

Qualitative Voting*

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Abstract

Can we devise mechanisms where voters can express the intensity of their preferences when monetary transfers are forbidden? We answer this question in two stages.

First, as opposed to the *classical* voting system (one person - one decision - one vote), we propose a *new* voting system where agents are endowed with a given number of votes that can be distributed freely between a prearranged number of issues that have to be approved or dismissed. Its essence relies on allowing voters to express the intensity of their preferences in a simple and applicable manner. This voting system is optimal in a well-defined sense: in a setting with two voters, two issues and uniform independent priors, Qualitative Voting Pareto dominates Majority Rule and achieves the only ex-ante incentive compatible optimal allocation. The result holds true whenever we introduce a third player as long as the valuations towards the issues differ sufficiently.

And second, in a setting with I players and N issues, we generally show that a social choice function is implementable only if it does not undertake interpersonal comparisons of utility (it should only be contingent on the voters' relative valuations between the issues). Following this characterisation we find the impossibility of implementing strategy-proof mechanisms that are sensitive to the voters' intensities of preferences and satisfy the unanimity property. We end by dropping the unanimity property and providing an infinite set of social choice functions that satisfy some appealing properties.

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“The history of economic institutions shows a great deal of change, facilitating economic activities that would have earlier been impossible. No similar development and change has occurred in the political system; yet the need for such facilitation is undoubtedly equally great” James Coleman (1970)

1 Introduction

1.1 Motivation

Voting is the paradigm of democracy. It reflects the will of taking everyone’s opinion into account instead of imposing, by different means, the decision of a particular individual. At its root lies the belief that people should be allowed to freely cast their votes and, above all, they should be treated equally.¹ Consequently, as opposed to many economic situations, voting is considered a situation where no side payments are allowed so that agents are treated in an ex-ante identical position and wealth effects play no role.

Despite the adequacy of different particular rules to different settings, Majority Rule (MR, hereafter) is almost the uniquely used. From an economist’s perspective, and given that most of our work is built on the diverse behaviour of individuals with different marginal propensities to consume, produce, etc., the main concern is that MR does not capture the intensity of voters’ preferences. Just as we contemplate the importance of the *willingness to pay* in the provision of public goods, we should take into account the *willingness to influence* in a voting situation. An increase in the overall efficiency should follow.

The answer to this criticism has always been founded on an argument of equality: if we were to treat differently a very enthusiastic voter from a very apathetic one, equality would no longer hold.² Nevertheless this reasoning is too narrow. In this paper we show that we can build a very simple voting rule that allows voters to express intensity and reach a strictly Pareto superior allocation than the one achieved by MR; moreover, we characterise what can be implemented in multidimensional settings with no transfers.

Following Coleman’s quote, we ultimately want to stimulate the current debate around the

¹See, for instance, Locke (1690).

²See Spitz (1984).

development that should occur in our political institutions to better represent and govern our societies. We want to consider voting systems where the concept of decision *preferred by most members* is replaced by decision *most preferred by members*; we want votes to have an embedded *quality* which is somehow associated to the intensity of the voters' preferences; ultimately, we want to show under which circumstances the strategic interactions between voters do not undermine the gains we expect from them expressing their *willingness to influence*.

In a setting with a closed agenda of N issues that have to be approved or dismissed, we first propose a *Qualitative Voting* rule (QV, hereafter) that allows voters to simultaneously and freely distribute a given number of votes among the issues. In this way we are providing voters with a broader set of strategies than the classical “one person – one decision – one vote” and we are preserving the equality argument inherent in any voting procedure given that all individuals are endowed with the same ex-ante voting power.

Essentially, QV introduces two main improvements with respect to the usual voting rules. On the one hand, it answers the classical debate in the political science literature on “the problem of intensity” allowing strong minorities to decide over weak majorities. Secondly, it allows voters to trade off their voting power, adding more weight to the issues they most care about, and unlocks conflict resolution situations.

The latter intuition is best captured by the following situation: imagine two voters with opposing views in two issues but such that the first (second) voter mostly cares about the outcome on the first (second) issue. QV allows each of them to decide on their most preferred issue and hence non-cooperatively coordinate on the only Pareto optimal allocation that yields a strictly positive utility to both voters (in the sense that each one wins his most preferred issue and loses the least preferred one whenever). We can devise many different instances in which such situations occur and where side payments may not be possible (or may be forbidden): a divorce settlement, an international dispute, a bilateral agreement in arms/pollution reduction, a country having the two chambers governed by opposing parties,³

³The US Congress and Senate have repeatedly been in a situation where one chamber had a Republican majority and the other a Democratic one. Consequently, many bills have been vetoed by one chamber so that decisions have not been easily made. QV could have made the decision process more efficient allowing each party to support those bills which its electorate felt more strongly about. Money and Tsebelis (1997) claim that the gains we expect from the use of QV may already be observed through the existence of committees: “One essential assumption of distributive theories of Congress is that the policy space is multidimensional. This is how committee chairs and members extract gains from trade. They give up their positions in the less important dimension in order to gain in the more important one, their

a clash between the management and the union of a particular firm, etc...⁴

The goal of the first part of the present paper is not only to compare QV to MR but, moreover, we want to assess its optimality. Hence we undertake a mechanism design approach that allows us to characterise the optimal allocations out of the implementable ones.

In a setting with two voters, two issues and independent uniform priors on the voters' preferences, Theorem 1 tells us that QV reaches the only ex-ante optimal allocation. Moreover, Theorem 2 establishes that the result holds true whenever we introduce a third voter as long as the valuations towards the issues differ sufficiently. The introduction of a third voter yields a departure from the pure conflict resolution situation so that we can assess the optimality of allowing minorities to decide over weak majorities. Theorem 2 tells us then that it is ex-ante optimal to resolve the divergent issues through QV as long as the minority feels much stronger than the sum of the majoritarian preferences.

Corresponding examples follow the statement of both theorems in order to illustrate the results and shed some light into the applicability of QV into the real world.

The dependence of the results on the independent uniform priors is proved to be critical in Section 3.4: the more deterministic the priors are, the more strategically voters react and, consequently, the more difficult it is to achieve a truthful revelation of preferences. Hence, the strategic interactions between individuals may lead to the non-existence of pure-strategy equilibria in the game induced by QV. This does not undermine the first part of this article: there are some situations in which one can strictly Pareto improve the allocation achieved by MR through a simple mechanism we have called QV.

The drawback above leads us to the second half of this article where we assess which voting rules or general mechanisms are *robust* –i.e. are implementable given any specification of the priors. In our setting *robustness* is equivalent to strategy-proofness, hence we move in the second half of the article from Bayesian Nash implementation to dominant strategy

own jurisdiction.”

⁴Our setting can be reinterpreted as an extension of the *Colonel Blotto Game* (two colonels are fighting over some regions and have to decide how to divide their forces; the one with larger forces wins the region and the winner of the battle is the one with the most won territory) taking into account that now the colonel is not indifferent between winning two different regions. Hence the payoff of the game is not only contingent on how many regions he has won or lost but precisely on which regions he has won or lost. Myerson (1993) refers also to the *Colonel Blotto Game* when analysing the incentives for candidates to create inequalities among voters by making heterogeneous campaign promises.

implementation. That is, we move from a situation of incomplete information where players just know the priors from which their opponent's preferences are drawn to a setting where players know their opponent's preferences.

We first characterise the set of strategy-proof mechanisms and present our main contribution to the mechanism design literature: any implementable mechanism should only be contingent on the voters' relative intensity between the issues. That is, it cannot treat very enthusiastic voters better than apathetic ones in the sense that absolute valuations cannot play a role for it to be incentive compatible. In other words, a mechanism is implementable only if it does not carry direct interpersonal comparison of utility; ultimately, as the mechanism is to aggregate individual preferences, it needs to do some comparison across individuals but based on the primary intrapersonal one.

Following the characterisation of all possible mechanisms we impose the usual conditions to further identify the mechanisms that are sensitive to the voters' intensity of preferences -we call them *qualitative* mechanisms. This course of action leads to a very negative result: there are no *qualitative* mechanisms that are strategy-proof and satisfy the unanimity property.⁵

The key intuition for this result lies on the fact that any strategy-proof mechanism that satisfies the unanimity property needs to be insensitive to the voters' intensities of preferences on those issues where unanimous wills exist. Consider now a strategy-proof *qualitative* mechanism that satisfies the unanimity property. It needs to be sensitive to the voters' intensities of preferences for some particular profiles but it cannot be so on those issues where unanimous wills exist. This places a very *asymmetric* behaviour on the sensitiveness to preferences' intensities and triggers the fact that such mechanisms cannot be strategy-proof.

We then proceed by dropping the unanimity requirement (alternatively we can restrict the set of preferences so that no unanimous wills exist) and we distinguish an infinite set of strategy-proof *qualitative* mechanisms satisfying the usual properties of anonymity and neutrality that are ex-post incentive efficient.

In the remaining of this section we review the existing literature and relate our model to this earlier work. Next, the paper is organised as follows: Section 2 introduces the model,

⁵In our setting, *unanimity* requires an issue to be approved or dismissed with certainty whenever all players wish so.

Section 3 analyses the indirect mechanism QV and its optimality in a setting with uniform and independent priors, Section 4 provides the general analysis of the intensity problem in a scenario robust to any specification of the priors (strategy-proof) and, finally, Section 5 concludes.

1.2 Related literature

The fact that any implementable mechanism needs only to rely on the relative valuations between the issues stresses that intensity of preferences can play a role in voting games only when we move away from unidimensional settings. Furthermore, QV arises as a way to allow voters to trade-off their voting power. Thus its gains come precisely from non-homogeneous preferences across issues.⁶ Accordingly, our work belongs to a wider set of models with two key features: heterogeneous preferences and a multidimensional setting. In fact we are dealing with a simple comparative advantage argument, the key question is how to implement it: in the same way that each country should specialise in the production of the good in which it is relatively more productive, QV allows voters to decide on that issue they relatively care more about.

The two papers most closely related to ours are Jackson and Sonnenschein (2003) and Casella (2003). Jackson and Sonnenschein (2003) show that linking decisions normally leads to Pareto improvements. More specifically, they present a simple rule that achieves the ex-ante efficient allocation and that induces truthful revelation as we increase the number of decisions. Such rule is very simple in the sense that it just requires voters to match their voting profiles to the frequency of preferences across decisions according to the underlying distribution of preferences. The key differences with our work is that they provide an efficiency limiting result for a particular indirect mechanism and their action space depends on the prior distribution of preferences. Instead, we provide an indirect mechanism which does not depend on the prior distribution and characterise its optimality on a very particular setting.⁷

⁶Bowen (1943) has already pointed out that MR is an efficient mechanism whenever the intensity of the voters' preferences is distributed symmetrically. In a similar way, Philipson and Snyder (1996) analyse an organised vote market and show that its efficiency gains (with respect to MR) are larger the more heterogeneous the preferences are.

⁷Assessing trade-offs between issues and extracting all possible gains from differences is also one of the main concerns of the negotiation analysis and the international relations literatures. See for instance Keeney and Raiffa (1991). Closer in spirit to our work, Shepsle and Weingast (1994, pg 156) assert that "The political solution is to create an institutional arrangement for exchanging support that is superior to a spot market". Likewise, Levy (2004) models political parties as being able to exploit the gains from differing relative valuations in a multidimensional policy space.

Casella (2003) proposes a system of *Storable Votes* to be used in situations where voters have to decide over the same binary decision repeatedly over time. Such a voting system is proved to Pareto dominate MR in a particular setting. Our framework is different in the sense that voters simultaneously cast all their votes and know their full preference profile at the time of voting (no time dimension). Moreover, we undertake a mechanism design analysis which allows us not only to compare two particular voting rules but also to characterise all implementable allocations and, from them, identify the optimal ones. Our impossibility result on implementing voting rules that are sensitive to the voter’s intensity of preferences (whenever we require *robustness* and unanimity) generalises also her conjecture that gains we expect from *Storable Votes* may hold as long as priors are not too polarized.

Most of the literature on mechanism design without transfers (and most of the literature on voting) is built on a setting with ordinal utilities and where one alternative has to be selected out of many, i.e. a setting of electing representatives.⁸ Within that literature, QV has a flavour of a *scoring rule* though there is a crucial distinction:⁹ a *scoring rule* is used to elect one representative out of many, instead QV deals with a situation where N independent issues have to be approved or dismissed. Our setting is one of a repeated binary election thus we are implicitly restricting the domain of preferences (see the example in page 41).

There is also an incipient literature on considering the *intensity problem* in different settings with no transfers. Eliaz, Ray and Razin (2004) analyse how voters may abstain from an election depending on their relative aversion towards disagreement; Borgers and Postl (2004) show in a setting where two agents have to elect a representative out of three that no efficient mechanism exists; and, finally, Abdulkadiroglu (2004) provides an improved mechanism for the allocation of indivisible goods where intensity of preferences can be elicited and the allocation achieved is at least as good as the one achieved by *random serial dictatorship*.

Our result on the impossibility of implementing strategy-proof qualitative mechanisms that satisfy the unanimity property parallels the literature on social choice (e.g. Arrow 1951), on

⁸The main references are Gibbard (1973) and Satterthwaite (1975). In essence, these works are a formal treatment of the Arrow’s Impossibility Theorem from a mechanism design perspective. We defer further discussion to this strand of the literature to Section 4.2 after the presentation of our impossibility result.

⁹“In a *scoring rule*, each voter’s ballot is a vector that specifies some number of points that this voter is giving to each of the candidates (or parties) that are competing in the election. These vote-vectors are summed over all voters, to determine who wins the election ”, Myerson (1999), pg 673-674.

implementation (e.g. Gibbard 1973, Satterthwaite 1975) and on the allocation of indivisible objects (e.g. Zhou 1990).

The literature on alternatives to MR is related to our work insofar as it provides mechanisms which capture the intensity of the voters preferences but their complexity undermines its applicability. On the one hand, Tideman and Tullock (1976) develops an application of the *Clarke-Groves* mechanism to a voting framework. Needless to say, this requires monetary transfers and thus fails to satisfy the equality property. On the other hand, Hylland and Zeckhauser (1979) propose a *Point Voting Rule* to be used for the contribution to public goods, with perfectly divisible points.¹⁰ They focus on providing an (arbitrary) social choice function that induces the truthful revelation of preferences. Actually, this function belongs to the set of strategy-proof functions we provide in Section 4.3.

When we imagine a way in which politicians give more weight to a particular position we immediately think of *logrolling* or *vote trading*. This occurs whenever two voters bilaterally agree on voting against one's position on some non salient issues which are salient for the other voter. The result is that both voters will have gained support on their salient issues at the cost of losing non-salient ones. The relationship and gains of QV with respect to that particular way of expressing the *willingness to influence* are shortly discussed in Section 3.4.

2 The general model

A *voting game* is defined as a situation where I voters have to dismiss or approve N issues and no monetary transfers are allowed. Voters privately know their preference profile across the N issues and the prior distributions from which these preferences are drawn are common knowledge (note that this allows for deterministic priors, i.e. commonly known preferences). From a mechanism design perspective this is a multidimensional problem with multilateral asymmetric information and no transfers.

Voters and issues are denoted $i \in \{1, 2, \dots, I\}$ and $n \in \{1, 2, \dots, N\}$, respectively. Voter i 's valuation towards issue n is θ_n^i . The preference vector of voter i is $\theta^i = (\theta_1^i, \dots, \theta_N^i)$. At the

¹⁰Brams and Taylor (1996) propose a *Point Voting Rule* (the *Adjusted Winner Procedure*) that is essentially our voting system in a setting of a conflict resolution. Their weakness, though, is that they do not take into account the strategic interactions and restrict players to be truthful on their cast votes.

moment, we impose no restriction on the range of possible types (i.e. $\theta^i \in \Theta \subseteq \mathbb{R}^N, \forall i = 1 \div I$) or on their prior distribution.

Preferences should be interpreted as follows: a positive type ($\theta_n^i > 0$) wishes the approval of the issue, a negative one ($\theta_n^i < 0$) wishes its dismissal and its absolute value ($|\theta_n^i|$) captures the intensity of the preference towards that particular issue.

Voter i 's payoff on a given voting procedure n is described as follows,

$$\begin{cases} \theta_n^i & \text{if the issue is approved} \\ -\theta_n^i & \text{if the issue is dismissed} \end{cases}$$

and the total payoff is the sum of the individual payoffs across the N voting procedures.¹¹

An allocation is a N -tuple of probabilities that corresponds to the probability of approving each of the N issues. The set of allocations is defined as $\mathcal{X} = \{(p_1, \dots, p_N) : p_1, \dots, p_N \in [0, 1]\}$ where p_n is the probability of issue n to be approved. Hence, a voter with preferences θ^i obtains the following utility $p \in \mathcal{X}$:

$$u(p, \theta^i) := \sum_{n=1}^N p_n \theta_n^i + (1 - p_n) (-\theta_n^i) = \sum_{n=1}^N (2p_n - 1) \theta_n^i.$$

Note that we are in a setting of private values where each agent's utility depends only on his own type and utilities are multilinear.

3 A new voting rule

In this section we describe the outcome of QV –a particular voting rule that allows voters to distribute freely a certain number of votes between a prearranged number of issues. Our goal is two sided. We first want to compare the welfare properties of QV and MR. Second, we want to assess the optimality of the allocation achieved by QV when compared to all implementable ones.

¹¹The definition of the payoff is implicitly assuming that issues are independently valued. That is, there are no complementarities between the issues. Nevertheless, provided that issues are independently valued, results can be extended to any linear transformation of the payoffs.

The strategy space defined by QV are mappings from preference profiles to *voting profiles* $\mathcal{V}$

$$\mathcal{V} := \left\{ (v_1, v_2) \in \{-V, \dots, -1, 0^-, 0^+, 1, \dots, V\}^2 : |v_1| + |v_2| = V \right\}$$

so that a positive (negative) vote indicates the wish towards the approval (dismissal) of the issue.¹²

QV also defines a particular way to aggregate the cast votes. An issue is approved whenever the sum of votes on that issue is strictly greater than zero, dismissed whenever it is strictly negative and a tie breaking rule is applied whenever the sum of votes is equal to zero. The importance of the tie breaking rule is explained in Section 3.1.2. At the moment assume that ties are resolved applying the usual MR (i.e. the will of the majority of voters is implemented and if no majority exists a fair coin is tossed). Briefly,

$$\begin{cases} v_n^1 + \dots + v_n^I > 0 & \Rightarrow \text{The issue is approved} \\ v_n^1 + \dots + v_n^I < 0 & \Rightarrow \text{The issue is dismissed} \\ v_n^1 + \dots + v_n^I = 0 & \Rightarrow \text{MR is applied}^{13} \end{cases}$$

for every $n = 1 \div N$.¹⁴

In order to assess the optimality of QV we need to simplify our analysis. We consider a setting with (i) two or three voters ($I = 2, 3$), (ii) two issues ($N = 2, n \in \{1, 2\}$), (iii) two valuations ($\theta_n^i \in \{\pm 1, \pm \theta\}, \theta \in (0, 1)$)¹⁵ and (iv) uniform and pairwise independent priors:

$$\begin{cases} \Pr(\theta_n^i = 1) = \Pr(\theta_n^i = -1) = \Pr(\theta_n^i = \theta) = \Pr(\theta_n^i = -\theta) = \frac{1}{4} \\ \text{Pairwise independence across issues and voters.} \end{cases}$$

We define the set of a voter's preference profiles as $\Theta := \{\pm 1, \pm \theta\} \times \{\pm 1, \pm \theta\}$

¹²By means of a small abuse of notation, the action space is defined so that investing zero votes is informative about the wish of the voters' preferences towards the approval or dismissal of the issue (i.e. 0^+ and 0^- have positive and negative *sign*, respectively).

¹⁴Note that MR is just a particular case of QV with $V = 0$.

¹⁵Note that without loss of generality and in order to simplify the notation we have assumed the high issue to take a value equal to one. The analysis is totally analogous to the more general setting where $\theta_n^i \in \{\pm \bar{\theta}, \pm \underline{\theta}\}, \bar{\theta} > \underline{\theta} > 0$.

3.1 The indirect mechanism

3.1.1 The two voters' case

The two voters' case introduces the main effect of QV as voting rule. It allows voters to trade-off their voting power. Specifically, when we consider two issues it allows voters to rank the issues and reach the only ex-ante optimal allocation. The next example best captures this intuition:

Example: Two friends, Anna ($i = 1$) and John ($i = 2$), are to go out on a Friday night and have decided they will first go for dinner and then to the movies. It is their first date so, above all, they want to be together even if they do not come to an agreement. Anna wants to see a horror film and would like to have dinner in a new Italian restaurant while John prefers a comedy film and eating sushi in a Japanese restaurant. Following the previous notation we could define issue 1 being the film decision (where p_1 would be probability of seeing horror film and $1 - p_1$ the probability of seeing comedy film) and issue 2 being the restaurant decision (where p_2 would be the probability of Italian restaurant and $1 - p_2$ the probability of Japanese restaurant). If they vote on each of the issues nothing is decided and they have to stay at home (we assume that option is not optimal for either of them). Additionally, suppose that Anna really cares about which restaurant to go to and John, instead, cares more about the film (i.e. $\theta^1 = (\theta, 1)$ and $\theta^2 = (-1, -\theta)$). It seems sensible that, as being good friends, each of them will give up on their least preferred option. That is, they will both go to the Italian restaurant and then to the comedy film yielding an overall utility of $(1 - \theta) > 0$.

From a game theoretic perspective, they are both coordinating on the only Pareto optimal allocation that yields a strictly positive utility to both players (in the sense that each one wins his/her most preferred issue and loses the least preferred one). QV is precisely a mechanism that allows voters to non-cooperatively coordinate on the only ex-ante optimal outcome. We turn now into the rigorous analysis of the two voters case.

Voters are endowed with $V > 0$ votes that can be freely distributed between the two issues. We assume that V is even so that voters can evenly split the votes between the two issues if necessary. The submitted votes can have a positive or negative value capturing the will of the voter towards the approval or dismissal of the issue.

The uniform and independent priors on the opponent's preferences imply that it is a dominant strategy to truthfully declare the true sign of the preferences. Notice that in the case where a voter loses one of the issues he definitely wins the remaining one. This is because of the binary nature of our setting with only two issues. Losing an issue means having opposing preferences to the opponent on that issue and having invested fewer votes than he did. This implies that the voter at hand has invested more votes in the remaining issue. It can be easily proved that it is optimal to ensure that a voter does not lose his most preferred issue and consequently the optimal strategy for a voter who is not indifferent between the two issues is to invest all votes in his most preferred issue.

Instead, a voter who is indifferent between the two issues is also indifferent between playing any of the strategies. We therefore assume that he splits his votes evenly. The adoption of this strategy can be seen as the middle point between the strategies followed by the mixed types and allows to reach the Pareto optimal allocation.

The *night out* example presented above highlights the fact that QV allows any voter to concede on his least preferred issue and whenever that issue is strongly preferred by his opponent, the opponent's will is implemented. It follows immediately that such a voting system Pareto dominates the allocations achieved by MR. Moreover, when we analyse the direct mechanism we prove that QV is not only superior to MR but it reaches *the* optimal allocation. Below we formally characterise the equilibria of the indirect mechanism.

Without loss of generality, we analyse the optimal strategy of voter i whenever he has positive preferences. His payoff is:

$$\left(\frac{1}{2} + \frac{1}{2}\tilde{P}_1\left(v_1^i \mid \theta_1^j < 0\right)\right)\theta_1^i + \left(\frac{1}{2} + \frac{1}{2}\tilde{P}_2\left(v_2^i \mid \theta_2^j < 0\right)\right)\theta_2^i$$

where $v_2^i = V - v_1^i$.

The previous expression captures the property that unanimous preferences are implemented and $\tilde{P}_n(v_n)$ is the expected value of $(2p_n - 1)$ whenever voter i casts v_n votes. They are

defined as follows (conditional probabilities are omitted for notational simplicity):

$$\begin{aligned}\tilde{P}_1(v_1^i) &:= 2 \left(\Pr(v_1^i - |v_1^j| > 0) + \frac{1}{2} \Pr(v_1^i - |v_1^j| = 0) \right) - 1 \\ \tilde{P}_2(v_2^i) &:= 2 \left(\Pr\left(-\left(V - |v_1^j|\right) + v_2^i > 0\right) + \frac{1}{2} \Pr\left(-\left(V - |v_1^j|\right) + v_2^i = 0\right) \right) - 1.\end{aligned}$$

Simple calculations allow us to rewrite the payoff of voter i as

$$\frac{1}{2}\theta_2^i + \left\{ \Pr(v_1^j + v_1^i > 0) + \frac{1}{2} \Pr(v_1^j + v_1^i = 0) \right\} (\theta_1^i - \theta_2^i).$$

Voter i wants to maximise the expression inside the curly brackets whenever $\theta_1^i > \theta_2^i$ (i.e. $v_1^i = V$).¹⁶ Instead, he wants to minimise it when $\theta_1^i < \theta_2^i$ (i.e. $v_1^i = 0^+$). Finally, he is indifferent on which strategy to play whenever he is indifferent between the two issues, $\theta_1^i = \theta_2^i$. In the latter case we assume that he splits evenly his voting power (note that this strategy can be seen as the limiting strategy of non-indifferent players and allows us to achieve the Pareto optimal allocation).

Summing up, the equilibrium strategies for a player with positive preferences are as follows:

$$\begin{aligned}\text{if } \theta_1^i &> \theta_2^i \text{ then } v^i = (V, 0^+) \\ \text{if } \theta_1^i &= \theta_2^i \text{ then } v^i = \left(\frac{V}{2}, \frac{V}{2}\right) \\ \text{if } \theta_1^i &< \theta_2^i \text{ then } v^i = (0^+, V)\end{aligned}$$

Hence, the allocation achieved by QV can be described as follows: whenever voters rank equally both issues or whenever both voters are indifferent, ties occur; instead, if voters rank issues differently, the individual that is not indifferent wins its preferred issue.

3.1.2 The three voters' case

We depart now from a pure conflict resolution situation and consider a setting with three voters. In the previous analysis any voter tried to counteract the votes invested by his opponents —i.e. cast a positive number of votes if the player wishes the approval and negative

¹⁶ v_1^i is equal to V because player i wants to set v_1^i strictly higher (if possible) than the absolute value of his opponent's invested votes on the first issue. Taking into account that player j plays accordingly, it follows that the only equilibrium has non-indifferent players investing all their voting power on their preferred issue.

if he wishes the dismissal. This effect is still in place now but we have an additional element: in some situations some voters may not be pivotal.

In the case of only two voters the tie breaking rule had no welfare effects. Instead, with three voters the tie breaking rule plays a crucial role and has important welfare effects. We will assume that in case of ties issues should be decided through the usual MR. We defer the discussion about the optimal voting rule to the end of this subsection once we have described characterised the equilibrium of the game.

Voters are endowed again with an even number of votes V . Provided the uniform and independent priors it is still a dominant strategy to declare the correct sign of their preferences. We will now focus on *symmetric* pure strategy equilibrium. *Symmetry* should be interpreted as usual in voting theory: the three voters play the same strategy.

We want to focus on the set of final allocations reached in equilibrium rather than the set of different equilibria. For this purpose we introduce the term *essential* as an equivalence class of equilibria that reach the same allocation -notice that given the nature of our game there are many situations where some votes are not pivotal and hence can be placed anywhere without affecting the outcome.

The following Lemma proves first that the strategy followed by any voter is independent of the labelling of the issues. That is, the strategy of a non-indifferent voter is summarised by a parameter $\gamma \in \{0, 1, \dots, V\}$ which should be interpreted (together with the corresponding positive or negative sign) as the number of votes invested in his most preferred issue. The votes invested in his least preferred issue are $(V - \gamma)$ or $(\gamma - V)$ depending on whether he desires the approval or dismissal of it. Secondly, the Lemma proves that in a symmetric equilibrium voters who are indifferent should divide equally their votes.

Lemma 1 *In a setting with two issues, three voters and uniform and independent priors, any symmetric pure strategy equilibrium satisfies the following two properties:*

1. *Non-indifferent voters use **essentially** the same strategy. That is, they invest the same number of votes in their most preferred strategy.*

2. Indifferent voters **essentially** split their votes evenly. That is, they invest $\frac{V}{2}$ votes on each issue

The proof (which is provided in the appendix) mostly relies on showing that an equilibrium with an *unbalanced* behaviour cannot be sustained. Imagine, for instance, that there exists an equilibrium where indifferent voters cast more votes on the first issue: $\gamma_{ind}^* > V/2$. Then any voter is better off by deviating and playing, for instance, the complementary strategy where he invests $\gamma_{ind} = V - \gamma_{ind}^*$ in the first issue. In this way, a voter shifts some votes from the first issue to the second and increases his pivotability.

Given the setting described above, an equilibrium to our game is uniquely defined by a number $\gamma^* \in \{0, \dots, V\}$. The independent and uniform priors imply that the number of votes invested on a high valued issue should be at least as big as the number of votes invested on an issue whenever the voter is indifferent, i.e. $\gamma^* \geq \frac{V}{2}$. The next Proposition tells us which are *essentially* the three equilibria that one can find.

Proposition 1 *In a setting with two issues and three voters, there are **essentially** three symmetric pure strategy equilibria. These are:*

$$\begin{aligned} \gamma^* &= V, & \gamma_{ind}^* &= \frac{V}{2} & \text{-all votes into preferred issue.} \\ \gamma^* &= \frac{V}{2}, & \gamma_{ind}^* &= \frac{V}{2} & \text{-equivalent to MR} \\ \text{for } \theta = \frac{1}{2}: & \gamma^* = \frac{3}{4}V, & \gamma_{ind}^* &= \frac{V}{2} \end{aligned}$$

where γ^* is the number of votes invested by non-indifferent voters in the most preferred issue and γ_{ind}^* is the number of votes invested by indifferent voters in issue one.

The proof of the proposition is quite tedious and is left to the appendix. Its difficulties lie on the *essential* aspect of it. This is because we can devise many possible combinations of votes where no individual is better off by deviating but where some votes are not pivotal and hence can be placed in any of the issues. The fact that these votes are not pivotal implies no changes on the final allocation.

The first equilibrium is the equilibrium we observed in the two voters case where non-indifferent voters invest all their votes in their preferred issue so that strong minorities impose

their will over weak majorities. The second equilibrium replicates the MR allocation. For future reference they will be called Equilibrium QV (EqQV) and Equilibrium MR (EqMR), respectively.

Finally, the third equilibrium can be seen as a mid point between the other two where a member of a majority that feels stronger about the remaining issue just needs an indifferent voter to overcome a strong minority (instead of a voter with strong preferences as would be the case in the EqQV). The non-divisibility of the votes may imply that this equilibrium (and only this one) may not exist. Note that this equilibrium is not very relevant given that it only holds for a particular value of θ .¹⁷

Two relevant aspects are left to be considered. On the one hand, the fact that the Proposition holds for any number of votes indicates that it may also hold whenever we consider votes to be perfectly divisible.¹⁸ On the other hand, the Proposition shows that QV has multiple equilibria and one of them replicates the outcome reached by MR. Henceforth we focus our attention on the first equilibrium. It does not seem worth it to propose a slightly more complicated voting system than the traditional MR if it just replicates the same allocation and introduces no strictly positive gains.¹⁹

The Tie Breaking Rule

We said above that in the three voter's case the tie breaking rule plays a crucial role and has important welfare effects. Consider how pivotal is a voter under MR. Given the uniform priors assumption, a voter observes his will being implemented on any issue with probability $\frac{3}{4}$ since the issue can only be dismissed if the remaining two voters are opposed to him –that event has probability $\frac{1}{4}$. Imagine now, that the tie breaking rule under QV is the toss of a fair coin, i.e. the issue is approved with probability $\frac{1}{2}$. This implies that any voter becomes much less pivotal ($\frac{1}{2} < \frac{3}{4}$) than he was under MR and it can be shown that QV is no longer

¹⁷This equilibrium disappears whenever we consider the continuous valuation of the issues (see Section 3.4). There are two reasons for this to be the case: (1) the relative intensity for which it holds has measure zero in the continuous case (given uniform preferences) and (2) the strategy followed by indifferent players is crucial for this equilibrium to hold and these voters have in general zero measure in the continuous case.

¹⁸In Section 3.4 below, we show that in the case with continuous valuation of issues and perfectly divisible votes, the EqQV and EqMR are the only equilibria.

¹⁹The multiplicity of equilibria when analysing different mechanisms is usually eluded by selecting the best equilibrium in each possible situation. Note that this approach would benefit our analysis because MR would never be able to do better than QV given that the latter also contemplates the allocation reached by the former. Therefore, focussing on the first equilibrium makes our optimality analysis more difficult.

optimal: MR does better.

The optimal tie breaking rule relies on preserving the pivotability of any player in case of ties in an incentive compatible way. In case of ties, issues should be decided through the usual MR. QV becomes a voting rule that allows issues to be decided on the grounds of the total intensity of preferences. In case the intensity of preferences is not decisive, the issue is approved on the basis of overall support (MR). QV happens to be a natural extension of the usual voting rule where voters declare their position with respect to the approval or dismissal of an issue and then invest extra votes to reflect their *willingness to influence*.

3.2 The direct mechanism

We want now to characterise the optimality properties of QV in the previous setting (i.e. two issues and two or three voters). In order to do that we first need to characterise the whole set of implementable mechanisms in our setting.

The *Revelation Principle* allows us, without any loss of generality, to restrict the analysis to the study of *direct revelation mechanisms*. A *direct revelation mechanism* is a function (p) that maps any revelation of the agents types into an allocation. Such mapping is known as a Social Choice Function (SCF).

$$p : \Theta^I \rightarrow \mathcal{X}$$

$$\text{i.e. } p(\theta^1, \dots, \theta^I) = (p_1(\theta^1, \dots, \theta^I), p_2(\theta^1, \dots, \theta^I)).$$

As is standard in the literature, we want to focus our analysis on the set of SCFs that preserve unanimous wills, have no systematic tendency towards the approval or dismissal of any of the issues (neutrality) and treat all individuals in the same manner (anonymity). Moreover, given that we are in a multidimensional setting we want to extend these properties accordingly. On the one hand, we want the SCF to be *neutral across issues* in the sense that it should be invariant with respect to the particular labelling on each of the remaining issues (i.e. the sign on the remaining issues should not affect) and, on the other hand, we want every issue to be treated analogously. It will be useful to define a SCF as being *reasonable* whenever it

satisfies the previous five properties.

Definition 1 A SCF $p : \Theta^I \rightarrow X$ is *reasonable* if and only if it satisfies

1. **Unanimity:** $p_n(\theta^1, \dots, \theta^I) = \begin{cases} 1 & \text{if } \theta_n^i > 0, \forall i = 1 \div I \\ 0 & \text{if } \theta_n^i < 0, \forall i = 1 \div I \end{cases}, \forall n = 1 \div N.$
2. **Anonymity:** $p_n(\theta^1, \dots, \theta^I) = p_n(\theta^{\sigma(1)}, \dots, \theta^{\sigma(I)}), \forall n = 1 \div N, \forall \sigma \in S_I.$
3. **Neutrality:** $p_n(\theta^1, \dots, \theta^I) = 1 - p_n(-\theta^1, \dots, -\theta^I), \forall n = 1 \div N.$
4. **Neutrality across issues:** $\forall n = 1 \div N$ and $\xi_m^j \in \{+1, -1\}, \forall j = 1 \div I, m = 1 \div N$

$$p_n(\theta^1, \dots, \theta^I) = p_n((\xi_1^1 \cdot \theta_1^1, \dots, \theta_n^1, \dots, \xi_N^1 \cdot \theta_N^1), \dots, (\xi_1^I \cdot \theta_1^I, \dots, \theta_n^I, \dots, \xi_N^I \cdot \theta_N^I)).$$
5. **Symmetry across issues:** $\forall n = 1 \div N, \forall \sigma \in S_N,$

$$p_n(\theta^1, \dots, \theta^I) = p_{\sigma(n)}((\theta_{\sigma(1)}^1, \dots, \theta_{\sigma(N)}^1), \dots, (\theta_{\sigma(1)}^I, \dots, \theta_{\sigma(N)}^I)).$$

where S_k denotes the set of all possible permutations of k elements.

It is trivial to check that the set of *reasonable* SCFs that are implementable is not empty. For instance MR is one of them.

3.2.1 Implementable mechanisms

We want to characterise all Bayesian Nash implementable allocations. Thus, we are interested in the SCFs that induce truthful revelation at the interim stage –the point where each agent privately knows his own type (but only holds beliefs on his opponents’ types) and he has to reveal his type in the direct mechanism or cast his votes in the indirect mechanism. The *interim utility* of a voter that declares $\hat{\theta}^i$ while his type is θ^i , is defined as:

$$u(\hat{\theta}^i, \theta^i) := E_{\theta_{-i}} \left\{ u(p(\hat{\theta}^i, \theta^{-i}), \theta^i) \right\}$$

where, $\theta^{-i} := (\theta^1, \dots, \theta^{i-1}, \theta^{i+1}, \dots, \theta^I)$. Note that this is simply his expected utility taking into account that his opponents truthfully reveal their type. To simplify the notation let us

also define the *interim prospect* on issue n as:²⁰

$$P_n(\hat{\theta}^i) := E_{\theta^{-i}} \left\{ 2p_n(\hat{\theta}^i, \theta^{-i}) - 1 \right\}.$$

Hence, the interim utility is: $u(\cdot, \theta) = P_1(\cdot)\theta_1 + P_2(\cdot)\theta_2$.

In order to characterise all implementable SCFs we just need to impose the *Incentive Compatibility* constraints (IC) according to which it should be optimal for each voter to reveal his true type. Restricting the analysis to the set of *reasonable* SCFs, together with the uniform and independent priors we assumed, imply that we just need to analyse the ICs from the perspective of a positive valued issue. That is, we just need to look at the interim prospects of approving the first issue whenever the declarations are $(1, \theta)$, $(\theta, 1)$, $(1, 1)$ and (θ, θ) . The utilities of each of the three types of voter given truthful revelation are:

- A non-indifferent type: $P(1, \theta) \cdot 1 + P(\theta, 1) \cdot \theta$
- A high type: $P(1, 1) \cdot 1 + P(1, 1) \cdot 1$
- A low type: $P(\theta, \theta) \cdot \theta + P(\theta, \theta) \cdot \theta$

The next Proposition tells us which are the conditions that any *reasonable* SCF should satisfy in order to be implementable.

Proposition 2 *A reasonable SCF $p : \Theta^I \rightarrow X$ is implementable if and only if the next four conditions are satisfied*

$$\begin{array}{ll} 1. & P(1, 1) = P(\theta, \theta) \quad 3. & P(1, 1) \geq \frac{P(\theta, 1) + P(1, \theta)}{2} \\ 2. & P(1, \theta) \geq P(\theta, 1) \quad 4. & P(1, 1) \leq \frac{P(\theta, 1)\theta + P(1, \theta)}{1 + \theta}. \end{array}$$

The proof of the Proposition is an immediate consequence of imposing the conditions for truthtelling. For instance, the first condition is a consequence of requiring that a high type does not have an incentive to deviate by declaring he is a low type together with a low type not having incentives to deviate by declaring he is a high type. The rest of the conditions follow from considering the remaining deviations.

²⁰Note that the interim prospect is the expectation of a linear transformation of the SCF, hence it is not a well defined probability. In particular, its domain lies on $[-1, 1]$.

There are a few interesting things to say about the previous result which will be generalised in Section 4.1. First of all, observe that the SCF treats exactly in the same way an enthusiastic and an apathetic voter ($P(1, 1) = P(\theta, \theta)$).²¹ This highlights the fact that the first best allocation (the one that maximizes the sum of ex-ante utilities) can never be achieved since it requires interpersonal comparisons of utility. That is, it requires favouring those voters with stronger preferences and this can never be incentive compatible.

The remaining three conditions imply that the interim utilities should be convex. In particular, they require the interim prospect on an issue to be weakly increasing on the declaration on that issue, i.e. $P(1, 1) > P(\theta, 1)$ and $P(1, \theta) > P(\theta, \theta)$.

Finally, note that the Proposition holds for any number of voters as long as they are deciding over two issues.

3.2.2 Is qualitative voting optimal?

From the viewpoint of the designer of the mechanism it is reasonable to ask if the voting rule he would like to implement is the best one under the “veil of ignorance”. That is, if by weighting all the possible combinations of types (given the prior distributions of them) the voting rule reaches the best possible allocation.

As Holmstrom and Myerson (1983) first pointed out, “the proper object for welfare analysis in an economy with incomplete information is the decision rule, rather than the actual decision or allocation ultimately chosen [...] a decision rule is efficient if and only if no other feasible decision rule can be found that may make some individuals better off without ever making any other individuals worse off.” In our setting this means that we do not have to compare the set of final allocations but the set of implementable mappings from preference

²¹The *symmetry across issues* property plays a relevant role for this result to hold true. The next example shows that dropping such property may be critical in the case with discrete preferences:

There is only one voter ($i = 1$), and there only two issues ($n = 1, 2$). The player's valuation θ_1^1 and θ_2^1 are stochastically independent and uniformly distributed on $\{1, 2\}$. The following SCF is strategy-proof but is not HD0 because it allocates a different outcome to the players (1, 1) and (2, 2):

$$\begin{array}{llll} p_1(1, 1) = 1 & p_2(1, 1) = 0 & p_1(0, 1) = 0 & p_2(0, 1) = 1 \\ p_1(1, 0) = 1 & p_2(1, 0) = 0 & p_1(2, 2) = 0 & p_2(2, 2) = 1 \end{array}$$

I am indebted to Tilman Borgers for bringing this fact to my attention.

In Section 4.1 we show that the “equal treatment of proportional voters” holds in general whenever we have a continuous support.

profiles to allocations (i.e. implementable SCFs). It would be useless to provide a welfare analysis regardless of incentive compatibility because strategic manipulation of privately held information will almost surely lead to a different allocation than the expected one.

Henceforth we adopt the criteria that any optimality analysis is made out of the set of implementable SCFs. We denote this set $\mathcal{P}$ (i.e. $\mathcal{P} := \{p : \Theta^I \rightarrow X : p \text{ is implementable}\}$).

The welfare criteria we are interested in is the set of SCFs that reach a Pareto optimal allocation at the ex-ante stage.²² First, a Definition for the ex-ante utility for voter i given the SCF p :

Definition 2 *Given an implementable SCF, $p \in \mathcal{P}$, $u^i(p) := E_{\theta^i} \{E_{\theta^{-i}} \{u(p(\theta^i, \theta^{-i}), \theta^i)\}\}$*

Definition 3 *An **ex-ante efficient** SCF $p : \Theta^I \rightarrow X$ is an implementable SCF such that there does not exist any other implementable SCF such that makes some voters better off without worsening off any other, i.e.*

$$p \text{ is } \mathbf{ex\text{-}ante \text{ efficient}} \Leftrightarrow \nexists \hat{p} \in \mathcal{P} \text{ such that } u^i(\hat{p}) \geq u^i(p) \text{ for all } i = 1 \div I \\ \text{and } u^i(\hat{p}) > u^i(p) \text{ for some } i \in \{1, \dots, I\}.$$

Definition 4 *A mechanism is said to be **optimal** if its associated direct revelation mechanism is reasonable and ex-ante efficient.*

It is essential to consider SCFs that are ex-ante efficient so that they are stable in the sense that voters will never want to jointly deviate and jointly choose a different decision rule. This argument also holds for the interim stage: we want mechanisms to be robust once agents privately know their types. It can be proved that ex-ante efficiency implies interim efficiency, hence our welfare criteria will also imply the stability of the voting rule at the interim stage.

The *night out* example described above illustrates that MR is in some cases not interim efficient. In that example, John and Anna had incentives to concede on their least preferred issue and both go to the Italian restaurant and the comedy film. It follows that MR is not

²²Our definition of ex-ante efficiency corresponds to the notion of *ex-ante incentive efficient* in Holmstrom and Myerson (1983).

ex-ante efficient and that both friends may unanimously agree on resolving their dissenting issues through alternative methods.

The assumption that the intensity of the preferences towards each issue can only take two values (θ and 1) becomes now crucial. It allows us to write the interim prospects in terms of a finite number of parameters and, given that we restricted the analysis to reasonable SCFs, the number of parameters is treatable. The optimal SCFs are simply those that maximise the ex-ante utility of any single voter subject to the four constraints in Proposition 2. The detailed analysis of the resulting linear program is left to the appendix.

Theorem 1 *In a setting with two issues and two voters, QV is optimal. Moreover, MR is not optimal.*

QV is replicating the only ex-ante efficient and reasonable SCF but it is not the only indirect mechanism that can do so; QV is just one possible alternative and no other mechanism can do better.

In the three voters case we have seen that QV reaches two equilibria: one that replicates the MR outcome and one that allows strong minorities to decide over weak majorities. The next Theorem tells us when is the second equilibria ex-ante efficient.

Theorem 2 *In a setting with two issues and three voters, whenever the values of the various issues are “different enough” (i.e. $\theta \in (0, \frac{1}{3})$), QV is ex-ante optimal. Moreover, in that case MR is not optimal.*

What do we mean by issues being “different enough”? Recall that when we described the simplified model we denoted the relative valuation of a low issue with respect to a high one as θ . The Theorem above is telling us that QV is optimal whenever the valuation of the high issue is at least three times the one of the low issue – $\theta \in (0, \frac{1}{3})$. In other words, it is optimal to implement the will of an enthusiastic minority as long as the majority does not oppose the preference of the minority too strongly – agents want to commit to use such a rule before knowing their preferences so that their possibly strong views are not silenced by indifferent

majorities.²³

The main argument for proposing an alternative voting rule to allow voters to express their *willingness to influence* the final decision implicitly assumed that gains can only be possible as long as voters differed on which issue is the most relevant. Theorem 2 reinforces this idea and shows precisely that the optimality of QV relies now on a particular range of values of the parameter θ in contrast to the case with only two voters.

3.3 Two examples

3.3.1 The two voters' case: *conflict resolution*

A more realistic version of the *night out* example may take the shape of a conflict resolution situation. In this case, two parties that have agreed on all concurring issues are to resolve on some dissenting ones. In this context it seems sensible not to expect the amicable behaviour we observed in the previous example. Now, parties may see any concession as a loss and (given the sequential nature of bargaining) may never truthfully declare their preferred alternatives leading to the deferring of any decision.²⁴

Imagine a family enterprise that, after being badly managed for two generations, is in a very delicate situation and decides to hire a manager or CEO to redirect their business. The new CEO's team carries out a comprehensive analysis of the situation and concludes that the image of the firm has to be updated and two proposals are made. On the one hand a restyling of the logo will change the consumer's perception of their brand at a very low cost. On the other hand, a structural improvement of their main product line would also be beneficial to consumers' perceptions and, furthermore, it will gain the attention of the press.

The owners are against any change in their product because this is, from their point of view, the essence of their business. Similarly, they cannot contemplate a restyling of their logo

²³In the interval $\theta \in (\frac{1}{3}, \frac{1}{2})$ the allocation achieved by the third equilibrium replicates the optimal allocation –note though that the third equilibrium is only exists for $\theta = \frac{1}{2}$. For $\theta \in (\frac{1}{2}, 1)$ MR achieves the optimal allocation. Proofs are provided in the appendix. Note that the costs of the incentive compatibility are captured precisely in the interval $\theta \in (\frac{1}{3}, \frac{1}{2})$: from an ex-ante perspective (and regardless of incentive constraints) it is optimal for a strong minority to decide over a weak majority when $\theta \in (\frac{1}{3}, \frac{1}{2})$.

²⁴The social psychology literature has largely focussed on the problem of people not declaring what they perceive as less important because there exists the risk that they will lose that issue without any compensation. See for instance Rubin *et al* (1986).

because it was designed by one of their ancestors and they feel emotionally attached to it.

The negotiations between both parties are at a deadlock and, as was highlighted before, any concession is seen as a loss. Furthermore, the parties rank the issues differently. The CEO realises that the first policy is interesting given its low costs but it will have no persistent effect on the public and he sees the latter as the essential move to refloat the firm. Instead, the family owners realise that something has to change but would not like to be unfaithful to their ancestor so, above all, want to keep their logo. This is a *Prisoner's Dilemma* situation: whatever the opponent does any voter is always better off by not conceding and declaring both issues to be equally important (it is *dominant* to do so). And, as it is always the case, the unique equilibrium is a Pareto dominated one.

QV allows the voters to unlock the negotiation and non-cooperatively choose the Pareto optimal allocation. Let us analyse its logic: the CEO and the family are endowed with V votes each and invest all votes in their preferred issue. The reason being that, given the binary nature of the situation, winning one issue implies losing the remaining one. Hence, the optimal strategy is to make sure that the most preferred issue is not lost

Note that a particular feature of the conflict resolution situation (voter's preferences are opposed) with two issues is that it is robust to any possible prior in the voters' preferences –i.e. it is dominant for a non-indifferent voter to invest all his voting power on his preferred issue. In other words, Theorem 1 is strategy-proof whenever both voters have opposing preferences.

3.3.2 The three voters' case: *a committee meeting*

Imagine now a religious association which is composed of three factions with the same voting power at the annual committee. In that committee they need to update the association's position in two major biological scientific advances: human cloning and the use of stem cells. Imagine that each of the members of the committee has no clue about their opponents' preferences but privately know their own. The most progressive faction has no strong position on any of the issues but it is mostly in favour of both. Each of the other two strongly opposes one of the two issues and recognises that the positive aspects of the other one outweighs their

moral prejudices and hence favours it. The next diagram captures their position:

	Human cloning	Use of stem cells
F1	<i>agree</i>	<i>agree</i>
F2	<i>strongly disagree</i>	<i>agree</i>
F3	<i>agree</i>	<i>strongly disagree</i>

If they vote through MR, both issues are approved: a weak majority imposes its will over a strong minority. Is that situation *optimal*? We have just shown that from an ex-ante perspective (i.e. before voters know what they are going to vote) the MR outcome may not be optimal. If the difference between the strength of the *strongly disagree* and the *agree* positions is wide enough, it is optimal to allow the enthusiastic minorities decide over the apathetic majorities. QV is again a system where agents are able to increase the probability of winning their preferred issue investing all their votes on that issue.

Following the analysis above, the first faction evenly splits its votes, the second invests all of them in the first issue and the third does the same in the second one (as depicted in the table below).

	Human cloning	Use of stem cells
F1	$\frac{V}{2}$	$\frac{V}{2}$
F2	$-V$	0
F3	0	$-V$

The outcome is now the opposite to the one before, both issues are dismissed and the overall welfare is strictly higher than the one obtained through MR.

3.4 Discussion

The equilibrium of the voting game is not driven by the non-divisibility of points or the binary nature of preferences. Whenever we consider preferences to belong to the interval $[-1, 1]$ with independent and uniform priors, voters still follow the described strategy: they invest all votes in their most preferred issue or, only in the case with three voters, they evenly

split their votes.²⁵

Conversely, all optimality analysis rested heavily on the binary nature of the preferences and the uniform and pairwise independent priors. It seems natural to relax the latter assumptions and check whether the main optimality results are affected by such a change. A more precise knowledge of the opponents' preferences may lead to the non-existence of pure strategy equilibria in the game induced by QV. The intuition is the following. For the voting profiles to be an equilibrium in a complete information framework, no voter should invest a single vote in an issue he is going to lose; consequently, a single vote should be sufficient to win any issue and overcoming the single vote invested by an opponent will occur almost surely.²⁶ Hence, relaxing the priors may lead to some critical problems in the applicability of QV and in its optimality properties.^{27,28}

Briefly, we have seen that more skewed priors may lead to voters becoming more strategic. Consequently, it is more difficult to achieve truthful revelation of preferences and such interaction may outweigh the welfare gains we expect from the use of QV. This contrasts with the behaviour we observe under MR where voters always declare truthfully their type. In other words, MR is robust to any possible specification on the preferences' prior distributions.

It is largely the above observation that leads us to the general analysis of the *intensity problem* in Section 4 –we characterise the implementable SCFs that allow voters to express the intensity of their preferences under any specification of the priors.

There are a couple of aspects related to QV that we should discuss before proceeding to the study of robust mechanisms in *voting games*. On the one hand, we should comment how QV relates to the most usual way political parties express the intensity of their preferences, i.e. logrolling. And, on the other hand, we should also comment on the importance of the agenda

²⁵Formal proofs of these statements can be found in the Appendix.

²⁶In general it is also true that the situation where ties occur in all issues is not an equilibria.

²⁷This may contrast with the intuition derived from Cremer and McLean (1988) that correlation allows the attainment of an efficient allocation. The result does not follow in our setting because correlation enhances the strategic interaction between individuals without introducing *penalties associated with lying* (recall that we are not allowing transfers). Jackson and Sonnenschein (2003) provide an example that illustrates how the correlation on the intensity between the issues affects the gains we expect from linking decisions: perfect positive correlation collapses the problem into a one-dimensional one; conversely, perfect negative correlation is the best possible scenario for QV.

²⁸Milgrom and Weber (1985) ensures the existence of pure strategy equilibrium in the general game with I issues, N players and V votes as long as the informational structure of the game satisfies some reasonable conditions. Whenever we consider the more general case with perfectly divisible votes Simon and Zame (1990) generally characterise the conditions under which equilibria exists.

in our setting.

Logrolling is defined as the exchanging of votes among legislators to achieve the approval or dismissal of the issues that are of interest to one another. Heuristically we could say that QV is related to logrolling in the same way monetary economies are related to barter. It eases the ways through which agents can express their *willingness to influence* given that it does not require a double coincidence of wants. Furthermore, it seems reasonable to expect that this increased freedom in the available strategies should prevent agents trading their votes because under QV a vote for an issue can never be seen as something useless since it can be unilaterally moved to a more relevant issue.

The problem of modelling theoretically such phenomena relies on the fact that it usually occurs in a situation where a certain knowledge of the opponent's preferences exists but there is still scope for the understatement of one's preferences and, of course, the violation of the agreement once it is made.²⁹ The latter can be easily overcome through some kind of reputation argument but the former generates major difficulties and remains an area of interest for future research.

The selection of the agenda is shown to be an important matter that arises when analysing QV and is one of the most important problems that arises in any negotiation. The introduction of a new bill can drastically change the action taken by a particular individual, as is the case with QV. Namely, how, by who and when should the issues be selected? There is a clear incentive to manipulate the agenda in order to induce particular outcomes and bundle issues that benefit some particular groups.³⁰ Nevertheless, the literature lacks tractable models of agenda setting given the somehow dubious knowledge of the opponent's preferences that is needed to correctly manipulate it. In our case, we need to rely on those cases in which the agenda is exogenous (e.g. the goods to be split in a divorce settlement) and also in those situations in which after some unmodelled negotiations an agenda agreed by all voters is

²⁹The lack of a satisfactory theoretical treatment of *logrolling* supports such an assertion. The most relevant work is by Wilson (1969) where agents interact in an exchange economy framework with votes being tradeable and perfectly divisible.

³⁰As an example of the scope of such a problem see Metcalfe (2000). In the context of criminalising bribery at an international level between OECD countries, he shows how the setting of the agenda monopolised the negotiations for twelve years. He also emphasizes the perverse effect that the introduction of a *divisive* issue has in a negotiation: it creates a conflict between two factions that strongly disagree on the outcome of such issue and prevents any agreement being reached on the remaining ones. In a different setting Dutta *et al.* (2003) define and prove the existence of an *equilibrium for agenda formation* when one alternative has to be selected out of many -other studies on agenda setting can be found therein.

reached.

4 The general analysis of *the intensity problem*

All results from the previous section have rested on the assumption of uniform and pairwise independent priors. It seems natural to relax such assumptions and check whether the main optimality results are affected by such change. Section 3.4 above has called attention to the fact that a more precise knowledge of the opponents' preferences may lead to non-existence of equilibrium in the game induced by QV. Relaxing the priors may also lead to some critical problems in the optimality properties of QV –this is indeed a long standing critique to the whole literature on Bayesian Nash implementation.

Driven by the fact that MR induces truthful revelation given any possible specification of the priors, we want to characterise the set of SCFs that are *robust* to any specification of the priors and are also sensitive to the voters' intensity of preferences.³¹

Bergemann and Morris (2004) show that requiring a SCF to be *robust* to any specification of the priors (interim implementation for all possible type spaces) in private value environments is equivalent to *ex-post implementation* and is also equivalent to *dominant strategy implementation* or *strategy-proofness*. Hence in the remaining of the section we use the standard notion of strategy-proof. In a setting with I players and N issues, we generally show that a social choice function is implementable only if it does not undertake interpersonal comparisons of utility (it should only be contingent on the voters' relative valuations between the issues). Following this characterisation we find the impossibility of implementing strategy-proof (or *robust*) mechanisms that are sensitive to the voters' intensities of preferences and satisfy the unanimity property.

The impossibility result is congruent with both literatures on social choice and implementation. The former has exposed the impossibility of producing rational aggregators (in the sense that the social preference relation is transitive) whenever we consider universal preference domains. The latter has shown that the strategic interaction between voters that

³¹A voting game has been defined as a situation where N independent binary decisions have to be made. Trivially, it is always optimal for any voter to truthfully reveal whether he wishes the approval or dismissal of each of the issues. See Dasgupta and Maskin (2003) for a further defense of the robustness of MR in a standard Social Choice framework.

arises from the fact that individual's preferences are not publicly observable also leads to impossibility results (e.g. Gibbard-Satterthwaite Theorem).

At the end of this Section we drop the unanimity requirement and provide an infinite set of SCFs that satisfy some appealing conditions.

4.1 Implementability result

Without any loss of generality we restrict the analysis to the study of *direct revelation mechanisms*. That is, *Social Choice Functions* (SCF) that map any possible preference profile into an allocation.

$$p : \Theta^I \rightarrow \mathcal{X}$$

$$\text{i.e. } p(\theta^1, \dots, \theta^I) = (p_1(\theta^1, \dots, \theta^I), \dots, p_N(\theta^1, \dots, \theta^I)).$$

We want to characterise all feasible allocations under the *universal domain assumption* -the SCFs that induce truthful revelation when $\Theta = \mathbb{R}^N$. It will be useful to define the *prospect* of issue n being approved for the present case of dominant strategy implementation analogously as in the previous section:

$$P_n(\theta) := 2p_n(\theta) - 1, \theta \in \Theta^I$$

Hence, the indirect utility of a type θ^i who declares being $\hat{\theta}^i$ whenever the remaining voters truthfully reveal their type is:

$$\begin{aligned} u(\hat{\theta}^i, \theta) & : = u(p(\hat{\theta}^i, \theta^{-i}), \theta^i) \\ & = \sum_{n=1}^N P_n(\hat{\theta}^i, \theta^{-i}) \cdot \theta_n^i = P(\hat{\theta}^i, \theta^{-i}) \cdot \theta^i \end{aligned}$$

where, $\theta^{-i} := (\theta^1, \dots, \theta^{i-1}, \theta^{i+1}, \dots, \theta^I)$.

Strategy-proof mechanisms are those SCFs that satisfy *Incentive Compatibility* constraints (IC) -i.e. it should be optimal for each voter to reveal his true type given any profile of preferences:

$$\theta^i \in \arg \max_{\hat{\theta}^i \in \Theta} u(\hat{\theta}^i, \theta), \quad \forall i = 1 \div I, \forall \theta \in \Theta^I$$

We define $u^i(\theta) := u(\theta^i, \theta)$ as the utility of voter i in equilibria.

It follows that a strategy-proof SCF needs to satisfy the necessary first and second order conditions for truthtelling for all voters:

$$\begin{cases} \frac{\partial}{\partial \hat{\theta}^i} u(\hat{\theta}^i, \theta) \Big|_{\hat{\theta}^i = \theta^i} = 0 \\ \frac{\partial^2}{\partial (\hat{\theta}^i)^2} u(\hat{\theta}^i, \theta) \Big|_{\hat{\theta}^i = \theta^i} \text{ is negative semidefinite} \\ \text{for } i = 1, \dots, I. \end{cases}$$

The next Proposition is just an extension of the usual technique used in one-dimensional screening problems due to Mirrlees (1971). It is the first step to simplify the first and second order conditions above in order to characterise all the implementable allocations.

Proposition 3 *The SCF $p : \Theta^I \rightarrow \mathcal{X}$ is strategy-proof if and only if the voters' induced utilities are convex and its gradients are equal to the interim prospects.*

$$i.e.: \theta^i \in \arg \max_{\hat{\theta}^i \in \Theta} u(\hat{\theta}^i, \theta) \iff \begin{cases} \nabla_{\theta^i} u^i(\theta) = P(\theta) \text{ for all } \theta \in \Theta^I & (1) \\ u^i \text{ is convex on } \theta^i \in \Theta \text{ for all } \theta \in \Theta^I & (2) \end{cases}$$

where $\nabla_{\theta^i} u^i(\theta) := \left(\frac{\partial u^i(\theta)}{\partial \theta_1^i}, \dots, \frac{\partial u^i(\theta)}{\partial \theta_N^i} \right)$.

Proof. Sufficiency. The envelope Theorem directly implies that $\frac{\partial u(\hat{\theta}, \theta)}{\partial \theta} \Big|_{\hat{\theta}^i = \theta^i} = \nabla_{\theta^i} u^i(\theta) = P^i(\theta)$. Given that the FOC is satisfied for all $\theta \in \Theta^I$ it can be differentiated wrt to θ yielding: $\frac{\partial^2 u(\theta^i, \theta)}{\partial \hat{\theta}^2} + \frac{\partial^2 u(\theta^i, \theta)}{\partial \hat{\theta} \partial \theta} = 0$. The SOC implies that the first matrix is negative semidefinite, hence the second one should be positive semidefinite.

Necessity. One can easily reverse the previous reasoning to get the local conditions. We just need to prove that the conditions are global. A continuously differentiable function $u^i : \Theta^I \mapsto \mathbb{R}$ is convex on θ^i iff $u^i(\theta) \geq u^i(\hat{\theta}) + \nabla_{\theta^i} u^i(\hat{\theta}) (\theta - \hat{\theta})$, $\forall \theta, \hat{\theta} \in \Theta$. Using (1) and

the Definition of $u^i(\theta)$ we can get the global condition:

$$\begin{aligned} u^i(\theta) &\geq u^i(\hat{\theta}) + P(\hat{\theta})(\theta - \hat{\theta}), \forall \theta, \hat{\theta} \in \Theta^I \\ P(\theta) \cdot \theta &\geq P(\hat{\theta}) \cdot \hat{\theta} + P(\hat{\theta}) \cdot (\theta - \hat{\theta}), \forall \theta, \hat{\theta} \in \Theta^I \\ P(\theta) \cdot \theta &\geq P(\hat{\theta}) \cdot \theta, \forall \theta, \hat{\theta} \in \Theta^I \\ u^i(\theta, \theta) &\geq u^i(\hat{\theta}, \theta), \forall \hat{\theta}, \theta \in \Theta^I \end{aligned}$$

■

An analogous result can be found in Rochet and Chone (1998). Note that the convexity condition implies that the prospects are, *ceteris paribus*, weakly increasing in the type of each voter on the relevant issue (i.e. $\frac{\partial P_n(\theta)}{\partial \theta_n^i} \geq 0, \forall n, i$). Moreover, the convexity condition implies that the utility function of each player is differentiable almost for all preference profile. Hence we require no regularity condition on the set of solutions of our problem but instead these are derived from the IC constraints.³²

Note that condition (1) together with the Definition of $u^i(\cdot)$ imply that any implementable SCF should satisfy the following linear first-order partial differential equation:

$$\nabla_{\theta^i} u^i(\theta) \cdot \theta = u^i(\theta).$$

Euler's Theorem implies that the former equality is satisfied if $u^i(\cdot)$ is homogeneous of degree one on $\theta^i \in \Theta$ (HD1, i.e. $u^i(\theta^1, \dots, \lambda \cdot \theta^i, \dots, \theta^N) = \lambda \cdot u^i(\theta)$, $\lambda \in \mathbb{R}, \lambda > 0$). Furthermore, Euler's Theorem on homogeneous functions is invertible, that is, only homogeneous functions of degree one satisfy the equation.³³ The next result follows:

Theorem 3 *The SCF $p : \Theta^I \rightarrow \mathcal{X}$ is strategy-proof if and only if the voters' induced utilities are HD1 and convex on their own preferences. That is, $u^i(\theta)$ is HD1 and convex on θ^i for all $\theta \in \Theta^I$.*

The homogeneity of degree one on the interim utilities implies that the interim prospects

³²See Rochet (1985) for a detailed proof of the fact that implementability implies differentiability for almost all preference profiles. Note that condition (1) in Proposition 3 should be stated in terms of "for almost all $\theta^i \in \Theta^i$ —thus all following results hold only with probability one.

³³See Lemma 4 in the Appendix.

are homogeneous of degree zero (HD0).³⁴ This means that all proportional types are treated equally, are bunched. In other words, the interim prospects are only sensitive to the relative valuation between the issues. The result can be interpreted as implying that there cannot be any direct interpersonal comparison of utilities and any aggregation procedure should be preceded by an intrapersonal one. Intuitively, an apathetic voter and an enthusiastic one are essentially treated in the same manner. This extends the equality argument embedded on any voting game and presented in the introduction: not only it is the case that wealth effects can play no role in a voting game, but neither can the preference endowment of each individual. Whilst the former argument is an axiomatic one (imposed by ethical or practical reasons) the latter is an equilibrium result, a necessary condition for the voting game to be implementable.

Note that we consider a multidimensional mechanism design problem with multilateral asymmetric information without transfers. The main difficulty (and main contribution with respect to the existing literature) lies in the fact that transfers are not allowed. Consequently, we introduce an endogeneity problem in the sense that we can no longer associate a high transfer to a high type declaration in order to induce truthful revelation of the preferences. Therefore, when a voter declares that an issue is highly preferred, the SCF should not only increase the probability of winning that issue but the associated cost should be formulated in terms of a decrease in the probability of him winning any other issue.³⁵ Intuitively, this complicates the analysis. However, as opposed to what one would expect, having no transfers simplifies the analysis because the first order partial differential equation that arises from imposing truthful revelation (IC) is now explicitly solvable.

To illustrate such a property, imagine a setting with only two issues. For a SCF to be implementable it should only depend on the direction of the preference vector and should be invariant to its modulus.³⁶ As a result, we have reduced the dimensionality of our problem to one dimension. Furthermore, if the setting is unidimensional, the HD0 implies that interim

³⁴The partial derivative of a HD1 function is a HD0 function.

³⁵Precisely, this intuition is at the heart of the particular voting rule we described in Section 3: QV endows agents with a given number of votes such that whenever an agent wishes to strengthen his position on a particular issue he does so at the cost of lowering his voting power on the remaining ones.

³⁶Such concept becomes clearer if we consider polar coordinates. In that setting, the interim prospects should only care about the angular coordinate (angular coordinates if the setting has more than two issues) and neglect the radial coordinate.

prospects should be invariant with respect to the intensity of the preferences and should only depend on its sign (i.e. whether the voter wants the approval or the dismissal of the issue).

Hence, the argument usually endorsed by political scientists that “the introduction of an intensity dimension attacks political equality in ways not permissible within the context of democratic theory” is questionable.³⁷ Intensity can be taken into account as long as we broaden the usual limits and we bundle together the voting of more than one issue. In other words, allowing agents to express the intensity of their preferences whenever they vote goes hand-in-hand with the argument of analysing voting games in multidimensional settings.

It is worth saying that Theorem 3 applies to any general setting as long as voters have quasi-linear von Neumann Morgenstern utilities (i.e. cardinal utilities) and there are no transfers. Hence it does apply to electing representatives, allocating private goods to individuals... We may, in each case, add extra feasibility constraints on the sum of probabilities across individuals or issues. Note also that our result is stronger than the standard result that voters’ incentives in any game are not changed if the von Neumann Morgenstern utilities are multiplied by a constant. Indeed it is the case from the IC constraints that the expected utility of a voter when he declares θ^i or $\lambda \cdot \theta^i$ ($\lambda > 0$) coincides, yet this is a weaker statement than requiring the SCF on *every* single issue to remain unchanged when the voter’s declaration is multiplied by a positive scalar.³⁸ Moreover, our result is not solely a necessary condition for implementability but, together with the convexity condition, it is also a characterisation of *all* implementable SCFs.

It is clear from Theorem 3 that increasing the number of issues should relax the implementability constraints which in turn allow us to reinterpret the main result in Jackson and Sonnenschein (2003) - incentive costs diminish as we increase the number of issues we consider and first best can be arbitrarily approached. The HD0 result implies that the SCF can only be sensitive to declarations that have one lower dimension than the preference space. Consequently, the constraints that truthtelling impose on the implementable SCFs are less binding the higher the dimensionality of the preference space; at the limit, these constraints

³⁷Spitz (1984), pg 30.

³⁸It could be the case that the allocation (i.e. the n -dimensional vector of probabilities) changes when we multiply the declaration of player i by a positive scalar though keeping his expected payoff constant. This may ease the achievement of truthtelling of player j and/or could have an effect on the ex-ante total welfare achieved by a particular SCF.

tend not to bind and the first best can be arbitrarily approached.

Theorem 3 also extends naturally to a Bayesian Nash implementation setting. We just need to be aware that the conditions are then imposed on the interim utilities rather than the ex-post ones. It is worth pointing out that in that case the Theorem is general in the sense that it allows for any prior on the opponents preferences. That is, it allows for correlation between issues, individuals, etc. The drawback is that it would be stated in terms of the interim prospects and consequently the necessary and sufficient conditions for a SCF to be implementable critically depend on such priors.

Before we state two immediate consequences of Theorem 3 in the form of corollaries we want to stress the fact that the implementable SCFs have a very *balanced* structure. This is best captured by applying Schwarz's Theorem (the order of the differentiation does not alter the result) to voter i 's induced utilities:

$$\frac{\partial P_n(\theta)}{\partial \theta_m^i} = \frac{\partial P_m(\theta)}{\partial \theta_n^i} \quad \left\{ \begin{array}{l} \forall n, m = 1 \div N \\ \forall i = 1 \div I \\ \forall \theta \in \Theta^I \end{array} \right.$$

That is, the marginal change of issue n 's prospect to a variation on voter i 's preference on issue m should coincide with the correspondent change on issue m 's prospect to a variation on issue n . Note that this should hold for every preference profile and any voter.

Corollary 1 *The utilitarian first best allocation can never be reached.*

The utilitarian first best allocation requires approving an issue whenever the sum of utilities is higher than zero and dismissing it whenever it is lower than zero; needless to say, this requires interpersonal comparisons of utilities thus cannot be truthfully implementable.

Theorem 3 also implies that any voter is indifferent between declaring his own preferences or declaring his own preferences normalised by, say, the L_1 norm (i.e. such that the sum of the absolute value of its components adds up to 1). This line of reasoning leads to the following Corollary.

Corollary 2 *Any strategy-proof mechanism in a multidimensional setting with no transfers can be replicated by a point-voting mechanism where voters are endowed with a given number of votes that can be distributed freely among the issues.*

This last result can be read as a *taxation principle* in our environment. Just in the same way that the *revelation principle* allowed us to restrict our attention to direct revelation mechanisms we can now go back from any direct mechanism to an indirect mechanism where players are endowed with one perfectly divisible point that can be split among the issues.

A way to move from the direct mechanism to This Corollary offers a new explanation for the existence of fiat money in the following sense: any mechanism we can devise in a setting with no monetary transfers can be replicated by a mechanism where we introduce a numeraire that has no value. It has no value, first, because it does not enter the utility function of agents and, second, because it is useless outside the framework where it is defined (i.e. it can only be used to express the voters' preferences in a particular voting game). Hence, the only possible use it may have is on smoothing transactions, on allowing the mechanism to elicit the voters' intensities of preferences when deciding which allocation to implement. Money is in our model a *useless* token that plays three main roles: (1) allows the mechanism to compare the voters' valuation (*unit of account*); (2) allows agents to trade-off their voting power among the issues (if we gave our model a temporal reinterpretation this could be considered the usual *storage of value* property); and (3) allows agents to extract gains from their different relative valuations towards the issues (*medium of exchange*).

The analogy with prices allows to a better understanding of Theorem 3 above. Just in the same way as a consumer requires that his marginal rate of substitution equals the price ratio of goods when he maximizes his utility, the ratio between the allocated votes on each issue should be equal to the relative valuation between them.

4.2 Impossibility result

We have characterised all strategy-proof SCFs as those that induce indirect utilities that are HD1 and convex. Trivially we can see that the set of such functions is not empty. Indeed,

any voting rule that is not sensitive to the voters' intensity of preferences is implementable. MR, dictatorial rules or rules that implement a particular allocation regardless of the voters declared preferences induce truthtelling. Obviously, we would like to impose the minimal requirements on the set of SCFs we want to analyse so as to avoid the latter ill-behaved rules.

In this section we pay no attention to the usual requirements of anonymity and neutrality (i.e. invariance of the mechanism with respect to the labelling of individuals and issues) but, instead, we require unanimous wills to be implemented.

The *unanimity* condition is a very mild requirement but leads to the central result of this section, an impossibility result. This condition is also known in the social choice literature as a weak form of efficiency; it requires an issue to be approved (alt. dismissed) with certainty when all player wish so.

Definition 5 *The SCF $p : \Theta^I \rightarrow \mathcal{X}$ satisfies the unanimity property if*

$$p_n(\theta^1, \dots, \theta^I) = \begin{cases} 1 & \text{if } \theta_n^i > 0, \forall i = 1 \div I \\ 0 & \text{if } \theta_n^i < 0, \forall i = 1 \div I \end{cases}, \forall n = 1 \div N.$$

We also define a mechanism as being *qualitative* whenever it is sensitive to the voters' intensities of preferences –i.e. it implements different allocations when some players vary the intensity of their preferences (but do not vary their wish towards the approval or dismissal of any of the issues). Note, for instance, that MR is not qualitative in the sense that it is only sensitive to the sign of the voters' preferences and it is not sensitive to the particular relative intensities.

Definition 6 *The SCF $p : \Theta^I \rightarrow \mathcal{X}$ is qualitative if there exists two preference profiles $(\theta, \tilde{\theta} \in \Theta^I)$ such that $\text{sign}(\theta) = \text{sign}(\tilde{\theta})$ and $p(\theta) \neq p(\tilde{\theta})$.³⁹*

The following Lemma shows that without loss of generality we can restrict our attention to those qualitative mechanisms that are sensitive to the intensity of the preference of a

³⁹The operator *sign* should be interpreted as a vector of minus ones, zeros and ones according to the sign of each coordinate.

particular voter on a single issue.

Lemma 2 *If $p : \Theta^I \rightarrow \mathcal{X}$ is qualitative then there exists a voter j , an issue m and two preference profiles $(\theta, \tilde{\theta} \in \Theta^I)$ such that*

$$\begin{cases} \exists j, m \text{ such that } \text{sign}(\theta_m^j) = \text{sign}(\tilde{\theta}_m^j) \\ \theta_n^i = \tilde{\theta}_n^i, \forall i \neq j, n \neq m \\ p(\theta) \neq p(\tilde{\theta}) \end{cases}$$

Proof. Consider a qualitative voting mechanism p such that $\text{sign}(\theta) = \text{sign}(\tilde{\theta})$ and $p(\theta) \neq p(\tilde{\theta})$. We now propose an iterative process where we change the initial preference profile θ into $\tilde{\theta}$ by varying at each stage a single value. That is, we vary a particular θ_n^i at each stage.

We know that the allocation achieved by the two preference profiles differs. Therefore, during the iterative process we observe the SCF modifying the implemented allocation at least once. In other words, the allocation will change when we vary the preference of a particular individual on a single issue. ■

The key intuition of this section lies on the fact that any strategy-proof SCF that satisfies the unanimity property needs to be insensitive to the voters' intensities of preferences on those issues where unanimous wills exist. In case the opposite occurs, the SCF can no longer be strategy-proof: any voter has incentives to *save* resources on that issue where unanimous wills exist thus strengthening his position on the remaining issues.

Lemma 3 *If $p : \Theta^I \rightarrow \mathcal{X}$ is strategy-proof and satisfies the unanimity property then, whenever $\text{sign}(\theta_n^i) = \text{sign}(\theta_n^j) \forall i, j = 1 \div I$,*

$$p((\theta_1^i, \dots, \theta_n^i, \dots, \theta_n^i), \theta^{-i}) = p\left(\left(\theta_1^i, \dots, \underset{(n)}{x}, \dots, \theta_n^i\right), \theta^{-i}\right), \forall x \in \mathbb{R} \text{ such that } \text{sign}(x) = \text{sign}(\theta_n^i).$$

Proof. Without loss of generality we assume that at $\tilde{\theta} \in \Theta^I$ all voters wish the approval of issue one, in particular voter i has preferences $\tilde{\theta}_1^i > 0$. Given that there are unanimous wills on issue one, the probability of approving it is one and should not change whenever voter i

slightly varies the strength of his preference towards that issue,

$$\left. \frac{\partial p_1(\theta)}{\partial \theta_1^i} \right|_{\theta=\tilde{\theta}} = 0.$$

Moreover, that probability should still remain unchanged whenever voter i varies his declaration on any of the remaining issue. Hence, given Schwarz's Theorem, we have that the next equalities should hold:

$$\left. \frac{\partial p_1(\theta)}{\partial \theta_n^i} \right|_{\theta=\tilde{\theta}} = \left. \frac{\partial p_n(\theta)}{\partial \theta_1^i} \right|_{\theta=\tilde{\theta}} = 0, \forall n = 2 \div N.$$

Therefore $\left. \frac{\partial p(\theta)}{\partial \theta_1^i} \right|_{\theta=\tilde{\theta}} = (0, \dots, 0)$ for all $\tilde{\theta}_1^i > 0$. ■

The proof is just using extensively the fact that the probability of approving an issue where unanimous wills exist can not change as long as unanimous wills are in place.

The Lemma implies that whenever unanimous wills exist, the implementability conditions should apply to the remaining declarations. That is, if agents are deciding over N issues and there are unanimous wills on one issue then the implementability conditions should apply to the remaining $N - 1$ issues. In particular, if there are two issues and voters unanimously agree on one, no intensities of preferences can be considered at all because the HD0 applies to the single remaining issue.

Consider now a strategy-proof qualitative mechanism that satisfies the unanimity property. It needs to be sensitive to the voters' intensities of preferences for some particular profiles but it cannot be so on those issues where unanimous wills exist. This places a very *asymmetric* behaviour on the sensitiveness to the intensity of the preferences and triggers the fact that such mechanisms cannot be strategy-proof. The next example sheds some light on this reasoning.

Example: Imagine a situation with two issues, two voters and a strategy-proof qualitative mechanism. We can find three positive parameters $(a, b, \alpha > 0)$ such that $\theta = ((a, 1), (-b, -1))$ and $\tilde{\theta} = ((\alpha, 1), (-b, -1))$ implement a different allocation (i.e. $p(\theta) = p(\tilde{\theta})$). Without loss of generality assume that $a > \alpha$ thus $p_1(\theta) > p_1(\tilde{\theta})$ (hence $p_2(\theta) < p_2(\tilde{\theta})$).

Now consider the preference profile $\varphi = ((a, -1), (-b, -1))$. Note that it only changes the sign of player 1's preferences on issue 2. The SCF needs now to satisfy unanimous wills hence $p_2(\varphi) = 0$. The SCF is now left with evaluating the voters' preferences on only one issue, hence it can no longer be sensitive to their intensities. In other words, $p((x, -1), (-y, -1)) = (\bar{p}_1, 0)$ for any $x, y > 0$.

Trivially, for x large enough and $y = b$ we need $\bar{p}_1$ to be at least as high as $p_1(\theta)$ for the SCF to be strategy-proof -i.e. $\bar{p}_1 \geq p_1(\theta)$. Instead, for y large enough and $x = \alpha$ we need $\bar{p}_1$ to be at least as low as $p_1(\tilde{\theta})$ for the SCF to be strategy-proof -i.e. $\bar{p}_1 \leq p_1(\tilde{\theta})$.

Indeed, the condition $p_1(\theta) > p_1(\tilde{\theta})$ tells us that the former two conditions cannot be satisfied.

The proof of the following (impossibility) Theorem is basically an extension of the former example to the general case with I voters and N issues.

Theorem 4 *There exists no strategy-proof qualitative SCF that satisfies the unanimity property.*

Proof. We prove the Theorem by induction on the number of issues.

When $N = 1$ there exists no strategy-proof qualitative mechanism (note that the unidimensional case makes no use of the unanimity property).

Suppose now that there exists no strategy-proof qualitative mechanism for $N = k$ and, instead, suppose the opposite for $N = k + 1$.⁴⁰ That is, imagine that there exists a strategy-proof mechanism that satisfies the unanimity property and is sensitive to the voters intensity of preferences in the $(k + 1)$ -dimensional case:

$$\exists \theta, \varphi \in \Theta \text{ such that } \text{sign}(\theta) = \text{sign}(\varphi) \text{ and } p(\theta) \neq p(\varphi)$$

where no unanimous wills are present in any of the issues of θ or $\tilde{\theta}$.

By Lemma 2 we can assume without loss of generality that θ and φ differ only on the valuation of voter one in the first issue, i.e.

$$\begin{cases} \theta = ((a, \theta_2^1, \dots, \theta_N^1), \dots, \theta^I) \\ \varphi = ((b, \theta_2^1, \dots, \theta_N^1), \dots, \theta^I) \end{cases}$$

⁴⁰The remaining of the proof consists on showing that the proposed SCF cannot be qualitative and strategy proof and satisfy the unanimity property when $N = k + 1$. Thus by contradiction we show that the inductive argument is satisfied.

where $a > b \geq 0$. Strategy-proofness implies that $p_1(\theta) > p_1(\varphi)$.

Define the set of voters that wish the dismissal of the first issue as $J := \{j \in I : \theta_1^j < 0\} \subsetneq I$. This set is not empty given that there are no unanimous wills. Denote $k := \#J$ and $J = \{j_1, \dots, j_k\}$.

We consider now an iterative process on the elements of J similar to the one described in the proof Lemma 2 where we sequentially switch the negative valuations towards the first issue of voters in J into neutral valuations.

Define the following preference profiles where voter j_1 's preference towards the first issue is set to zero:

$$\begin{cases} \bar{\theta} = \left((a, \theta_2^1, \dots, \theta_N^1), \dots, (0, \theta_2^{j_1}, \dots, \theta_N^{j_1}), \dots, \theta^I \right) \\ \bar{\varphi} = \left((b, \theta_2^1, \dots, \theta_N^1), \dots, (0, \theta_2^{j_1}, \dots, \theta_N^{j_1}), \dots, \theta^I \right). \end{cases}$$

Strategy-proofness implies that $p_1(\bar{\theta}) \geq p_1(\theta)$ and $p_1(\bar{\varphi}) \geq p_1(\varphi)$. Three things can happen: $p_1(\bar{\theta}) > p_1(\bar{\varphi})$, $p_1(\bar{\theta}) = p_1(\bar{\varphi}) < 1$ and $p_1(\bar{\theta}) = p_1(\bar{\varphi}) = 1$.

If $p_1(\bar{\theta}) > p_1(\bar{\varphi})$ or $p_1(\bar{\theta}) = p_1(\bar{\varphi}) < 1$ we move into the second stage of the iterative process now with j_2 and starting from the resulting preference profiles $\bar{\theta}$ and $\bar{\varphi}$ from the precedent stage. We keep on repeating the process until we reach the third possible scenario where $p_1(\bar{\theta}) = p_1(\bar{\varphi}) = 1$. The unanimity property ensures that such allocation is achieved and the process should end in at most k ($k < \infty$) stages (say it ends in stage κ).

The fact that $p_1(\theta) > p_1(\varphi)$ implies that $p_1(\bar{\varphi}) > p_1(\varphi)$. Moreover, the inductive hypothesis implies the allocation achieved by $\bar{\theta}$ and $\bar{\varphi}$ coincide with the allocation where the first issue is approved and the intensity of preferences are not taken into account. This is immediate if $k = \kappa$. Instead, if $k < \kappa$ we can finish our iterative process by switching the preference of all voters that wish the dismissal on the first issue and hence end up in a situation where unanimous wills towards the first issue exist (note that the SCF does not change in this process). Hence, by Lemma 3 the achieved allocation is equivalent to one where the dimensionality is reduced in one and hence the inductive hypothesis applies and the SCF can only consider the sign of the preferences.

We now show that there exists a particular preference profile of voter j_κ for which truthtelling

cannot be an equilibria.

Once again, strategy-proofness implies that $p_1(\bar{\varphi}) > p_1(\varphi)$ and that for least one $m \in \{2, \dots, N\}$, one of the two following conditions holds

$$\begin{cases} p_m(\bar{\varphi}) < p_m(\varphi) \text{ and } \theta_m^{j_\kappa} > 0 \\ p_m(\bar{\varphi}) > p_m(\varphi) \text{ and } \theta_m^{j_\kappa} < 0. \end{cases}$$

Imagine $p_m(\bar{\varphi}) < p_m(\varphi)$ and $\theta_m^{j_\kappa} > 0$. We can find preference profiles for voter j_κ for which he has no incentives to truthtell.

Given that the allocation achieved by $\bar{\varphi}$ does not take into account the intensity of the preferences, for any profile of voter j_κ such that the sign of his preferences does not change the SCF implements the same allocation. In particular, the SCF should implement the same allocation for any positive value of voter j_κ 's preference on issue m . Trivially, for a big enough value it cannot be optimal to truthtell given that $p_m(\bar{\varphi}) < p_m(\varphi)$.

An analogous argument applies whenever $p_m(\bar{\varphi}) > p_m(\varphi)$ and $\theta_m^{j_\kappa} < 0$. ■

We have seen that any mechanism that is robust to any possible specification on the priors and satisfies the unanimity property can not take into account the intensity of the voters preferences. This impossibility result is congruent to the social choice literature and, most particularly, the Gibbard-Satterthwaite (G-S) Theorem where we see that the strategic interactions between individuals do not allow to propose mechanisms that are implementable in dominant strategies and satisfy some appealing properties.

A version of the G-S Theorem states that in an election with three or more outcomes and where we assume a universal domain in the voters' preferences, the only strategy-proof and onto SCFs are dictatorial. This result is more restricting than ours because we only claim the impossibility of implementing qualitative mechanisms. The reason why we get a distinct result is because we are implicitly restricting the domain of preferences. This is best captured by considering the following example. Consider a voting game with two issue and map all

possible outcomes into the G-S framework with four alternatives:

$$\left\{ \begin{array}{l} \text{Alternative } A \text{ is defined as approving both issues} \\ \text{Alternative } B \text{ is defined as approving the first issue and denying the second.} \\ \text{Alternative } C \text{ is defined as denying the first issue and approving the second.} \\ \text{Alternative } D \text{ is defined as denying both issues} \end{array} \right.$$

Trivially, not all strict preferences can be assumed in the set of outcomes $\{A, B, C, D\}$ -e.g. the strict preference $A \succ D \succ B \succ C$ can never be observed. Most related to our work, Hylland (1980) proved in an unpublished paper that even in the case of *cardinal* and unrestricted preference profiles the random dictatorship was the only strategy-proof mechanism that satisfied the unanimity property.

Given the impossibility of implementing qualitative mechanisms that are strategy-proof and satisfy the unanimity property we are left with the question of which mechanism may be optimal. May's Theorem (1952) could be extended to our setting if we added a stronger condition than unanimity, namely, *positive responsiveness*.⁴¹ In that case we obtain that the only SCF that is strategy-proof, anonymous, neutral and positive responsive is MR.

In spirit to the literature following Arrow's Impossibility result we should see which ways there are to overcome our result. One way to get through this result is relaxing the equilibrium criteria from dominant strategies to Bayesian Nash. We have seen in Section 3 that this particular line of research proves to be very successful and we can characterise some situations where there exists a very simple mechanism (QV) that allows the expression of interest by the voters and is not only superior to the MR but also optimal.

The second way to provide some positive results consists on not requiring the unanimity property. This immediately implies that we are not able to achieve ex-post efficiency (regardless of Incentive Compatibility), yet we are able to characterise an infinite set of strategy-proof SCFs. The next subsection develops this question.

⁴¹ "By this [*positive responsive*] we mean that if the group decision is indifference or favorable to x , and if the individual preferences remain the same except that a single individual changes in a way favorable to x , then the group decision becomes favorable to x ." May (1952) pg 682.

4.3 A way to overcome the impossibility result

We restrict our analysis to the two voters case to ease the presentation. At the end of the section we present how our main result extends to the general case with I voters.

As indicated above, we now drop the unanimity requirement to avoid the impossibility result. Equivalently we could restrict the domain of preferences such that there are no unanimous wills or assume that there has been a previous stage where all unanimous wills have been implemented.

In order to proceed in a meaningful way and avoid trivial mechanisms such as constant or dictatorial ones we have to require further conditions. Recall that the neutrality condition requires no systematic tendency towards the approval or dismissal of any of the issues ($p_n(\theta^1, \theta^2) = 1 - p_n(-\theta^1, -\theta^2)$, $\forall n = 1 \div N$). In the two voters case, any neutral qualitative mechanism satisfies a rationality constraint in the sense that it never does worst than MR.⁴² Showing the latter follows from the implementability conditions:

$$u(\theta^i, \theta) \geq u(\hat{\theta}^i, \theta) = P(\hat{\theta}^i, \theta^j) \cdot \theta^i, \forall \hat{\theta}^i, \theta^i, \theta^j \in \Theta.$$

Whenever voter i declares the opposite preference than voter j ($-\theta^j$), neutrality implies that the SCF should implement the MR outcome. Thus, for every possible preference profile the allocation implemented by truthfully reporting the type is at least as good as the one achieved by MR.

From the observation above we know that the utility induced by any neutral and implementable SCF should be non-negative since MR never generates a negative payoff. We also know that any implementable SCF generates utilities that are convex and HD1. Hence any neutral and strategy-proof SCF induces a seminorm in the space of preferences of each voter.—i.e. the indirect utilities are functions defined in the space of preferences of each voter that satisfy the nonnegative property ($u(\theta^i, (\theta^i, \theta^j)) \geq 0$), the scaling property ($u(a \cdot \theta^i, (a \cdot \theta^i, \theta^j)) = |a| \cdot u(\theta^i, (\theta^i, \theta^j))$) and the triangular (or subadditive) property ($u(\theta^i, (\theta^i, \theta^j)) + u(\varphi^i, (\varphi^i, \theta^j)) \geq u(\theta^i + \varphi^i, (\theta^i + \varphi^i, \theta^j))$). Hence the pair $(\Theta, u(\cdot, (\cdot, \theta^j)))$

⁴²In the two players scenario MR is analogous to Unanimity: it implements unanimous will when they are in place and ties issues when players have opposing views.

is a seminormed space for all $\theta^j \in \Theta$ (and for all $i, j \in \{1, 2\}, i \neq j$). Guided by this idea and the fact that the implementable SCFs should have a well defined structure as suggested by Schwarz's Theorem, we can propose the following infinite countable set of implementable SCFs:

$$P_n(\theta^1, \theta^2) = \frac{1}{2} \left(\left(\frac{\theta_n^1}{\|\theta^1\|_k} \right)^{k-1} + \left(\frac{\theta_n^2}{\|\theta^2\|_k} \right)^{k-1} \right)$$

where k is any positive even number and $\|\cdot\|_k$ denotes the usual k -norm.⁴³ The prospect of approving any issue is well defined between -1 and 1 . We can extend this set for any real number k greater than one just by carefully adapting the previous formula to avoid complex solutions:

$$P_n(\theta^1, \theta^2) = \frac{1}{2} \left(\text{sign}(\theta_n^1) \cdot \left(\frac{|\theta_n^1|}{\|\theta^1\|_k} \right)^{k-1} + \text{sign}(\theta_n^2) \cdot \left(\frac{|\theta_n^2|}{\|\theta^2\|_k} \right)^{k-1} \right), k > 1. \quad (\Omega)$$

Besides strategy-proofness, this infinite uncountable set of strategy-proof functions satisfy some appealing properties such as neutrality, anonymity, symmetry across issues and neutrality across issues.⁴⁴ It is also interesting to observe that the SCF tends to the MR outcome whenever we let $k \rightarrow 1$ -in that case, the exponent $(k - 1)$ tends to zero thus tends to put equal weight on all issues and voters. Instead, whenever $k \rightarrow \infty$, the SCF tends to be almost equivalent to a SCF that requires voters to rank issues and only uses the information about the highest ranked issue -i.e. puts a weight equal to one on the most preferred issue of any voter and zero on the remaining issues.

There are still a couple of properties worth mentioning. On the one hand, the defined SCFs are *ex-post incentive efficient* in the sense of Holmstrom and Myerson (1983). That is, there is no implementable SCF that makes some voters better off without worsening off some other voters.⁴⁵

On the other hand, we know that any linear combination of strategy-proof mechanisms is also strategy-proof. Moreover, any convex combination of SCFs characterised by (Ω) is also neutral, anonymous, symmetric across issues, neutral across issues and ex-post incentive

⁴³The k -norm on $\mathbb{R}^N$ is defined for any real number $k \geq 1$ as follows: $\|x\|_k = \left(|x_1|^k + \dots + |x_N|^k \right)^{\frac{1}{k}}$.

⁴⁴See Definition 1 for a precise definition of these terms.

⁴⁵Note that the concept of ex-post incentive efficiency differs from the one of ex-post efficiency. As was highlighted above, the sole fact of not satisfying the unanimity property implies that ex-post efficiency can not be achieved.

efficient. This leads us to conjecture that the whole set of SCFs satisfying the previous properties is described by (Ω) . We have found no counterexample nor formal proof of such statement.

In case we proceeded by restricting the set of preferences such that no unanimous wills exist, we should have dropped the coefficient $\frac{1}{2}$ in front of (Ω) given that the value of the term inside the brackets is now included in $[-1, 1]$ instead of $[-2, 2]$.⁴⁶ Finally the analysis extends trivially to a setting with I voters defining:

$$P_n(\theta^1, \dots, \theta^I) = \left(\text{sign}(\theta_n^1) \cdot \left(\frac{|\theta_n^1|}{\|\theta^1\|_k} \right)^{k-1} + \dots + \text{sign}(\theta_n^I) \cdot \left(\frac{|\theta_n^I|}{\|\theta^I\|_k} \right)^{k-1} \right), k > 1.$$

5 Conclusion

In the first half of this article we have proposed an alternative to the usual voting rule which is simple and allows voters to express their *willingness to influence*. A mechanism which seems the most natural extension to MR and that is proved to be not only superior to MR but also a mechanism that achieves the best possible allocation and induces truthful revelation of the voters' preferences in some general settings. Its essence relies on almost allowing for transferable utilities without introducing money; players can freely move their voting power across issues to strengthen their position in some issues. Following our initial quote we have extended the use of purely economic concepts into the political system. Yet, the results in the second half of the paper show that such development is not exempt from difficulties and we have shown that it is impossible to allow the willingness to influence to play a role in general settings where unanimity is satisfied. James Coleman best captured, once again, the rationale behind our reasoning:

“Clearly a system of power that was parallel in all respects to a monetary system could not be devised, because of the different nature of private goods and public policies. Yet it is equally absurd to believe, as the lack of political innovations

⁴⁶In case that we restrict the preferences in the two voters' case so that they have opposing preferences, QV replicates the allocation of the provided SCF in (Ω) when $k \rightarrow \infty$. In particular, QV is strategy-proof and ex-post incentive efficient.

*seems to imply, that political power must be as different from economic power in its organizations as is the case in existing political systems.”*⁴⁷

The main findings of this article can be summarised through its four theorems: (1) QV unlocks conflict resolution situations allowing each of the opponents to trade off their voting power between the various divergent issues; (2) in a situation with more than two voters, QV allows *very* enthusiastic minorities to decide on those issues that the majorities are mostly indifferent towards; (3) whenever a public decision has to be made and no transfers are allowed, only the agents’ relative intensities between the issues can be considered; and (4) there exists no mechanism that sensitive to the voters’ intensity of the preferences, satisfies the unanimity property and is *robust* to any specification of the priors.

The driving force on our results and our main contribution to the existing literature relies on forbidding any kind of transfers between voters. This has been assumed the ground of an *equal* argument so that no endowment effects can ever play a role in voting games. Furthermore, we have extended such a concept when analysing QV by imposing the *anonymity* property: any aggregating device should not benefit any particular individual. Finally, departing from these axiomatic properties we derived an equilibrium result (Theorem 3) that concludes that a further condition of *equality* has to be satisfied: no direct interpersonal comparison of utility can be undertaken. It is not solely because a voter values one issue more strongly that he should be given more voting power on it. In other words, *preference endowments* should not play a role either.

Precisely, the *equality* argument in the three forms expressed above is crucial to ensure the stability of any aggregating mechanism as it is stated in the following quote:

“I do not believe, and I never have believed, that in fact men are necessarily equal or should always be judged as such. But I do believe that, in most cases, political calculations which do not treat them *as if* they were equal are morally revolting.”⁴⁸

Likewise, Dahl (1956) endorses the view that intensity of the voters’ preferences should be taken into account in order to ensure the stability of political institutions.

⁴⁷ Coleman (1970) pg 1082.

⁴⁸ Robbins (1938) pg 635.

Given how the complexity of the problem escalates when we consider more general settings, we are actually working on experimenting with QV in a more complex setting with diverse issues and voters to realise how people may react to different information structures.⁴⁹ It seems sensible to expect that, the more issues or voters, the more dispersed the information about the opponents' preferences will be. Consequently, similar results to the ones stated in this paper should follow. This is congruent with the notion that voters may not be able to react rationally to some complex situations given their lack of time, knowledge or aptitude to do so. Hence we may observe a less strategic misrepresentation of preferences and voters may use their private information almost truthfully. Be that as it may, this and further considerations are left for further analysis.

QV also introduces a new ingredient in the debate around the institutional adjustment that should occur in an extended European Union (EU). "There is a widespread conviction that the system established by the Treaty of Rome cannot function effectively in a Union of 25 to 30 members"⁵⁰. *Logrolling* is a common feature in the present EU where countries (almost publicly) exchange their votes depending on the issue at hand. In an enlarged EU such system can not be efficient and QV could be the answer. Countries would be free to intensify their votes regardless of side agreements and, hence, obscure side payments.

⁴⁹ See Hortala-Vallve (2004).

⁵⁰ <http://europa.eu.int>

6 Appendix

Proof of Part 1 of Lemma 1.

Given the uniform and independent priors we can restrict our attention without loss of generality to voters with positive preferences.

Assume that there is an equilibrium where non-indifferent voters use different strategies. That is, where a voter that prefers the first issue invests γ on his preferred issue and a voter that prefers the second issue invests w on his preferred issue ($\gamma \neq w$). Finally, an indifferent voter invests γ_{ind} on the first issue (without loss of generality we assume that $\gamma_{ind} \geq \frac{V}{2}$). Once again, the described priors imply that $\gamma, w \geq \frac{V}{2}$ and $\gamma \geq \gamma_{ind}, w \geq V - \gamma_{ind}$. We now show that the described equilibrium cannot be so because an indifferent voter always has incentives to deviate.

Any voter can face thirty six possible situations on each issue depending on the strategy played by both his opponents. In some situations the votes cast by his opponents are higher or equal than zero in which case, regardless of his strategy, the issue is approved. Similarly, if the invested votes are smaller or equal than $-V$ the issue is dismissed. The table below depicts such situations with a positive and negative sign, respectively. The remaining cells capture the total number of votes cast by voters two and three:

ISSUE 1						
γ	+	+	+	+	+	+
γ_{ind}	$\gamma_{ind} - \gamma$	+	+	+	+	+
$(V - w)$	$V - w - \gamma$	$V - w - \gamma_{ind}$	+	+	+	+
$-(V - w)$	$-V + w - \gamma$	$-V + w - \gamma_{ind}$	$-2(V - w)$	+	+	+
$-\gamma_{ind}$	-	-	$-V + w - \gamma_{ind}$	$V - w - \gamma_{ind}$	+	+
$-\gamma$	-	-	$-V + w - \gamma$	$V - \gamma - w$	$\gamma_{ind} - \gamma$	+
	$-\gamma$	$-\gamma_{ind}$	$-(V - w)$	$(V - w)$	γ_{ind}	γ

ISSUE 2						
w	+	+	+	+	+	+
$(V - \gamma_{ind})$	$V - w - \gamma_{ind}$	+	+	+	+	+
$(V - \gamma)$	$V - w - \gamma$	$\gamma_{ind} - \gamma$	+	+	+	+
$-(V - \gamma)$	$-V + \gamma - w$	$-2V + \gamma + \gamma_{ind}$	$-2(V - \gamma)$	+	+	+
$-(V - \gamma_{ind})$	$-V + \gamma_{ind} - w$	$-2(V - \gamma_{ind})$	$-2V + \gamma + \gamma_{ind}$	$\gamma_{ind} - \gamma$	+	+
$-w$	-	$-V + \gamma_{ind} - w$	$-V + \gamma - w$	$V - \gamma - w$	$V - w - \gamma_{ind}$	+
	$-w$	$-(V - \gamma_{ind})$	$-(V - \gamma)$	$(V - \gamma)$	$(V - \gamma_{ind})$	w

We can now compute the final allocation in each possible situation whenever voter one follows the three possible strategies. That is, whenever he invests $(\gamma_{ind}, V - \gamma_{ind})$, $(\gamma, V - \gamma)$ or $(V - w, w)$. In

order to compute the expected interim payoffs we define the following parameters:

$$\begin{aligned}
a = 1 & \Leftrightarrow V - \gamma - w + \gamma_{ind} \geq 0 & b = 1 & \Leftrightarrow -2V + 2w + \gamma_{ind} > 0 \\
a = -1 & \Leftrightarrow V - \gamma - w + \gamma_{ind} < 0 & b = -1 & \Leftrightarrow -2V + 2w + \gamma_{ind} \leq 0 \\
\\
c = 1 & \Leftrightarrow 2V - w - 2\gamma_{ind} \geq 0 & d = 1 & \Leftrightarrow 2V - \gamma - w - \gamma_{ind} \geq 0 \\
c = -1 & \Leftrightarrow 2V - w - 2\gamma_{ind} < 0 & d = -1 & \Leftrightarrow 2V - \gamma - w - \gamma_{ind} < 0 \\
\\
e = 1 & \Leftrightarrow -V + 2\gamma - \gamma_{ind} > 0 & B = 1 & \Leftrightarrow -2V + \gamma + 2w > 0 \\
e = -1 & \Leftrightarrow -V + 2\gamma - \gamma_{ind} \leq 0 & B = -1 & \Leftrightarrow -2V + \gamma + 2w \leq 0 \\
\\
C = 1 & \Leftrightarrow 2V - 2\gamma - w \geq 0 \\
C = -1 & \Leftrightarrow 2V - 2\gamma - w < 0
\end{aligned}$$

Weighting each possible situation by its probability⁵¹ we have that the expected payoffs when playing the three possible strategies are

$$\left\{ \begin{array}{l} \Pi(\gamma_{ind}, V - \gamma_{ind}) \cdot 64 = 58 + 2a + b + 4c + 2d + e \\ \Pi(\gamma, V - \gamma) \cdot 64 = 60 - 4a + 4d - 3e + B + 2C \\ \text{and} \\ \Pi(w, V - w) \cdot 64 = 63 + 4a - 4b - 4c - 4d - 2B - C \end{array} \right.$$

Now we just need to consider all possible combinations of parameters to realise whether it is strictly better to deviate. The following inequalities show that not all parameter combinations are possible⁵²

$$\begin{aligned}
\underbrace{V - \gamma - w + \gamma_{ind}}_a &\geq \underbrace{2V - 2\gamma - w}_C \geq \underbrace{2V - \gamma - w - \gamma_{ind}}_d \geq \underbrace{2V - w - 2\gamma_{ind}}_c \\
\underbrace{-2V + \gamma + 2w}_B &\geq \underbrace{-2V + 2w + \gamma_{ind}}_b
\end{aligned}$$

Whenever $a = -1$ an indifferent voter is strictly better off by playing $(\gamma, V - \gamma)$. Hence, for the proposed strategies to be an equilibrium a should be equal to one.

Repeating the previous reasoning for $d = -1$ we can also see that an indifferent voter has incentives to deviate by playing $(V - w, w)$. Thus, $C = d = 1$.

Now assume that $c = -1$. In that case, the expected interim payoffs are equal to $\Pi(\gamma_{ind}, V - \gamma_{ind}) \cdot 64 = 58 + b + e$ and $\Pi(w, V - w) \cdot 64 = 66 - 4b - 2B$. Note that it is not strictly better to deviate only when $b = e = B = 1$. It can be easily shown that $d = 1$ and $b = 1$ imply that $w > \gamma$, but $d = 1$ and $e = 1$ imply that $V + \gamma - w - 2\gamma_{ind} > 0$. The latter inequality cannot hold when $w > \gamma$. Hence,

⁵¹ Given the uniform and independent priors, all columns (alternatively rows) occur with probability $\frac{1}{8}$ except columns two and five which occur with probability $\frac{1}{4}$.

⁵² For instance, $a = -1 \Rightarrow C = d = c = -1$.

in equilibrium, $a = C = d = c = 1$.

Suppose now that $B = 1$. First note that in that situation e should be equal to -1 because $e = 1$ implies (together with $d = 1$) that $w > \gamma$ and this is not compatible with $V + \gamma - w - 2\gamma_{ind} > 0$ (this inequality results from combining $C = 1$ and $B = 1$). Thus $e = -1$. Nevertheless, in that situation a non-indifferent voter that prefers issue 2 is better off by deviating and playing $(\gamma_{ind}, V - \gamma_{ind})$. Hence, in equilibrium $B = b = -1$.

$e = 1$ implies, as before, that a non-indifferent voter that prefers issue 2 is better off by deviating and playing $(\gamma_{ind}, V - \gamma_{ind})$. And finally, $e = -1$ achieves an allocation which is identical to have all voters splitting evenly their voting power (hence it is essentially a situation where $\gamma = w = \gamma_{ind}$ and all values are close enough to $\frac{V}{2}$, i.e. $2V - 3\gamma \geq 0$) ■

Proof of Part 2 of Lemma 1.

The proof is analogous to the previous one. Assume that there is an equilibrium (γ, γ_{ind}) such that indifferent voters do not evenly split their voting power. That is, such that it reaches a different allocation to $(\gamma, \frac{V}{2})$. Without loss of generality we assume that $\gamma_{ind} > \frac{V}{2}$. Given that the only equilibrium with $\gamma = \gamma_{ind}$ is $(\frac{V}{2}, \frac{V}{2})$ we have that $\gamma > \gamma_{ind} > \frac{V}{2}$. As before, the uniform and independent priors allow us to do our analysis from the perspective of voter one and we assume that he has positive preferences (i.e. he desires the approval of both issues).

The table below depicts the thirty six possible situations that a voter can face on each issue depending on the strategy played by both his opponents.

ISSUE 1						
γ	+	+	+	+	+	+
γ_{ind}	$\gamma_{ind} - \gamma$	+	+	+	+	+
$(V - \gamma)$	$V - 2\gamma$	$V - \gamma - \gamma_{ind}$	+	+	+	+
$-(V - \gamma)$	-	$-V + \gamma - \gamma_{ind}$	$-2(V - \gamma)$	+	+	+
$-\gamma_{ind}$	-	-	$-V + \gamma - \gamma_{ind}$	$V - \gamma - \gamma_{ind}$	+	+
$-\gamma$	-	-	-	$V - 2\gamma$	$\gamma_{ind} - \gamma$	+
	$-\gamma$	$-\gamma_{ind}$	$-(V - \gamma)$	$(V - \gamma)$	γ_{ind}	γ
ISSUE 2						
γ	+	+	+	+	+	+
$(V - \gamma_{ind})$	$V - \gamma - \gamma_{ind}$	+	+	+	+	+
$(V - \gamma)$	$V - 2\gamma$	$\gamma_{ind} - \gamma$	+	+	+	+
$-(V - \gamma)$	-	$-2V + \gamma + \gamma_{ind}$	$-2(V - \gamma)$	+	+	+
$-(V - \gamma_{ind})$	-	$-2(V - \gamma_{ind})$	$-2V + \gamma + \gamma_{ind}$	$\gamma_{ind} - \gamma$	+	+
$-\gamma$	-	-	-	$V - 2\gamma$	$V - \gamma - \gamma_{ind}$	+
	$-\gamma$	$-(V - \gamma_{ind})$	$-(V - \gamma)$	$(V - \gamma)$	$(V - \gamma_{ind})$	γ

We can now compute the final allocation in each possible situation whenever voter one follows the proposed strategy and whenever he unilaterally deviates and invests $(V - \gamma_{ind})$ votes in the first issue. As noted in the main text, we want to consider a deviation where voter one, realising that both his opponents invest more voting power on the first issue, deviates and casts more votes on the second one. Furthermore, the considered deviation does not change his payoff when he faces non-indifferent voters. In order to compute the expected interim payoffs we define the following parameters:

$$\begin{aligned}
a = 1 &\Leftrightarrow V - 2\gamma + \gamma_{ind} \geq 0 & c = 1 &\Leftrightarrow 2V - \gamma - 2\gamma_{ind} \geq 0 \\
a = -1 &\Leftrightarrow V - 2\gamma + \gamma_{ind} < 0 & c = -1 &\Leftrightarrow 2V - \gamma - 2\gamma_{ind} < 0 \\
b = 1 &\Leftrightarrow -2V + 2\gamma + \gamma_{ind} > 0 & d = 1 &\Leftrightarrow -2V + 3\gamma_{ind} \leq 0 \\
b = -1 &\Leftrightarrow -2V + 2\gamma + \gamma_{ind} \leq 0 & d = -1 &\Leftrightarrow -2V + 3\gamma_{ind} > 0
\end{aligned}$$

Weighting each possible situation by its probability we have that the expected payoffs when non-deviating and deviating are respectively

$$\left\{ \begin{array}{l} \Pi := \frac{1}{64} [(27 - 2b + 4c - a) + (31 + 2a + b)] \\ \text{and} \\ \Pi_d := \Pi + \frac{1}{64} [8 - 4c + 4d]. \end{array} \right.$$

Now we just need to consider all possible combinations of parameters to realise whether it is strictly better to deviate.

Whenever $d = 1$ or $c = -1$ it is strictly better to deviate. Instead, when $d = -1$ and $c = 1$ both strategies yield the same expected payoff. Nevertheless in that situation some of the hypotheses are violated (for $b = -a = 1$ and $b = -a = -1$, (γ, γ_{ind}) is *essentially* equal to $(\gamma, \frac{V}{2})$; instead, when $a = b = 1$, (γ, γ_{ind}) does not constitute an equilibrium because a non-indifferent voter that prefers issue one is strictly better off playing γ_{ind} votes on the first issue; finally, the case $a = b = -1$ can never happen). ■

Proof of Proposition 1.

Given that indifferent voters invest $\frac{V}{2}$ votes in each issue we have that all possible combinations of cast votes in any of the issues by two voters that follow the strategy $(\gamma, \frac{V}{2})$ are depicted in the matrix below:

γ	+	+	+	+	+	+
$\frac{V}{2}$	$\frac{V}{2} - \gamma$	+	+	+	+	+
$(V - \gamma)$	$V - 2\gamma$	$\frac{V}{2} - \gamma$	+	+	+	+
$-(V - \gamma)$	-	$-\frac{3}{2}V + \gamma$	$-2(V - \gamma)$	+	+	+
$-\frac{V}{2}$	-	-	$-\frac{3}{2}V + \gamma$	$\frac{V}{2} - \gamma$	+	+
$-\gamma$	-	-	-	$V - 2\gamma$	$\frac{V}{2} - \gamma$	+
	$-\gamma$	$-\frac{V}{2}$	$-(V - \gamma)$	$(V - \gamma)$	$\frac{V}{2}$	γ

As we did before, we define the following four parameters:

$$\begin{aligned}
a = 1 & \Leftrightarrow \bar{v} \geq 2\gamma - V & c = 1 & \Leftrightarrow \bar{v} > \frac{3}{2}V - \gamma \\
a = -1 & \Leftrightarrow \bar{v} < 2\gamma - V & c = -1 & \Leftrightarrow \bar{v} \leq \frac{3}{2}V - \gamma \\
b = 1 & \Leftrightarrow \bar{v} \geq \gamma - \frac{V}{2} & d = 1 & \Leftrightarrow \bar{v} > 2V - 2\gamma \\
b = -1 & \Leftrightarrow \bar{v} < \gamma - \frac{V}{2} & d = -1 & \Leftrightarrow \bar{v} \leq 2V - 2\gamma
\end{aligned}$$

where $\bar{v}$ indicates the number of votes invested in issue one by the remaining voter. Without loss of generality we assume that this voter has positive preferences and strictly prefers the first issue.

$(\gamma, \frac{V}{2})$ is an equilibrium if and only if it is optimal for the remaining voter to invest exactly γ votes on the first issue (i.e. $\bar{v} = \gamma$ should be optimal).

The way to proceed is to define all possible cases so that the conditions that define the four parameters are well ordered. For instance, whenever $\gamma > \frac{5}{6}V$ we have that $0 \leq 2V - 2\gamma \leq \gamma - \frac{V}{2} \leq \frac{3}{2}V - \gamma \leq 2\gamma - V \leq V$ and it can easily be shown that $\bar{v} = \gamma$ is an optimal response for voter one. Hence, $(\gamma, \frac{V}{2})$ is a symmetric equilibrium as long as $\gamma \in (\frac{5}{6}V, V]$. This set of equilibria are *essentially* identical to $(V, \frac{V}{2})$.

A further analysis shows that there exists no symmetric equilibrium where $\gamma \in (\frac{3}{4}V, \frac{5}{6}V]$. The case in which $\gamma = \frac{3}{4}V$ implies that $0 < \gamma - \frac{V}{2} < 2V - 2\gamma = 2\gamma - V < \frac{3}{2}V - \gamma < V$ and a symmetric equilibrium can be sustained if and only if $\theta = \frac{1}{2}$. If $\theta < \frac{1}{2}$, voter one prefers investing more voting power on his preferred issue and, inversely, he prefers to split his votes more equally whenever $\theta > \frac{1}{2}$. Hence we conclude that $(\frac{3}{4}V, \frac{V}{2})$ is an equilibrium if and only if $\theta = \frac{1}{2}$. Moreover, that equilibrium can be sustained by any $\gamma \in (\frac{2}{3}V, \frac{3}{4}V]$.

Finally, $\gamma \in (\frac{V}{2}, \frac{2}{3}V)$ can constitute a symmetric equilibrium only when $\theta \geq \frac{1}{2}$; when $\theta < \frac{1}{2}$, a non-indifferent voter knows that by deviating and investing all of his voting power on his preferred issue he gains that issue when he is confronted with an indifferent voter and a low one (instead he loses it if he invests γ votes). This equilibrium reaches the same allocation as MR. In fact, $(\frac{V}{2}, \frac{V}{2})$ is trivially an equilibrium for any θ because any voter is equally pivotal with any number of votes (in particular with $\gamma = \frac{V}{2}$). ■

Proof of Theorem 1.

Any direct mechanism is defined by 512 parameters. That is, all possible combinations of both voters' types multiplied by the number of issues we are considering. Restricting the analysis to *reasonable* SCFs renders the problem tractable and simplifies the analysis into six parameters; we need to define the SCF only on a particular issue when both voters' preferences on that issue are opposed and this can be done regardless of the sign of the remaining issue.

More precisely, the *neutrality* property defines the value of the SCF whenever voters have analogous

preferences (i.e. whenever both voters coincide on how strongly they prefer each issue) and allows us to focus on positively valued issues (when the agent we analyse wants the approval of the issue). The *symmetry across issues* allows us to focus on a particular issue (say, issue one) and the *neutrality across issues* property reduces the possible types we have to analyse to four because the SCF has to be invariant with respect to the sign of the remaining issue. Finally, *unanimity* implies that we only have to consider the cases when the opponent wants the dismissal of issue one. The next table depicts the six parameters that uniquely define any SCF given the properties above:

$(1, \theta)$	$\frac{1}{2}$	A	B	C
$(\theta, 1)$		$\frac{1}{2}$	D	E
$(1, 1)$			$\frac{1}{2}$	F
(θ, θ)				$\frac{1}{2}$
	$(-1, \theta)$	$(-\theta, 1)$	$(-1, 1)$	$(-\theta, \theta)$

Note that these parameters are probabilities of approving an issue, hence they lie in the interval $[0, 1]$.

We define the interim prospects given the four possible declarations as $P(1, \theta)$, $P(\theta, 1)$, $P(1, 1)$ and $P(\theta, \theta)$. For instance, $P(1, \theta) = 2 \left\{ E_{\tilde{\theta}} \left(p \left((1, \theta), (\tilde{\theta}) \right) \right) \right\} - 1 = 2 \cdot \frac{1}{8} \left\{ \frac{1}{2} + A + B + C + 4 \right\} - 1$. The optimal (*reasonable* and ex-ante efficient) SCF is the one that maximises the ex-ante utility subject to the truthtelling constraints (Proposition 2) and the feasibility ones (the six parameters need to belong to the interval $[0, 1]$). The program reads as follows

$$\begin{aligned} & \max_{A, B, C, D, E, F \in [0, 1]} u^i(p) = 8[3 + A + C - D + F + (4 - A - B + E + F)\theta] \\ \text{subject to } & \begin{cases} \mathbf{1.} & -B + C - D + E + 2F - 1 = 0 \\ \mathbf{2.} & 2A + B + C - D - E - 1 \geq 0 \\ \mathbf{3.} & -6B - 2C - 6D - 2E + 4F + 6 \geq 0 \\ \mathbf{4.} & -A - 2B - C - D + F + 2 + (A - B - 2D - E + F + 1)\theta \leq 0 \end{cases} \end{aligned}$$

Solving this linear program we get that $A = C = B = 1, D = E = 0$ and $F = 1/2$. Note that this allocation is the same than the one achieved by QV, hence QV is optimal.⁵³ ■

Proof of Theorem 2.

Any direct mechanism is now defined by 8192 parameters. Restricting the analysis to *reasonable* SCFs renders the problem tractable and simplifies the analysis into 44 parameters belonging to the interval $[0, 1]$. The following tables define such parameters depending on the preferences of each individual. Note that given that we have three voters the final allocation should be a three dimensional table. Hence, in order to depict it we provide four tables each one corresponding to a different preference

⁵³Note that IC implies that players that are indifferent between the issues should be treated analogously at the interim stage whether they hold strong or weak preferences. We have now proved that this is not only the case at the interim stage but also at the ex-post stage. In other words, the optimal implementable SCF does not undertake ex-post interpersonal comparisons of utility.

profile of voter one (as we assume throughout, voter one has positive preferences towards both issues).

$\theta^1 = (1, \theta),$	$(1, \theta)$	A	B	C	D	1	1	1	1
	$(\theta, 1)$	E	F	G	H	1	1	1	1
	$(1, 1)$	I	J	K	L	1	1	1	1
	(θ, θ)	M	N	O	P	1	1	1	1
	$(-\theta, \theta)$	1-M	Q	R	S	P	L	H	D
	$(-1, 1)$	1-I	T	U	R	O	K	G	C
	$(-\theta, 1)$	1-E	V	T	O	N	J	F	B
	$(-1, \theta)$	1-A	1-E	1-I	1-M	M	I	E	A
		$(-1, \theta)$	$(-\theta, 1)$	$(-1, 1)$	$(-\theta, \theta)$	(θ, θ)	$(1, 1)$	$(\theta, 1)$	$(1, \theta)$
$\theta^1 = (\theta, 1),$	$(1, \theta)$	E	F	G	H	1	1	1	1
	$(\theta, 1)$	1-V	a	b	c	1	1	1	1
	$(1, 1)$	1-T	d	e	f	1	1	1	1
	(θ, θ)	1-Q	g	h	i	1	1	1	1
	$(-\theta, \theta)$	1-N	1-g	j	k	i	f	c	H
	$(-1, 1)$	1-J	1-d	l	j	h	e	b	G
	$(-\theta, 1)$	1-F	1-a	1-d	1-g	g	d	a	F
	$(-1, \theta)$	1-B	1-F	1-J	1-N	1-Q	1-T	1-V	E
		$(-1, \theta)$	$(-\theta, 1)$	$(-1, 1)$	$(-\theta, \theta)$	(θ, θ)	$(1, 1)$	$(\theta, 1)$	$(1, \theta)$
$\theta^1 = (1, 1),$	$(1, \theta)$	I	J	K	L	1	1	1	1
	$(\theta, 1)$	1-T	d	e	f	1	1	1	1
	$(1, 1)$	1-U	1-l	n	o	1	1	1	1
	(θ, θ)	1-R	1-j	p	q	1	1	1	1
	$(-\theta, \theta)$	1-O	1-h	1-p	r	q	o	f	L
	$(-1, 1)$	1-K	1-e	1-n	1-p	p	n	e	K
	$(-\theta, 1)$	1-G	1-b	1-e	1-h	1-j	1-l	d	J
	$(-1, \theta)$	1-C	1-G	1-K	1-O	1-R	1-U	1-T	I
		$(-1, \theta)$	$(-\theta, 1)$	$(-1, 1)$	$(-\theta, \theta)$	(θ, θ)	$(1, 1)$	$(\theta, 1)$	$(1, \theta)$
$\theta^1 = (\theta, \theta),$	$(1, \theta)$	M	N	O	P	1	1	1	1
	$(\theta, 1)$	1-Q	g	h	i	1	1	1	1
	$(1, 1)$	1-R	1-j	p	q	1	1	1	1
	(θ, θ)	1-S	1-k	1-r	s	1	1	1	1
	$(-\theta, \theta)$	1-P	1-i	1-q	1-s	s	q	i	P
	$(-1, 1)$	1-L	1-f	1-o	1-q	1-r	p	h	O
	$(-\theta, 1)$	1-H	1-c	1-f	1-i	1-k	1-j	g	N
	$(-1, \theta)$	1-D	1-H	1-L	1-P	1-S	1-R	1-Q	M
		$(-1, \theta)$	$(-\theta, 1)$	$(-1, 1)$	$(-\theta, \theta)$	(θ, θ)	$(1, 1)$	$(\theta, 1)$	$(1, \theta)$

Similarly to the proof of Theorem 1, we just need to compute the interim prospects in terms of these parameters and maximise the ex-ante utility of any of the voters subject to the truthtelling constraints. The interim prospects are proportional to:

$$\begin{aligned}
P(1, \theta) &= -9 + A + 2D + 2B + 2C + 2F + 2G + 2H + 2J + 2K + \\
&\quad + 2L + 2N + 2O + 2P + 2Q + 2R + S + 2T + U + V. \\
P(\theta, 1) &= 2 + 2E - B + 2G + 2H - 2J - 2N - 2Q - 2T - 2V + \\
&\quad + 2i + a + 2b + 2c + 2f + 2h + 2j + l + k + 2e. \\
P(1, 1) &= 9 + 2I + 2J + 2L - 2T + 2d + 2f - 2U - 2l + n + \\
&\quad + 2o - 2R - 2j + 2q - 2O - 2h - 2G - C - b + r. \\
P(\theta, \theta) &= 12 - 2L - 2f - 2R - 2j + 2O - D - 2H - 2Q - 2S + \\
&\quad + 2h + 2N - 2k + 2g - c - o + 2p + 2M + s - 2r.
\end{aligned}$$

The optimal (*reasonable* and ex-ante efficient) SCF is the one that maximizes the ex-ante expected utility subject to the truthtelling constraints and the feasibility ones (i.e. the forty parameters need to belong to the interval $[0, 1]$).

$$\text{subject to } \begin{cases} \max u^i(p) \\ P(1, 1) = P(\theta, \theta) \\ P(1, \theta) \geq P(\theta, 1) \\ P(1, 1) \geq \frac{P(\theta, 1) + P(1, \theta)}{2} \\ P(1, 1) \leq \frac{P(\theta, 1)\underline{\theta} + P(1, \theta)\bar{\theta}}{\bar{\theta} + \underline{\theta}}. \end{cases}$$

The end of the proof relies on writing the program in terms of the forty parameters and then, step by step, assuming whether or not any of the constraints is binding. Once this is done we are just left with some tedious (though trivial) linear programs. And it can be proved that for different values of θ the corner solution varies. More specifically, all parameters are equal to one except those specified below:

- $\theta \in (0, \frac{1}{3})$: $R = S = U = b = c = j = k = l = 0$.
- $\theta \in (\frac{1}{3}, \frac{1}{2})$: $Q = R = S = T = U = j = k = l = r = 0$.
- $\theta \in (\frac{1}{2}, 1)$: $Q = R = S = T = U = V = j = k = l = r = 0$.

A proper analysis of such allocations tells us that they coincide with the allocations achieved by the strategies where a non-indifferent voter invests V , $\frac{3}{4}V$ and $\frac{V}{2}$ votes on his preferred issue, respectively.

■

The equilibria with continuous preferences and divisible votes (2 players).

We restrict the analysis to pure strategy equilibrium. Theorem 3 in Section 4.1 tell us that the optimal strategy is only contingent on the relative intensities of the preferences and, moreover, it is well behaved (monotonic) with respect to them. In order to simplify the analysis we assume a uniform

distribution on the relative intensities rather than on the preferences themselves i.e.

$$\theta_n^i \in \{\pm 1, \pm \theta\} : \begin{cases} \Pr \{|\theta_1^i| > |\theta_2^i|\} = \Pr \{\theta_n^i > 0\} = \frac{1}{2} \\ \theta \sim U[0, 1] \\ \text{Pairwise independence across issues and voters.} \end{cases}$$

We analyse the equilibrium from the perspective of a voter with positive preferences. The interim expected payoff of voter i when he invests $v^i \in [0, V] \subset \mathbb{R}$ votes on the first issue is:

$$\begin{aligned} & \tilde{P}_1(v^i) \cdot \theta_1^i + \tilde{P}_2(V - v^i) \cdot \theta_2^i \\ \text{where, } & \begin{cases} \tilde{P}_1(v^i) = \Pr(v^i + v^j > 0 \mid \theta_1^j < 0) + \frac{1}{2} \Pr(v^i + v^j = 0 \mid \theta_1^j < 0) \\ \tilde{P}_2(1 - v^i) = \Pr(v^i + v^j < 0 \mid \theta_2^j < 0) + \frac{1}{2} \Pr(v^i + v^j = 0 \mid \theta_2^j < 0) \end{cases} \end{aligned}$$

Simple calculations allow us to rewrite the interim expected payoff of voter i as:⁵⁴

$$\frac{1}{2} \theta_2^i + \left\{ \Pr(v^i + v^j > 0) + \frac{1}{2} \Pr(v^i + v^j = 0) \right\} \cdot (\theta_1^i - \theta_2^i).$$

Hence, an indifferent voter is indifferent between playing any of the strategies (as was done in the binary case, we assume that he plays the undominated strategy $v^i = \frac{V}{2}$) and a non-indifferent voter (say he prefers issue one) wants to maximise the expression inside the curly brackets. In the case where $v^j(\cdot)$ induces an atomless distribution on $[0, V]$ it is dominant for voter one to set $v^i = V$. Otherwise, if the induced distribution on the invested votes by voter j on issue one is not atomless, v^i will always be strictly higher (if possible) than the absolute value of the lowest possible value of v^j . Thus, the only equilibrium has non-indifferent voters investing all their voting power on their preferred issue.

Finally note that the proof can also be applied to the case of continuous preferences and non-divisible votes. We just need to restrict the set of strategies of voter i . ■

The equilibria with continuous preferences and divisible votes (3 players).

The setting is analogous to the one described in the proof above. We just need to add the restriction that we focus our analysis on *symmetric* equilibrium (i.e. the three voters play the same strategy) and (as was done in Section 4.2) we further assume that voters behave equivalently regardless of the labelling or the sign of the issue.

This proof is a bit more complicated than the one above because now we need to consider whether each of them is in favour or against the approval of each of the issues in order to assign the appropriate sign to the cast votes. Once we take this into account we have that the interim prospects read as

⁵⁴ Conditional probabilities are omitted for notational simplicity.

follows ($v^j, v^k \geq 0$):

- $\tilde{P}_1(v^i) = \frac{1}{2} \Pr(v^j + v^k < v^i \mid \theta_1^j, \theta_1^k < 0) + \Pr(v^j - v^k \leq v^i \mid \theta_1^j, -\theta_1^k < 0) - \frac{1}{2}$
- $\tilde{P}_2(1 - v^i) = \frac{1}{2} \Pr(v^j + v^k > V + v^i \mid \theta_1^j, \theta_1^k < 0) + \Pr(v^j - v^k \leq V - v^i \mid \theta_1^j, -\theta_1^k < 0) - \frac{1}{2}.$

Note that the tie breaking rule is now playing a role because voter i just needs to equate the sum of his opponents votes whenever only one of them desires the dismissal of the issue. Given the assumption that voters play equivalently regardless of the sign of his preferences we have that v^j and $(1 - v^j)$ have the same induced distribution (the same can be said about voter k 's strategy). That implies that v^j is symmetrically distributed around $\frac{V}{2}$. In order to simplify the notation we define $X := v^j + v^k$ (which, accordingly, is symmetrically distributed around V i.e. $\Pr(X < k) = \Pr(X > 2V - k)$ for $k \in [0, 2V]$). Using such symmetry and the fact that $(v^j + (1 - v^k))$ is distributed as X , we can write the interim expected payoff for a voter that prefers issue one as follows

$$ct + \frac{1}{2} \Pr(X < v^i) \cdot \left\{ \frac{1}{2} - \theta \right\} + \Pr(X \leq V + v^i) \cdot \left\{ 1 - \frac{1}{2} \theta \right\}.$$

First note that whenever both opponents are splitting their voting power evenly (the case of MR), voter i is indifferent between playing any of the strategies. In particular $v^i = \frac{V}{2}$ is a best response. Hence, a symmetric equilibrium has all voters always splitting their voting power equally among both issues.

In the remainder of the proof we show that there exists only one more (and only one) equilibrium which corresponds to the one in which non-indifferent voters invest all their voting power on their preferred issue.⁵⁵

Any other equilibrium will have non-indifferent voters investing more than $\frac{V}{2}$ votes on their preferred issue. Consequently, any voter with $\theta \in [0, \frac{1}{2})$ clearly invests all his voting power on his preferred issue. Suppose now that there are some voters with $\theta \in [\frac{1}{2}, 1]$ such that $v^i < V$. Theorem 1 tell us that the optimal strategy is a well behaved function (decreasing with respect to θ) thus we can consider a parameter $\tilde{\theta} \in [\frac{1}{2}, 1]$ such that any voter with $\theta^+ > \tilde{\theta}$ invests strictly less votes on his preferred issue ($v^i(\theta^+) < V$) and any voter with $\theta^- < \tilde{\theta}$ sticks to the strategy $v^i = V$.

Given that both are acting optimally we have that the next two inequalities should hold:

$$\begin{aligned} (\Pr(X < V) - \Pr(X < v^i(\theta^-))) \cdot \left\{ \theta - \frac{1}{2} \right\} &\leq (\Pr(X \leq 2V) - \Pr(X \leq V + v^i(\theta^-))) \cdot \left\{ 2 - \theta^- \right\} \\ (\Pr(X < V) - \Pr(X < v^i(\theta^-))) \cdot \left\{ \theta^+ - \frac{1}{2} \right\} &\geq (\Pr(X \leq 2V) - \Pr(X \leq V + v^i(\theta^-))) \cdot \left\{ 2 - \theta^+ \right\} \end{aligned}$$

Given that the optimal function is decreasing we have that we should consider two possible cases: (1) the function is smooth at $\tilde{\theta}$ (i.e. $\lim_{\varepsilon \rightarrow 0} v^i(\tilde{\theta} + \varepsilon) = V$) and (2) there is a discontinuity (i.e. $\lim_{\varepsilon \rightarrow 0} v^i(\tilde{\theta} + \varepsilon) = \bar{v} < V$). Consequently, taking limits as θ^- and θ^+ tend to θ in the previous inequalities lead to two

⁵⁵The behaviour of indifferent voters does not need to be specified because they have zero measure. Nevertheless, it can be shown that their best response to any of the equilibria is splitting their voting power evenly.

possible equalities depending on the behaviour of the optimal strategy at $\tilde{\theta}$:

$$\begin{aligned} 1- & (\Pr(X < V) - \Pr(X < \bar{v})) \cdot \left\{ \tilde{\theta} - \frac{1}{2} \right\} = (\Pr(X \leq 2V) - \Pr(X < 2V)) \cdot \left\{ 2 - \tilde{\theta} \right\}. \\ 2- & (\Pr(X < V) - \Pr(X < \bar{v})) \cdot \left\{ \tilde{\theta} - \frac{1}{2} \right\} = (\Pr(X \leq 2V) - \Pr(X \leq V + \bar{v})) \cdot \left\{ 2 - \tilde{\theta} \right\}. \end{aligned}$$

Trivially, the first equality cannot be met because there is a positive measure of types playing the non-diversification strategy thus $\Pr(X = 2V) > 0$. The second case also leads to a contradiction given the following inequalities and the fact that one of them will always be strict:

$$2\tilde{\theta} - 1 \leq 2 - \tilde{\theta}$$

$$\Pr(X < V) - \Pr(X < \bar{v}) \leq 2 \cdot (\Pr(X \leq 2V) - \Pr(X \leq V + \bar{v})).$$

The second inequality needs some clarification. The term in brackets on the RHS accounts for all those cases in which both opponents are investing strictly more than $(V + \bar{v})$ votes (i.e. $X \in (V + \bar{v}, 2V]$). That is, those cases in which both voters have a type belonging to the interval $[0, \tilde{\theta})$. Hence this occurs with probability ρ^2 where $\rho := \Pr\left\{\theta \in [0, \tilde{\theta})\right\}$. Instead, the LHS accounts for those cases in which X belongs to $[\bar{v}, V)$. A necessary condition for that event is that none of the voters should invests V votes i.e. it occurs with a probability lower than $1 - \rho$. Given that θ is uniformly distributed, we know that $\rho \geq \frac{1}{2}$.

Finally, we just need to see that the second inequality is strict for $\rho > \frac{1}{2}$ and the first one is strict for $\rho = \frac{1}{2}$. ■

Lemma 4 $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is homogeneous of degree k if and only if the following first order partial differential equation is satisfied:

$$\nabla f(x) \cdot x = k \cdot f(x).$$

Proof. Sufficiency. (Euler's Theorem) Given that f is homogeneous of degree k we have that for all $\lambda > 0$ and all $x \in \mathbb{R}^N$ the following holds: $f(\lambda x) = \lambda^k \cdot f(x)$. Differentiating the equality with respect to λ we obtain:

$$x_1 \cdot \frac{\partial f}{\partial x_1}(\lambda x) + \dots + x_N \cdot \frac{\partial f}{\partial x_N}(\lambda x) = k \cdot \lambda^{k-1} \cdot f(x).$$

For $\lambda = 1$ we get our result.

Necessity.⁵⁶ Define $\xi(\lambda) := \lambda^{-k} \cdot f(\lambda x) - f(x)$ and differentiate such expression,

$$\xi'(\lambda) := -k \cdot \lambda^{-k-1} \cdot f(\lambda x) + \lambda^{-k} \cdot \left[x_1 \cdot \frac{\partial f}{\partial x_1}(\lambda x) + \dots + x_N \cdot \frac{\partial f}{\partial x_N}(\lambda x) \right].$$

Using the fact that $\nabla f(x) \cdot x = k \cdot f(x)$ we have that $\xi'(\lambda) = 0$. Hence, $\xi(\lambda)$ is constant. Moreover, $\xi(1) = 0$, thus $\xi(\lambda) = 0$ for all $\lambda > 0$ which proves that f is homogeneous of degree k . ■

⁵⁶This part of the proof is extracted from Martin J. Osborne webpage (www.chass.utoronto.ca/~osborne)

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