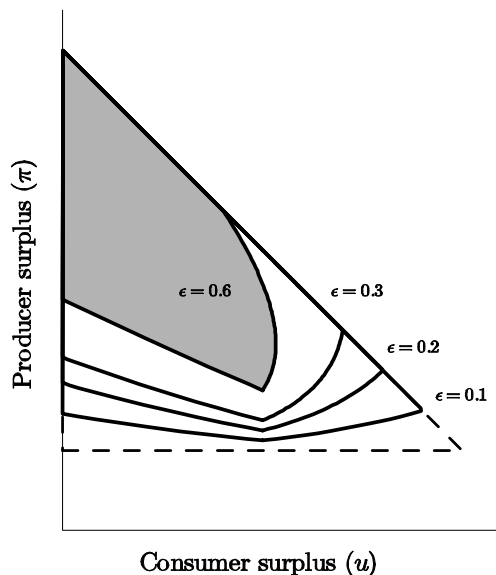
Figure 11: Surplus Set with a Continuum of Allocations

6   Conclusion

It was the objective of this paper to study the impact of additional information about consumers'
valuations on the distribution of surplus in a canonical setting of monopoly price discrimination. We
showed that additional information above and beyond the prior distribution can have a substantial
effect on consumer and producer surplus. In general, there are many directions in which welfare could
move relative to the benchmark of a unified market. We showed that while additional information
can never hurt the seller, it can lead social and consumer surplus to both increase, both decrease, or
respectively increase and decrease. Most notably, we establish the exact limits of these predictions
without any restrictions on the aggregate market and in particular the sharp boundaries do not rely
on any regularity or concavity assumption on the distribution of values or the profit function.

Exactly which form of market segmentation arises in practice is no doubt influenced by many
factors, which may include technological and legal limitations on how information can be collected
and used. In an age in which individuals are increasingly concerned about the preservation of privacy,
it is important to understand the welfare consequences that may result from the collection of data
on consumers' preferences. Policy discussion often assumes that this will favor producers and hurt
consumers. This may be a reasonable assumption if the data ends up in the hands of producers.
But this need not be the case. Our findings indicate that the relationship between efficiency and
information can only be understood in the context of how data will be used, and this crucially depends

on the preferences of those who collect the information. Thus, a natural and important direction for future research is to better understand which forms of price discrimination will endogenously arise, and for whose benefit.

7 Appendix

Proof of Lemma 2. We begin with the following four observations:

Fact 1: LC implies that optimality constraints are satisfied for all $i \geq l+1$.

Fact 2: UC implies that the non-negativity constraint is slack for some $i \geq l$.

Fact 3: If LC binds, then the non-negativity constraint is slack for some $i \geq l+1$.

Fact 4: If UC binds and $x_i > 0$ for some $i > l$, then v_l is not optimal.

As with X_k, any element $x \in X_{k,l}$ must satisfy the $2(K-1)$ non-negativity and optimality constraints. In addition, x must satisfy the lower and upper minimum output constraints. An extreme point of $X_{k,l}$ is characterized by a subset of at least $K-1$ of these constraints which are binding (see Simon (2011), Proposition 15.2). By Fact 1, we can drop all of the optimality constraints for $i \geq l+1$, leaving just UC, LC, non-negativity, optimality constraints for $i \leq l$. Let $m = |\{l, \ldots, K\} \setminus \{k\}|$. As before, non-negativity of x_i and optimality of v_i are mutually exclusive, so we can obtain at most $K - 1 - m$ binding optimality or non-negativity constraints for $i < l$, leaving m constraints to define the extreme point. We will establish by cases that the choice of the remaining active constraints must be equivalent to choosing at most one optimality constraint from $\{l+1, \ldots, K\}$ and at least one optimality constraint from $\{l, \ldots, K\}$, with the remaining binding constraints being non-negativity. Note that if $l = K$, then LC is always satisfied and can be dropped, and UC implies that v_K is an optimal price.

First, suppose that non-negativity is binding for x_l, i.e., $x_l = 0$. Note that this case can only arise if $l < K$. Then UC and LC are redundant and equivalent to the single linear restriction:

$$v^* \sum_{j \geq i^*} x_j = \sum_{j \geq l+1} v_j x_j.$$

By Fact 2, there must be exactly $m - 1$ binding non-negativity constraints for $i \geq l$. This implies that optimality binds for the single $i \geq l+1$ for which the non-negativity constraint on x_i need not bind.

Now suppose that $x_l > 0$. Then at most one of UC and LC can bind (the gap between them is strict). Suppose that it is LC. By Fact 3, some non-negativity constraint is slack for $i \geq l+1$. If more than one of these is slack, or if optimality is not binding for v_l, we could not reach the requisite number of binding constraints. Thus, exactly one non-negativity constraint is slack for $i \geq l+1$, and the optimality binds for v_l. Together with LC, this gives us the required m. As a result, optimality must be binding for the single $i \geq l+1$ for which non-negativity does not bind.

Alternatively, suppose $x_l > 0$ and UC binds. (Note that this case cannot arise when $l = k$, since then UC is implied by optimality of v_k, and can be dropped from the problem.) By Fact 4, non-negativity must bind for all $i \geq l+1$. Otherwise, neither optimality of v_l, non-negativity of x_l, nor non-negativity of x_i for $i \geq l+1$ binds, and we would only have UC plus at most $m-2$ non-negativity constraints for $i \geq l+1$, which is one short of the m required. We conclude that $x_i = 0$ for all $i \geq l+1$, so that UC implies that optimality binds for v_l.

Finally, it must be that either LC or UC binds. Otherwise, then clearly $x_l > 0$ and the only way to get to $K-1$ constraints is all non-negativity constraints are binding for $i \geq l+1$, and optimality binding for v_l, which contradicts that LC does not bind. ∎

Proof of Lemma 3. As X is metrizable in the Prokhorov metric, cl $\widehat{X}_v$ is the set of limits of convergent sequences in $\widehat{X}_v$. Take a weakly convergent sequence $x_k \in \widehat{X}_v$, and suppose that it converges to some $x \notin \overline{X}_v$. Then we can find some $v' \in \mathbf{supp}\, x$ such that $v'\, x\left([v', \overline{v}]\right) \neq \min \mathbf{supp}\, x$.

Claim: For any $v' \in \mathbf{supp}\, x$ and for any $\varepsilon > 0$, there exists a K such that for all $k \geq K$, we can find $v_k \in \mathbf{supp}\, x_k$ such that $|v_k\, x\left([v_k, \overline{v}]\right) - v'\, x\left([v', \overline{v}]\right)| < \varepsilon$. The measure x can have at most countably many mass points, so we can find $\delta > 0$ so that $x\left(\{v' - \delta, v' + \delta\}\right) = 0$ (i.e., both $[v' - \delta, v' + \delta]$ and $[v' - \delta, \overline{v}]$ are continuity sets of x) and $x\left([v' - \delta, v')\right) < \frac{\varepsilon}{2\overline{v}}$. By weak convergence, we must have that $x_k\left([v' - \delta, \overline{v}]\right) \to x\left([v' - \delta, \overline{v}]\right)$ and $x_k\left([v' - \delta, v' + \delta]\right) \to x\left([v' - \delta, v' + \delta]\right)$. Since $v' \in \mathbf{supp}\, x$, it must be that $x\left([v' - \delta, v' + \delta]\right) > 0$, so we can pick a K large enough so that for all $k \geq K$, $x_k\left([v' - \delta, v' + \delta]\right) > 0$ and $|x_k\left([v' - \delta, \overline{v}]\right) - x\left([v' - \delta, \overline{v}]\right)| < \frac{\varepsilon}{2\overline{v}}$. Let $v_k = \min \mathbf{supp}\, x_k \cap [v' - \delta, v' + \delta]$, which is non-empty because $x_k\left([v' - \delta, v' + \delta]\right) > 0$. Hence,

$$
\begin{aligned}
|v_k\, x_k\left([v_k, \overline{v}]\right) - v\, x\left([v', \overline{v}]\right)| &= |v_k\, x_k\left([v' - \delta, \overline{v}]\right) - v\, x\left([v', \overline{v}]\right)| \\
&\leq \left|v_k\, x_k\left([v' - \delta, \overline{v}]\right) - v'\, x\left([v' - \delta, \overline{v}]\right) + v'\frac{\varepsilon}{2\overline{v}}\right| \\
&\leq \left|(v_k - v')\frac{\varepsilon}{2\overline{v}}\right| + \left|v'\frac{\varepsilon}{2\overline{v}}\right| \\
&\leq \varepsilon
\end{aligned}
$$

which proves the claim.

Thus, we can find K large enough so that for $k \geq K$, there exist v_k and $v_k' \in \mathbf{supp}\, x_k$ such that $|v_k\, x_k\left([v_k, \overline{v}]\right) - \min \mathbf{supp}\, x| < \varepsilon$ and $|v_k'\, x_k\left([v_k', \overline{v}]\right) - v'\, x\left([v', \overline{v}]\right)| < \varepsilon$ where $\varepsilon = \left(|v'\, x\left([v', \overline{v}]\right) - \min \mathbf{supp}\, x|\right)/2$. But this means that $|v_k\, x_k\left([v_k, \overline{v}]\right) - v_k'\, x_k\left([v_k', \overline{v}]\right)| > 0$, which contradicts the assumption that $x_k \in \overline{X}_v$. ∎

Proof of Proposition 1B. The monopolist is indifferent to a pricing rule that puts probability one on v^* when $x \in \overline{X}_{v^*}$. Under such a rule, producer surplus is π^*. By definition of an extremal

segmentation, the maximum and minimum pricing rules are optimal, and the former results in a consumer surplus of zero and the latter results in all consumers purchasing the good, so that total surplus is w^*. ■

Proof of Theorem 1B. The argument for necessity is as in the finite case. For sufficiency, we first argue that there exists a uniform profit preserving extremal segmentation. Since simple measures are dense in X_{v^*}, we can find a sequence x_k of markets in X_{v^*} that converge to x^* in the weak topology. By Lemma 1, there exist extremal segmentations of x_k for every k, which we can identify with elements σ_k of $\Delta\left(\text{cl}\,\widehat{X}_{v^*}\right)$. Since $\text{cl}\,\widehat{X}_{v^*}$ is a closed subset of the compact set X, it is also a compact. By the Banach-Alaoglu Theorem, $\Delta\left(\text{cl}\,\widehat{X}_{v^*}\right)$ is compact, so σ_k has a convergent subsequence that converges to some $\sigma \in \Delta\left(\text{cl}\,\widehat{X}_{v^*}\right)$.

Claim: σ is a segmentation of x^*. For any continuous and bounded function f on $[0,\overline{v}]$, we have

$$
\begin{aligned}
\int_{v\in[0,\overline{v}]} f(v)\,x(dv) &= \lim_{k\to\infty} \int_{v\in[0,\overline{v}]} f(v)\,x_k(dv) \\
&= \lim_{k\to\infty} \int_{v\in[0,\overline{v}]} f(v) \int_{x'\in X} x'(dv)\,\sigma_k(dx') \\
&= \lim_{k\to\infty} \int_{x'\in X} \int_{v\in[0,\overline{v}]} f(v)\,x'(dv)\,\sigma_k(dx') \\
&= \int_{x'\in X} \int_{v\in[0,\overline{v}]} f(v)\,x'(dv)\,\sigma(dx')
\end{aligned}
$$

where the first line follows from weak convergence, the second line is the definition of an extremal segmentation, the third line is Fubini's Theorem, and the last line is again weak convergence, using the fact that $\int_{v\in[0,\overline{v}]} f(v)\,x'(dv)$ is a continuous function of x'. Hence, the measure $\int_{x'\in X} x'\sigma(dx')$ is a version of x^*.

To conclude, there exists a uniform profit preserving segmentation σ, under which minimum and maximum pricing rules are optimal and induce the points $(w^* - \pi^*, \pi^*)$ and $(0, \pi^*)$. As before, the segmentation σ' defined by $\sigma'(Y) = x^*\left(\{v|\,x^{\{v\}} \in Y\}\right)$ and $x^{\{v\}}$ is the Dirac measure on $\{v\}$, together with any optimal pricing rule, induces the welfare outcome $(0, w^*)$. Weighted averages $\sigma'' = \alpha\sigma + (1-\alpha)\,\sigma'$ and $\phi = \beta\underline{\phi} + (1-\beta)\,\overline{\phi}$ achieve every other surplus pair in the triangle. ■

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8 Supplemental Online Appendix

This Appendix provides a formal analysis of the examples of Section 5. The monopolist is selling to a market that consists of three types of consumers with valuations $v_k = k$ for $k \in \{1, 2, 3\}$. As before, x_k is the mass of consumers with valuation v_k, and $\sum_{k=1}^{3} x_k = 1$. Type k's utility from $q \in [0, 1]$ units of the good is:

$$v_k q + \varepsilon q (1 - q).$$

The monopolist offers a menu of quantity-price pairs, and we will solve three screening problems in this framework. The first considers what happens when the monopolist is restricted to choosing one of the three discrete output levels in $\{0, \frac{1}{2}, 1\}$. For this model, we will characterize the decomposition of the simplex into regions in which particular menus are optimal. For the second specification, we reduce the model to two valuations by imposing $x_3 = 0$. For this version of the problem, we will characterize the limits of welfare outcomes that can arise under segmentation. The third specification again has two valuations, but now the monopolist is allowed to choose any output $q \in [0, 1]$. For this specification, we again derive the surplus pairs that can arise under some segmentation of the market.

8.1 Finite Set of Allocations

We first consider a model in which the monopolist must choose $q \in \{0, \frac{1}{2}, 1\}$. Optimal incentive compatible menus can be identified with monotonic allocations (q_1, q_2, q_3) that specify the quantity sold to each type. In this setting, the usual "no distortion at the top" condition holds, so that $q_3 = 1$ in any optimal allocation. The corresponding transfers are pinned down by a binding individual rationality constraint for the low type, and binding downward incentive constraints for the higher types.

The following table lists the monotonic allocations that have no distortion at the top, and the corresponding transfers, expected profits, and expected consumer surpluses:

(q_1, q_2, q_3)	p_1	p_2	p_3	π	u
$(1,1,1)$	1	1	1	1	$x_2 + 2x_3$
$\left(\frac{1}{2},1,1\right)$	$\frac{1}{2}+\frac{\varepsilon}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3)$	$\frac{1}{2}x_2 + \frac{3}{2}x_3$
$(0,1,1)$	0	2	2	$2(x_2+x_3)$	x_3
$\left(0,\frac{1}{2},1\right)$	0	$1+\frac{\varepsilon}{4}$	$\frac{5}{2}$	$\left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3$	$\frac{1}{2}x_3$
$(0,0,1)$	0	0	3	$3x_3$	0
$\left(\frac{1}{2},\frac{1}{2},1\right)$	$\frac{1}{2}+\frac{\varepsilon}{4}$	$\frac{1}{2}+\frac{\varepsilon}{4}$	2	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\frac{1}{2}x_2 + x_3$

The six menus give rise to fifteen linear inequalities defining the markets in which one contract generates greater expected profit than the other. The following tabulates all of the inequalities, and reports the extreme points of the simplex that lie along the boundary of each region:

$(q_1,q_2,q_3) \succeq (q_1',q_2',q_3')$	Region	Extreme points
$(1,1,1) \succeq \left(\frac{1}{2},1,1\right)$	$1 \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3)$	$\left\{\left(\frac{2}{4-\varepsilon},\frac{2-\varepsilon}{4-\varepsilon},0\right),\left(\frac{2}{4-\varepsilon},0,\frac{2-\varepsilon}{4-\varepsilon}\right)\right\}$
$(1,1,1) \succeq (0,1,1)$	$1 \geq 2(x_2+x_3)$	$\left\{\left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right\}$
$(1,1,1) \succeq \left(0,\frac{1}{2},1\right)$	$1 \geq \left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3$	$\left\{\left(\frac{\varepsilon}{4+\varepsilon},\frac{4}{4+\varepsilon},0\right),\left(\frac{3}{5},0,\frac{2}{5}\right)\right\}$
$(1,1,1) \succeq (0,0,1)$	$1 \geq 3x_3$	$\left\{\left(\frac{2}{3},0,\frac{1}{3}\right),\left(0,\frac{2}{3},\frac{1}{3}\right)\right\}$
$(1,1,1) \succeq \left(\frac{1}{2},\frac{1}{2},1\right)$	$1 \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\left\{\left(\frac{4}{6-\varepsilon},0,\frac{2-\varepsilon}{6-\varepsilon}\right),\left(0,\frac{4}{6-\varepsilon},\frac{2-\varepsilon}{6-\varepsilon}\right)\right\}$
$\left(\frac{1}{2},1,1\right) \succeq (0,1,1)$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3) \geq 2(x_2+x_3)$	$\left\{\left(\frac{2}{4+\varepsilon},\frac{2+\varepsilon}{4+\varepsilon},0\right),\left(\frac{2}{4+\varepsilon},0,\frac{2+\varepsilon}{4+\varepsilon}\right)\right\}$
$\left(\frac{1}{2},1,1\right) \succeq \left(0,\frac{1}{2},1\right)$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3) \geq \left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3$	$\left\{\left(\frac{4}{6+\varepsilon},0,\frac{2+\varepsilon}{6+\varepsilon}\right),\left(0,\frac{4}{6-\varepsilon},\frac{2-\varepsilon}{6-\varepsilon}\right)\right\}$
$\left(\frac{1}{2},1,1\right) \succeq (0,0,1)$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3) \geq 3x_3$	$\left\{\left(\frac{6}{8+\varepsilon},0,\frac{2+\varepsilon}{8+\varepsilon}\right),\left(0,\frac{1}{2},\frac{1}{2}\right)\right\}$
$\left(\frac{1}{2},1,1\right) \succeq \left(\frac{1}{2},\frac{1}{2},1\right)$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x_1 + \frac{3}{2}(x_2+x_3) \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\left\{(1,0,0),\left(0,\frac{2}{6-\varepsilon},\frac{4-\varepsilon}{6-\varepsilon}\right)\right\}$
$(0,1,1) \succeq \left(0,\frac{1}{2},1\right)$	$2(x_2+x_3) \geq \left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3$	$\left\{(1,0,0),\left(0,\frac{2}{6-\varepsilon},\frac{4-\varepsilon}{6-\varepsilon}\right)\right\}$
$(0,1,1) \succeq (0,0,1)$	$2(x_2+x_3) \geq 3x_3$	$\left\{(1,0,0),\left(0,\frac{1}{3},\frac{2}{3}\right)\right\}$
$(0,1,1) \succeq \left(\frac{1}{2},\frac{1}{2},1\right)$	$2(x_2+x_3) \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\left\{\left(\frac{3}{4}-\frac{\varepsilon}{8},\frac{1}{4}+\frac{\varepsilon}{8},0\right),(0,0,1)\right\}$
$\left(0,\frac{1}{2},1\right) \succeq (0,0,1)$	$\left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3 \geq 3x_3$	$\left\{(1,0,0),\left(0,\frac{2}{6+\varepsilon},\frac{4+\varepsilon}{6+\varepsilon}\right)\right\}$
$\left(0,\frac{1}{2},1\right) \succeq \left(\frac{1}{2},\frac{1}{2},1\right)$	$\left(1+\frac{\varepsilon}{4}\right)x_2 + \frac{5}{2}x_3 \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\left\{\left(\frac{2}{4+\varepsilon},\frac{2+\varepsilon}{4+\varepsilon},0\right),\left(\frac{2}{4+\varepsilon},0,\frac{2+\varepsilon}{4+\varepsilon}\right)\right\}$
$(0,0,1) \succeq \left(\frac{1}{2},\frac{1}{2},1\right)$	$3x_3 \geq \left(\frac{1}{2}+\frac{\varepsilon}{4}\right)(x_1+x_2)+2x_3$	$\left\{\left(\frac{4}{6+\varepsilon},0,\frac{2+\varepsilon}{6+\varepsilon}\right),\left(0,\frac{4}{6+\varepsilon},\frac{2+\varepsilon}{6+\varepsilon}\right)\right\}$

These boundaries are depicted visually in Figure 12 for $\varepsilon = 0.6$. The simplex is shaded to indicate regions in which a given menu is optimal, which is the intersection of all of the regions in which that menu is pairwise superior to other menus. We can readily see from Figure 12 that there are exactly twelve extremal markets, which are extreme points in which a given menu is optimal. The resulting

regions where each one of the menus is optimal but without the pairwise boundaries are displayed in the text in Figure 7.
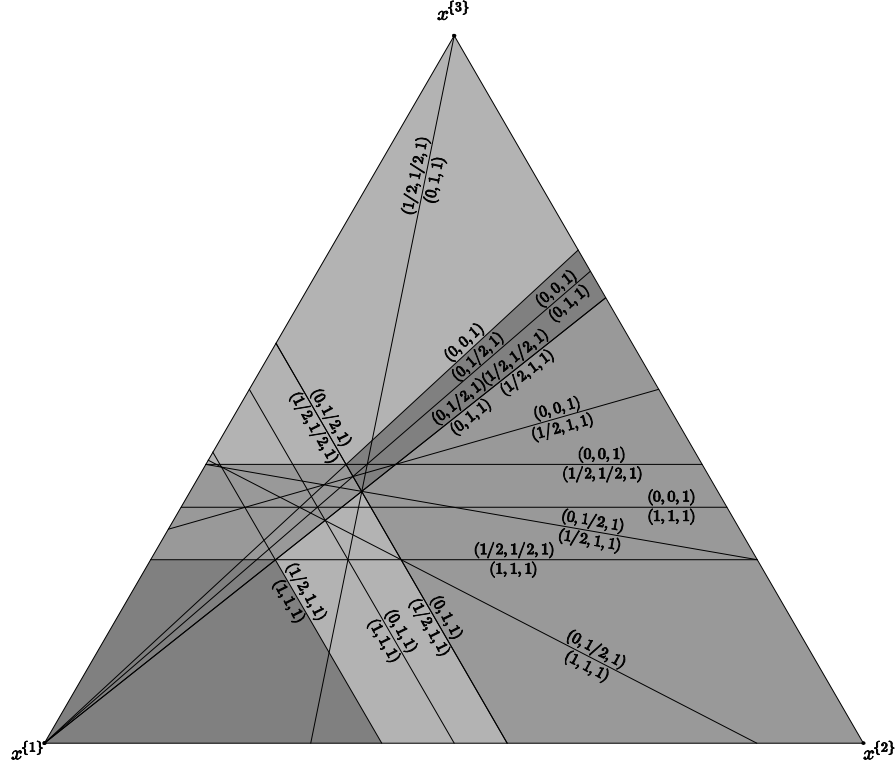

Figure 12: (Pairwise) Optimal Menus in the Probability Simplex

First, there are the three degenerate markets which have a single valuation in the support. Next, there are six extremal markets that have a support of two valuations, which are as follows:

$\mathbf{supp}\, x$	Optimal menus	Market
$\{1,2\}$	$\left\{(0,1,1),\left(\frac{1}{2},1,1\right)\right\}$	$\left(\frac{2}{4+\varepsilon},\frac{2+\varepsilon}{4+\varepsilon}0\right)$
$\{1,2\}$	$\left\{\left(\frac{1}{2},1,1\right),(1,1,1)\right\}$	$\left(\frac{2}{4-\varepsilon},\frac{2-\varepsilon}{4-\varepsilon},0\right)$
$\{2,3\}$	$\left\{(0,0,1),\left(0,\frac{1}{2},1\right)\right\}$	$\left(0,\frac{2}{6+\varepsilon},\frac{4+\varepsilon}{6+\varepsilon}\right)$
$\{2,3\}$	$\left\{\left(0,\frac{1}{2},1\right),(0,1,1)\right\}$	$\left(0,\frac{2}{6-\varepsilon},\frac{4-\varepsilon}{6-\varepsilon}\right)$
$\{1,3\}$	$\left\{(0,0,1),\left(\frac{1}{2},\frac{1}{2},1\right)\right\}$	$\left(\frac{4}{6+\varepsilon},0,\frac{2+\varepsilon}{6+\varepsilon}\right)$
$\{1,3\}$	$\left\{\left(\frac{1}{2},\frac{1}{2},1\right),(1,1,1)\right\}$	$\left(\frac{4}{6-\varepsilon},0,\frac{2-\varepsilon}{6-\varepsilon}\right)$

Finally, there are three extreme points with full support, $\mathbf{supp}\, x = \{1, 2, 3\}$:

Optimal menus	Market
$\left\{(1,1,1),\left(\frac{1}{2},\frac{1}{2},1\right),\left(\frac{1}{2},1,1\right)\right\}$	$\left(\frac{4}{8-\varepsilon}, \frac{4(4-\varepsilon)}{(12-\varepsilon)(8-\varepsilon)}, \frac{4-\varepsilon}{12-\varepsilon}\right)$
$\left\{(0,0,1),\left(0,\frac{1}{2},1\right),\left(\frac{1}{2},\frac{1}{2},1\right)\right\}$	$\left(\frac{2}{4+\varepsilon}, \frac{2(2+\varepsilon)}{(6+\varepsilon)(4+\varepsilon)}, \frac{2+\varepsilon}{6+\varepsilon}\right)$
$\left\{(0,1,1),\left(0,\frac{1}{2},1\right),\left(\frac{1}{2},\frac{1}{2},1\right),\left(\frac{1}{2},1,1\right)\right\}$	$\left(\frac{4}{8+\varepsilon}, \frac{4(4+\varepsilon)}{(12-\varepsilon)(8+\varepsilon)}, \frac{(8-\varepsilon)(4+\varepsilon)}{(12-\varepsilon)(8+\varepsilon)}\right)$

8.2 Binary Types

We now calculate the surplus set for the model of the previous subsection when $x_1 = x_2 = \frac{1}{2}$ and $x_3 = 0$. With binary types it is sufficient to identify the probability of the low type, and so we set $x_1 \triangleq x$, and $x_2 \triangleq 1 - x$. The menus can now be represented as:

(q_1, q_2)	p_1	p_2	π	u
$(1,1)$	1	1	1	$1-x$
$\left(\frac{1}{2},1\right)$	$\frac{1}{2}+\frac{\varepsilon}{4}$	$\frac{3}{2}$	$\left(\frac{1}{2}+\frac{\varepsilon}{4}\right)x + \frac{3}{2}(1-x)$	$\frac{1}{2}(1-x)$
$(0,1)$	0	2	$2(1-x)$	0

Note that the menu $\left(\frac{1}{2},1\right)$ is optimal in the aggregate market $x^* = \left(\frac{1}{2},\frac{1}{2}\right)$, and yields a profit of $\pi^* = 1 + \frac{\varepsilon}{8}$. There are three extremal markets for the simplex of markets consisting of types 1 and 2, which are:

Optimal menus	x
$(1,1)$	1
$\left\{(1,1),\left(\frac{1}{2},1\right)\right\}$	$\frac{2}{4-\varepsilon}$
$\left\{\left(\frac{1}{2},1\right),(0,1)\right\}$	$\frac{2}{4+\varepsilon}$
$(0,1)$	0

Let $\underline{x} = \left(\frac{2}{4-\varepsilon}, \frac{2-\varepsilon}{4-\varepsilon}\right)$ and let $\overline{x} = \left(\frac{2}{4+\varepsilon}, \frac{2+\varepsilon}{4+\varepsilon}\right)$. We will consider surplus pairs that can be derived from extremal segmentations. Note that within each region where a given menu is optimal, consumer and producer surplus are both linear functions of the proportion x of low valuation types. Hence, to maximize any weighted sum of u and π, it is always optimal to segment using extremal markets. Moreover, we know from the concavification argument that it is without loss of generality to consider segmentations that have only two segments. There are five extreme points of the surplus set, as derived below.

The only profit preserving extremal segmentation uses the extremal markets $\underline{x}$ and $\overline{x}$, since these are the only extremal markets for which the monopolist is indifferent to the uniform monopoly menu.

The weight on $\underline{x}$ can be derived from:

$$\frac{1}{2} = \sigma\left(\underline{x}\right)\frac{2}{4-\varepsilon} + \left(1 - \sigma\left(\underline{x}\right)\right)\frac{2}{4+\varepsilon} \Leftrightarrow \sigma\left(\underline{x}\right) = \frac{4+\varepsilon}{8}.$$

Any optimal menu choice by the monopolist will preserve profits of π^*. Among optimal menus, consumer surplus is maximized by offering menu $(1,1)$ in market $\underline{x}$ and by offering menu $\left(\frac{1}{2}, 1\right)$ in market $\overline{x}$. Consumer surplus is minimized by offering menu $\left(\frac{1}{2}, 1\right)$ in market $\underline{x}$ and offering $(0,1)$ in market $\overline{x}$. Thus, this segmentation gives rise to the extreme points:

$$\left(\frac{4+\varepsilon}{8}\frac{1}{2}\frac{2-\varepsilon}{4-\varepsilon}, 1 + \frac{\varepsilon}{8}\right), \left(\frac{4+\varepsilon}{8}\frac{2-\varepsilon}{4-\varepsilon} + \frac{4-\varepsilon}{8}\frac{1}{2}\frac{2+\varepsilon}{4+\varepsilon}, 1 + \frac{\varepsilon}{8}\right).$$

There are two binary extremal segmentations which can facilitate an efficient welfare outcome. The first is perfect price discrimination, i.e., segmentation into $x^{\{1\}}$ and $x^{\{2\}}$, which yields the surplus pair:

$$\left(0, \frac{3}{2}\right).$$

To attain efficiency, the monopolist must be indifferent to offering $(1,1)$ in any market such that $x_1 > 0$. The only other segmentation with this property has segments $\underline{x}$ and $x^{\{2\}}$. The weight on $\underline{x}$ must satisfy:

$$\frac{1}{2} = \sigma\left(\underline{x}\right)\frac{2}{4-\varepsilon} \Leftrightarrow \sigma\left(\underline{x}\right) = \frac{4-\varepsilon}{4}.$$

If the monopolist offers $(1,1)$ at $\underline{x}$, we obtain the efficient surplus point:

$$\left(\frac{1}{2} - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}\right).$$

This being one of only two welfare points that we can generate on the efficient frontier, it must be an extreme point of the surplus set. On the other hand, if the monopolist offers $\left(\frac{1}{2}, 1\right)$ at $\underline{x}$, we obtain the point:

$$\left(\frac{1}{4} - \frac{\varepsilon}{8}, 1 + \frac{\varepsilon}{4}\right).$$

Now, as ε becomes small, this consumer surplus converges to $\frac{1}{4}$, whereas the minimum consumer surplus at the uniform menu profit converges to $\frac{1}{8}$. As a result, it is impossible that this is an extreme point of the surplus set for ε sufficiently small.

To attain consumer surplus of zero, the monopolist must be indifferent to offering $(0,1)$ whenever $x_2 > 0$. Hence, there is one binary extremal segmentation (aside from perfect discrimination) which can facilitate consumer surplus of zero, namely segmenting into $x^{\{1\}}$ and $\overline{x}$. The weight on $\overline{x}$ must satisfy:

$$\frac{1}{2} = \sigma\left(\overline{x}\right)\frac{2+\varepsilon}{4+\varepsilon} \Leftrightarrow \sigma\left(\overline{x}\right) = \frac{4+\varepsilon}{4+2\varepsilon}.$$

If the monopolist offers the menu $(0,1)$ at $\overline{x}$, we obtain the point:

$$\left(0, 1 + \frac{\varepsilon}{4 + 2\varepsilon}\right).$$

Since this is one of only two possible welfare points that we can generate using binary extremal segmentations where consumer surplus is zero, it must be an extreme point of the surplus set. On the other hand, if the monopolist offers the menu $\left(\frac{1}{2}, 1\right)$ at $\overline{x}$, we obtain the welfare outcome:

$$\left(\frac{1}{4}, 1 + \frac{\varepsilon}{4 + 2\varepsilon}\right).$$

This consumer surplus is less than the maximum amount we can generate on the efficient frontier, which approaches $\frac{1}{2}$ as $\varepsilon \to 0$, as well as less than the maximum we can generate at uniform menu profit, which approaches $\frac{3}{8}$. Hence, for ε sufficiently small, this welfare point cannot be extreme either.

We conclude that the four binary extremal segmentations generate precisely five extreme welfare points, for ε sufficiently small, which are given by:

$$\left(\frac{4+\varepsilon}{8}\frac{1}{2}\frac{2-\varepsilon}{4-\varepsilon}, 1 + \frac{\varepsilon}{8}\right), \left(\frac{4+\varepsilon}{8}\frac{2-\varepsilon}{4-\varepsilon} + \frac{4-\varepsilon}{8}\frac{1}{2}\frac{2+\varepsilon}{4+\varepsilon}, 1 + \frac{\varepsilon}{8}\right),$$
$$\left(0, \frac{3}{2}\right), \left(\frac{1}{2} - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4}\right), \left(0, 1 + \frac{\varepsilon}{4 + 2\varepsilon}\right),$$

and illustrated in Figure 9.

8.3 A Continuum of Allocations

Now consider the model with $q \in [0, 1]$ and $v_k \in \{1, 2\}$. Optimal menus have $q_2 = 1$, and thus we define the quantity of low type for simplicity as $q_1 \triangleq q$, and as individual rationality binds for the low type, so $p_1 = q + \varepsilon q (1 - q)$. The benefit of the high type from pretending to be the low type is thus:

$$2q + \varepsilon q (1 - q) - p_1 = q.$$

Since the high type has a binding incentive constraint, his payment is simply $p_2 = 2 - q$. When there is a proportion $x_1 \triangleq x$ of low types, profits when allocating q to the low type are:

$$(q + \varepsilon q (1 - q)) x + (2 - q) (1 - x).$$

Differentiating with respect to q, the first order condition is:

$$0 = (1 + \varepsilon (1 - 2q)) x - (1 - x) \Leftrightarrow q(x) = \frac{1}{2} - \frac{1 - 2x}{2\varepsilon x}.$$

If this number is in $[0,1]$, then the optimal menu sells this quantity to the low type. If it is negative, for which the condition is:
$$\frac{1}{2} \leq \frac{1-2x}{2\varepsilon x} \Leftrightarrow x \leq \frac{1}{2+\varepsilon},$$
then since profits are concave in q, it is optimal to exclude the low type with an allocation of $q=0$. Note that when $x < \frac{1}{2}$, excluding the low type also yields a strictly higher payoff than pooling.

Similarly, if the solution to the first-order condition is greater than 1, for which the condition is:
$$-\frac{1}{2} \geq \frac{1-2x}{2\varepsilon x} \Leftrightarrow x \geq \frac{1}{2-\varepsilon},$$
then it is optimal to pool the high type and the low type at the efficient output. Note that when $x > \frac{1}{2}$, pooling yields strictly higher profit than exclusion. Hence, output is:

$$q(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < \frac{1}{2+\varepsilon}; \\ \frac{1}{2} - \frac{1-2x}{2\varepsilon x}, & \text{if} \quad \frac{1}{2+\varepsilon} \leq x < \frac{1}{2-\varepsilon}; \\ 1, & \text{if} \quad \frac{1}{2-\varepsilon} \leq x \leq 1. \end{cases}$$

Producer surplus is given by
$$\pi(x) = q(x)(1+\varepsilon(1-q(x)))x + (2-q(x))(1-x);$$
and consumer surplus is given:
$$u(x) = (1-x)q(x),$$
and illustrated in Figure 10.

We will solve for the surplus set when the aggregate market is $x^* = \frac{1}{2}$. We can write:
$$w_\lambda(x) \triangleq \lambda\pi(x) + u(x)$$
for $\lambda \in \mathbb{R}$. The support function of the surplus set at a given direction $(1,\lambda)$ is given by the concavification of $w_\lambda(x)$ at x^*. Similarly, the support function at directions $(-1,-\lambda)$ is given by the concavification of $-w_\lambda(x)$ at x^*.

Eastern frontier The concavification of w_λ at x^* falls into four "regimes", defined relative to three cutoff values of λ which are $\underline{\lambda} < \widehat{\lambda} < \overline{\lambda}$. We depict examples for each of these regimes in Figure 13. For each of four values of λ, we plot scaled producer surplus, consumer surplus, the sum, as well as their respective concavifications. The plots are scaled to show the relative magnitudes of $\lambda\pi$ and u, with λ decreasing as we progress downwards through the figure. For λ extremely large, i.e., in
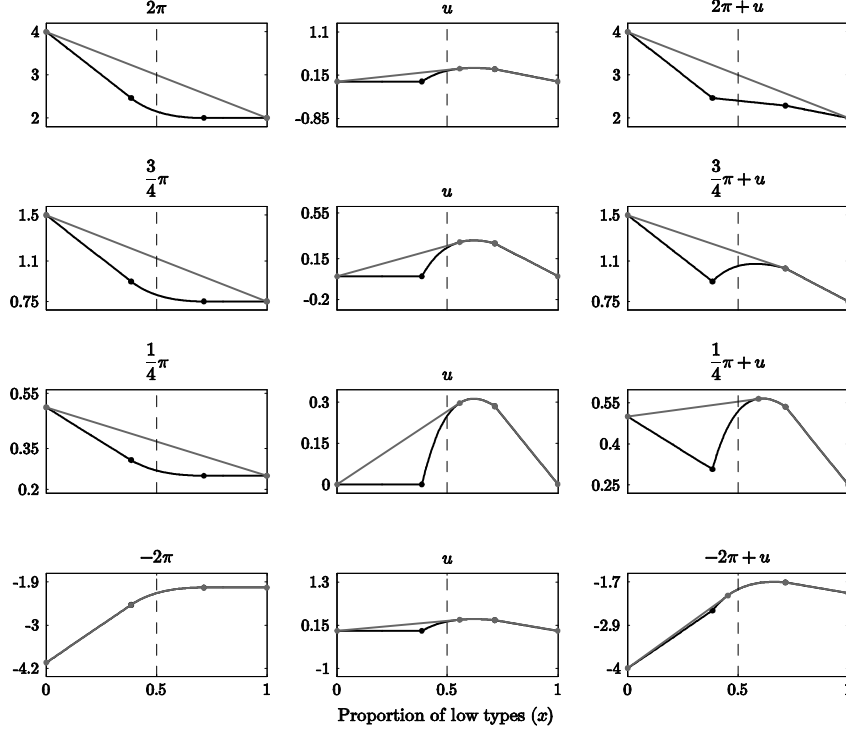
Figure 13: Examples for the concavification along the eastern frontier.

$\lambda \in \left(\overline{\lambda}, \infty\right)$, we are close to maximizing π. This is accomplished by perfect price discrimination, i.e., with $x = 0$ and $x = 1$, as depicted in the top row of Figure 13. For this range of λ, the extreme point of the welfare set is the efficient point where $u = 0$, i.e., the northernmost point in Figure 11.

As λ decreases, consumer surplus becomes more important, and the concavification of w_λ comes closer to the line segment between $x = \frac{1}{2-\varepsilon}$ and $x = 1$, where both u and π are linear. At a critical value $\overline{\lambda}$, the two coincide, in particular when:

$$\frac{1}{2-\varepsilon} w_\lambda (1) + \left(1 - \frac{1}{2-\varepsilon}\right) w_\lambda (0) = w_\lambda \left(\frac{1}{2-\varepsilon}\right).$$

The solution to this equation is given by $\overline{\lambda} = 1$. At this point, the concavification at $x = \frac{1}{2}$ becomes the line segment connecting $(0, w_\lambda (0))$ and $\left(\frac{1}{2-\varepsilon}, w_\lambda \left(\frac{1}{2-\varepsilon}\right)\right)$. In other words, for the next range of directions, it is optimal to segment between markets with $x = 0$ and $x = \frac{1}{2-\varepsilon}$. This case is depicted in the second row of Figure 13, where $\lambda = \frac{3}{4}$. Note that as λ decreases, this corresponds to the direction we are maximizing in rotating clockwise from due north. At some point, the optimum switches from the northernmost point to the easternmost point on the efficient frontier in Figure 11.

As λ decreases further towards zero, $w_\lambda (0) = 2\lambda$ falls relative to w_λ in $\left[\frac{1}{2+\varepsilon}, \frac{1}{2-\varepsilon}\right]$. Eventually, the tangent between $(0, w_\lambda (0))$ and the graph of w_λ moves from being at $\frac{1}{2-\varepsilon}$ to a point in $\left[\frac{1}{2}, \frac{1}{2-\varepsilon}\right]$. The

tangent just moves to the left of $\frac{1}{2-\varepsilon}$ when:

$$\frac{1}{2-\varepsilon} \lim_{x \uparrow \frac{1}{2-\varepsilon}} w'_\lambda(x) = \left(w_\lambda\left(\frac{1}{2-\varepsilon}\right) - w_\lambda(0) \right).$$

The solution is:

$$\widehat{\lambda} = \frac{3}{2} - \frac{1}{2\varepsilon}.$$

In this regime, the concavification at $x = \frac{1}{2}$ is given by the line connecting $(0, w_\lambda(0))$ and $(x(\lambda), w_\lambda(x(\lambda)))$, where $x(\lambda)$ solves:

$$x\, w'_\lambda(x) = w_\lambda(x) - w_\lambda(0) \Leftrightarrow x(\lambda) = \frac{2 - \lambda}{3 + \varepsilon(1 - \lambda) - 2\lambda}.$$

This is the case in the third row of Figure 13, where $\lambda = \frac{1}{4}$. For λ in this range, we trace out the curved portion of the eastern-southeastern frontier of Figure 11. The final cutoff $\underline{\lambda}$ is the solution to:

$$\frac{1}{2} w'_\lambda\left(\frac{1}{2}\right) = w_\lambda\left(\frac{1}{2}\right) - w_\lambda(0) \Leftrightarrow \underline{\lambda} = 1 - \frac{1}{\varepsilon}.$$

It is at this point that the tangent point from $(0, w_\lambda(0))$ to the graph of w_λ moves to the left of $\frac{1}{2}$, so that the concavification of w_λ at $x = \frac{1}{2}$ is just $w_\lambda\left(\frac{1}{2}\right)$, which is true for all $\lambda \in (-\infty, \underline{\lambda})$. In this range, the weight on minimizing producer surplus is so large that the solution is no segmentation, as in the bottom row of Figure 13. For this values, the direction $(1, \lambda)$ points southerly enough that the optimum is no information, i.e., the southern corner of Figure 11.

Western frontier For the western frontier, we find the concavification of $-w_\lambda(x)$ at $x = \frac{1}{2}$ for $\lambda \in \mathbb{R}$. As before, there are four regimes, and we depict examples in Figure 14. For λ sufficiently large, again we are close to minimizing π, and no segmentation is optimal, as in the top row. Note that large λ corresponds to maximizing a direction $(-1, -\lambda)$ close to due south, so again we are at the southern corner of Figure 11.

As λ decreases, u becomes relatively larger compared to π, and eventually the concavification at $x = \frac{1}{2}$ switches to the tangent line between $(1, w_\lambda(1)) = (1, \lambda)$ and the graph of $w_\lambda(x)$. Let $x(\lambda)$ again denote the point of tangency, which solves:

$$(1 - x)\, w'_\lambda(x) = 2\lambda - w_\lambda(x) \implies x(\lambda) = 1 / \left(1 - \frac{\sqrt{(1 - \varepsilon(6 - \varepsilon))\lambda(\lambda - 2)}}{\lambda - 2} \right).$$

The critical $\underline{\lambda}$ occurs when $x(\lambda) = \frac{1}{2}$, which is:

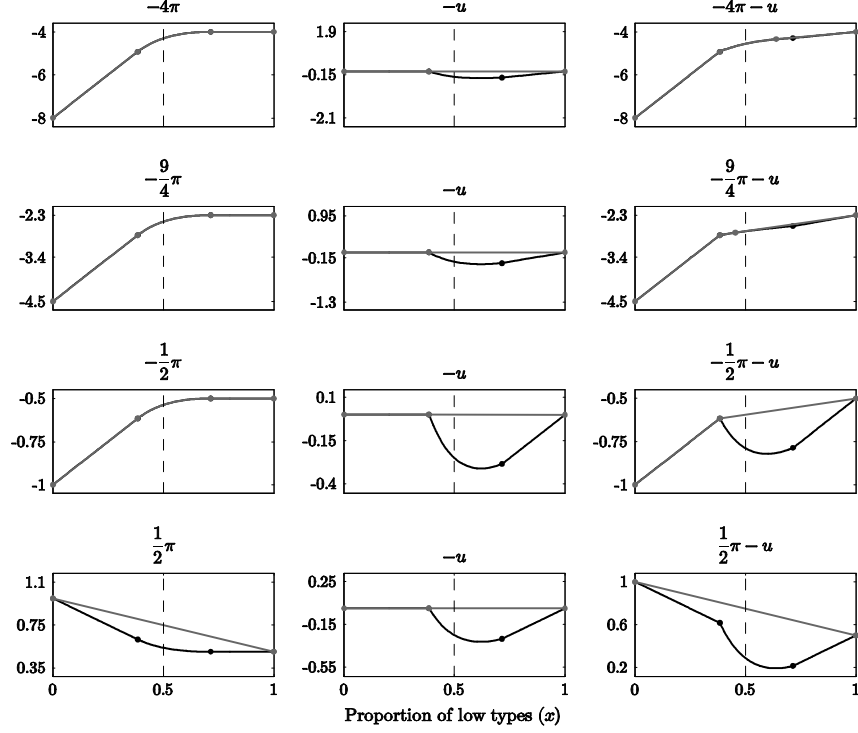$$\underline{\lambda} = \frac{2}{\varepsilon(6 - \varepsilon)}.$$

Figure 14: Examples for the concavification along the western frontier.

As λ increases above $\underline{\lambda}$, $x(\lambda)$ decreases from $\frac{1}{2}$ until it eventually hits $\frac{1}{2+\varepsilon}$. The critical λ is:

$$\widehat{\lambda} = \frac{(1+\varepsilon)^2}{4\varepsilon}.$$

For $\lambda \in \left[\underline{\lambda}, \widehat{\lambda}\right]$ the solution is to segment using markets $x(\lambda)$ and $x = 1$, as in the second row of Figure 14. Thus, in Figure 11, there is in fact a subtle curve to the southwestern frontier as $x(\lambda)$ moves smoothly from $\frac{1}{2}$ to $\frac{1}{2+\varepsilon}$. At $\widehat{\lambda}$, the regime changes to segmenting between $x = \frac{1}{2+\varepsilon}$ and $x = 1$, as in the third row of Figure 14. This generates the southernmost point along the western frontier where $u = 0$. This continues until we hit $\overline{\lambda}$ at which:

$$w_\lambda \left(\frac{1}{2+\varepsilon}\right) = \frac{1}{2+\varepsilon} w_\lambda(1) + \left(1 - \frac{1}{2+\varepsilon}\right) w_\lambda(0),$$

which occurs precisely at $\overline{\lambda} = 0$, when we are minimizing consumer surplus. Finally, for $\lambda \in \left(\overline{\lambda}, \infty\right)$, when we have negative weight on consumer surplus and a non-negative weight on producer surplus, the optimum is again perfect price discrimination, and we are back to the northernmost corner of Figure 11.