# Democratic Policy Making with Real-Time Agenda Setting. Part 1.*

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**Abstract**

We examine democratic policy-making in a simple institution with real-time agenda setting. Individuals are recognized sequentially. Once recognized, an individual makes a proposal, which is immediately put to a vote. If a proposal passes, it supercedes all previously passed proposals. The policy that emerges from this process is implemented. For some familiar classes of policy spaces with rich distributional politics, we show that the last proposer is effectively a dictator under a variety of natural conditions. Most notably, this occurs whenever a sufficient number of individuals have opportunities to make proposals. Thus, under reasonably general assumptions, control of the final proposal with real-time agenda setting confers as much power as control of the entire agenda.

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# 1 Introduction

Democratic institutions are characterized by a wide variety of rules and procedures for collective deliberation. A central objective of research in political economy is to identify the effects of these characteristics on policy outcomes, and on the distribution of political power (see Persson and Tabellini [2001] and Besley and Case [2001] for recent reviews of the literature). Previous research indicates that the rules governing agenda setting are especially critical.

The fundamental importance of agenda control was established by McKelvey [1976,1979], whose primary objective was to study voting intransitivities. McKelvey considered a model in which proposals are considered in a known, deterministic order established in advance of all voting (the agenda). The first policy is compared to the second based on majority rule, the winner is compared to the third, and so forth, with the ultimate victor enacted into law.[1] He assumed that voters are myopic, in the sense that they vote in favor of their preferred policy in every stage, irrespective of the implications for the final outcome. Under fairly weak conditions, he demonstrated that, for any two policies $p$ and $p'$, there is a finite agenda that starts with $p$ and implements $p'$. Thus, the ultimate outcome is completely determined by the agenda. Shepsle and Weingast [1984] subsequently established that the introduction of strategically sophisticated voting weakens McKelvey's result, but does not fundamentally alter the conclusion that the agenda plays a critical role in policy selection.[2]

In light of these results, it is important to understand the processes through which agendas are determined. The existing literature attacks this issue in three different ways. Some papers consider settings in which the agenda is selected by a single individual (see Shepsle and Weingast [1984], Banks [1985], Miller [1980], and Bloch and Rottier [2000]). Others assume that the agenda is determined as the result of strategic interaction among influential individuals (see Austen-Smith [1987], Banks and Gasmi [1987], Lockwood [1998], Penn [2001], and Dutta, Jackson, and Le Breton [2001]). An alternative approach is to finesse the problem by identi-

[1] This is known in the literature as a "forward" agenda. With a "backward" agenda, the policy emerging from the succession of pairwise comparisons is implemented only if it majority-defeats some exogenously given status quo (otherwise the status quo prevails).

[2] More precisely, they demonstrated that, for any two policies $p$ and $p'$, there is a finite agenda that starts with $p$ and implements $p'$ provided that $p$ does not *cover* $p'$. The policy $p$ covers the policy $p'$ if (1) $p$ defeats $p'$ by majority rule; and (2) every policy $p''$ that defeats $p$ also defeats $p'$.

fying outcomes that are robust, in the sense that they emerge whenever the policy in question is included in the agenda (see Shepsle and Weingast [1984] and Ferejohn, Fiorina and McKelvey [1987]). Not surprisingly (given the results described above), the existence of such robust outcomes is difficult to guarantee.[3] For the first approach, considerable political power is concentrated in the hands of the agenda setter. The implications of the second and third approaches for the distribution of power are less clear.

Despite this variety of approaches, the literature has adhered to the restrictive assumption that the sequence of proposals is known in advance of all voting ("advance" agenda setting). For many democratic institutions, the agenda is determined during the course of deliberation, in "real time," so that the dynamics of proposing and voting are interlocked.[4] Participants in these processes have the opportunity and incentive to make different proposals depending on the outcomes of previous votes.

The purpose of this paper is to examine real-time agenda setting in a simple democratic institution. In particular, we imagine that there is a group of individuals charged with making a collective choice. There is also an initial status quo policy, possibly one inherited from previous rounds of deliberation. Individuals are recognized sequentially in some predetermined order. Once recognized, an individual makes a proposal, which is immediately put to a vote. Individuals are permitted to condition both their proposals and their votes on all preceding events, including other proposals and votes. Passage of a proposal requires a simple majority. If a proposal passes, it becomes the new status quo and supersedes all previously passed proposals. The policy that emerges from this process is implemented.[5]

The institution described in the preceding paragraph differs from the standard model of agenda setting only with respect to the timing of proposals. This permits us to isolate the differences between real-time and advance agenda setting. In the sequel to this paper (Bernheim, Rangel, and Rayo [2002]), we explore the implications of real-time agenda setting for a wider (and more realistic) class of

[3]For institutions with forward agendas, such outcomes exist only if the policy space contains a Condorcet winner. However, for institutions with backward agendas and "pork-barrel" politics, existence is guaranteed, and the robust outcome benefits a bare majority of agents at the expense of a bare minority. See Shepsle and Weingast [1984] and Ferejohn, Fiorina, and McKelvey [1987].

[4]For example, in the U.S. Congress, when the queue of unvoted amendments reaches size four, a vote must be taken before further proposals can be made (see Oleszek [2001], chapter 5).

[5]In other words, we model institutions with forward agendas. In Bernheim, Rangel, and Rayo [2002], we consider insitutions with real-time agenda setting and backward agendas.

institutions.

We begin our analysis with the observation that real-time agenda setting always leads to the selection of a Pareto efficient outcome. Consequently, the bulk of this paper focuses on distribution. For some familiar classes of policy spaces that give rise to rich distributional politics, we show that the last proposer is effectively a dictator under a variety of natural conditions. Most notably, this occurs whenever a sufficient number of individuals have opportunities to make proposals. Thus, under reasonably general assumptions, control of the final proposal with real-time agenda setting confers as much political power as control of the entire agenda. Despite the fact that a majority is required to pass any particular proposal, the process can lead to an outcome that makes every member of the group worse off relative to inaction, save for the last proposer. Ironically, the last proposer need not have dictatorial powers unless a sufficient number of individuals participate in setting the agenda.[6] Accordingly, within the class of institutions considered in this paper, reforms that appear to be inclusive from a procedural perspective (by promoting "participatory democracy" or guaranteeing a "right to be heard") can have the unintended effect of concentrating political power.

This paper is also related to the literature on stochastic bargaining (see, e.g., Baron and Ferejohn [1989], Merlo and Wilson [1995], Diermier and Merlo [2000], Banks and Duggan [1998, 2000, 2001], and Eraslan [1998]), which similarly considers institutions in which the members of a group vote on proposals sequentially. As with real-time agenda setting, each proposal is made after the group has voted on the previous proposal, rather than in advance of all voting.[7] However, in contrast to the literature on agenda setting, once the group approves a proposal, the process terminates and the proposal is implemented. Consequently, these institutions do not allow for reconsideration of decisions once a policy receives majority approval. This is an important limitation in that, as a practical matter, policy makers can and do revisit previous decisions (indeed, every policy reform bill has this characteristic). The passage of a bill specifying the policy that is to prevail in some future year does not typically preclude a legislature from altering that policy by passing another "reform" bill before the year in question arrives; however, it does alter

[6] With a single proposal round, the outcome typically depends on the status quo. In particular, a status quo that is less desirable from the perspective of a majority of the voters leads to a better outcome for the proposer. See e.g. Romer and Rosenthal [1978].

[7] In the literature on stochastic bargaining, the order of proposers is determined randomly during the course of play.

the default option against which new alternatives are considered. The institutions studied in the agenda control literature (including the ones examined in this paper and its companion) are of interest in part because they allow for reconsideration. This distinction has an important formal counterpart. For models of stochastic bargaining, proposals may depend upon history for strategic reasons, but there is no state variable (unless amendments are permitted). To the extent one focuses on Markov-perfect equilibrium, members of the group might just as well announce their proposals in advance. In contrast, with real-time agenda setting, history always matters because there is a state variable (the current status quo).

The remainder of this paper is organized as follows. Section 2 lays out the basic model. Section 3 presents some general results, including the selection of Pareto efficient outcomes and Condorcet winners (where they exist). Section 4 specializes to a familiar policy space with rich distributional politics, and proves our dictatorship results. Section 5 extends the results to other policy spaces. We conclude in section 6 with a summary of our findings, and a discussion of results from Bernheim, Rayo, and Rangel [2002] concerning alternative institutions, rules, and procedures. To familiarize the reader with our analytic techniques, we include the proofs of several key propositions in the text. Other proofs are contained in the appendix.

## 2 The Model

Consider a decision-making body ("the group") consisting of $N$ individuals, labelled $l = 1, ..., N$, where $N \geq 5$. To avoid complications arising from tie votes, we assume for convenience that $N$ is odd. Let $M \equiv \frac{N+1}{2}$ denote the size of the smallest majority.

### 2.1 Policies and Payoffs

The group must select a policy $p \in P$, where $P$ denotes the set of feasible policies. Let $v_l(p)$ denote the payoff to individual $l$ if policy $p$ is implemented. Except where indicated, we impose the following two assumptions throughout:

**Assumpiton A1:** The policy space $P$ is finite.

**Assumption A2:** Individuals have strict preferences over policies: $p \neq p' \Rightarrow v_l(p) \neq v_l(p')$.

Assumptions A1 and A2 are relatively innocuous. Indeed, given A1, any failure of A2 is non-generic. We nevertheless acknowledge that these assumptions rule out some interesting and important cases, including the familiar "divide-the-dollar" problem. The analytics of the divide-the-dollar problem are considerably more complicated because one can exploit indifference to contrive elaborate history-dependent strategies. However, as we show in section 5, our central conclusions extend to that case largely intact.

## 2.2 Procedures for Collective Choice

The collective choice process consists of a sequence of $T$ "proposal rounds." Activity prior to each round $t$ establishes some "status quo" policy, $p_{t-1}$. Round $t$ begins when individual $i(t)$ is recognized. For now, both the initial status quo policy, $p_0$, and the order of recognition, $i : \{1, ..., T\} \rightarrow \{1, ..., N\}$, are predetermined and known to all individuals as of round 1. Recognition provides an individual with the opportunity to make a proposal, $p_t^m$, which can be any element of $P$. The proposal is then put to an immediate vote against $p_{t-1}$. If it receives majority approval ("passes"), it displaces $p_{t-1}$ as the status quo policy ($p_t = p_t^m$). If it does not pass, the status quo policy remains the same for the following round ($p_t = p_{t-1}$). Equivalently, one can think of any given proposal as adding to, deleting, or replacing portions of the prevailing status quo policy.[8] In effect, the round $t$ proposal consists of the *differences* between $p_{t-1}$ and $p_t^m$.

The ultimate fate of the policy that emerges from the last proposal round, $p_T$, is determined in some final stage of the collective choice process. As an example, consider an institution with a "backward" agenda: the final stage is a final up-or-down vote against an exogenous policy $p^E$; if $p_T$ wins it becomes law, otherwise

[8] It may at first seem odd to assume that a new proposal, once passed, displaces all policies previously passed. However, this assumption involves essentially no loss of generality. It is important to keep in mind that a policy (and therefore a proposal), as we have defined it, involves a complete description of *all* collective actions, and not merely the component actions pertaining to some particular subset of issues. To illustrate, consider the following example. Imagine that the government faces two choices: whether to build bombers, and whether to save whales. In each instance, there are two possibilities: build the bombers ($B$) or not ($NB$), and save whales ($S$) or not ($NS$). There are four possible policies: ($B, S$), ($NB$,$S$), ($B$,$NS$), and ($NB$,$NS$). Imagine also that the initial status quo policy ($p_0$) involves no action ($NB$,$NS$). If the first recognized individual wishes to propose to build bombers, he will propose ($B$,$NS$). If this passes, and if the second individual wishes to save the whales, she proposes ($B$,$S$). Though the second proposal, if passed, technically displaces the first, it is clear that indivdiuals are actually voting on the incremental component policy $S$.

$p^E$ is implemented.[9] In most of the paper we focus on the simplest possibility: $p_T$ is directly enacted into law. Nevertheless, for reasons that will become clear in the next section, it is analytically useful to allow for greater generality at the outset. For our purposes, we abstract from institutional details and simply assume that it is possible to derive some reduced form representation of the final stage, $\Omega : P \to P$. In other words, when the policy $p_T$ emerges from the final stage, the ultimate outcome is $\Omega(p_T)$. Obviously, this framework includes the special case of a *degenerate final stage*, wherein $p_T$ becomes law without further modification ($\Omega(p) \equiv p$).

### 2.3 Behavioral assumptions

Throughout our analysis, we assume that (1) individuals are strategically sophisticated, and (2) they always vote as if they are pivotal. We make the second assumption to deal with the familiar problem of indifference among non-pivotal voters, which otherwise gives rise to a vast multiplicity of equilibria. The equilibria that we rule out through the second assumption are unreasonable because agents cast votes that are contrary to their true preferences. Together, our two assumptions imply that individuals compare the continuation equilibrium if a proposal passes with the continuation equilibrium if it is defeated, and cast their vote for the option that yields the preferred continuation path. We also confine attention to pure strategy subgame perfect equilibria. Henceforth, the term "equilibrium" should therefore be construed as indicating a pure strategy subgame perfect equilibrium with the preceding characteristics.

## 3 Some general results

### 3.1 An equivalence result for final stages

Let $\Omega(P)$ denote the image of all points in $P$ under the mapping $\Omega$. Plainly, the final policy outcome must belong to the set $\Omega(P)$. Let $J \equiv \{j \mid j = i(t)$ for some $t = 1, ..., T\}$; this is the set of individuals who are recognized at least once.

Our first result establishes an extremely simple yet important equivalence principle:

[9] Institutions with final up-and-down votes (backward agendas) are considered, among others, by Ferejohn, Fiorina, and McKelvey [1987].

**Lemma 1:** *Consider a policy set $P$ satisfying A1 and A2. An institution with policy set $P$, initial status quo $p_0$, and final stage $\Omega$ yields the same equilibrium policy outcome as an otherwise identical institution with policy set $\Omega(P)$, initial status quo $\Omega(p_0)$, and a degenerate final stage.*

The proof of lemma 1 is completely straightforward, and, in effect, involves relabeling of branches and nodes in the extensive form of the game, as well as deletion of redundant branches. The lemma is useful because it implies that we can understand all institutions in this class (including those with backward agendas) by studying institutions with degenerate final stages (i.e., those with forward agendas). In particular, if one wishes to know the outcome generated by an institution with a non-degenerate final stage, one need only derive a reduced form mapping for the final stage ($\Omega$), and then consider an equivalent institution with a smaller policy space ($\Omega(P)$) and a degenerate final stage.

## 3.2 The recursive structure of equilibria

Lemma 1 is also important because it allows us to provide a useful recursive characterization of the equilibria for these models. This requires some additional notation.

For any $P' \subseteq P$ and $p' \in P'$ define

$$Z(p', P') \equiv \{q \in P' \mid \exists\ S \subseteq \{1, ..., N\} \text{ with } |S| \geq M \text{ and } v_l(q) \geq v_l(p') \text{ for all } l \in S\}.$$

This is the set of policies in $P'$ that (weakly) defeat $p'$ by majority rule. The use of weak inequalities here implies that $p' \in Z(p', P')$. However, in light of our genericity assumption, strict inequalities hold for all other $p \in Z(p', P')$. Next, define

$$\varphi_l(p', P') \equiv \arg \max_{q \in Z(p', P')} v_l(q).$$

This represents individual $l$'s most preferred element of the set $Z(p', P')$. Under assumptions A1 and A2, this function is well defined. Finally, define

$$\Phi_l(P') \equiv \varphi_l(P', P').$$

This is simply the image of the set $P'$ under the mapping $\varphi_l(\cdot, P')$.

Now we exhibit the recursion. Consider first the following institution:

**Institution #1:** $T$ proposal rounds, a recognition order $i(t)$ (for $t = 1, ..., T$), a policy space $P$, an initial status quo $p_0$, and a degenerate final stage.

Observe that, without altering the game in any substantive way, one can think of the final proposal round as part of the final stage. The policy that emerges from round $T-1$, $p_{T-1}$, then serves as the input for the final stage. For any particular $p_{T-1}$, solving this final stage is straightforward: $i(T)$ proposes the policy in $P$ she most prefers among those that (weakly) defeat $p_{T-1}$. In other words, $\Omega(p_{T-1}) = \varphi_{i(T)}(p_{T-1}, P)$. Lemma 1 tells us that this is in turn equivalent to the following institution:

**Institution #2:** $T-1$ proposal rounds, a recognition order $i(t)$ (for $t = 1, ..., T-1$), a policy space $\Phi_{i(T)}(P)$, an initial status quo $\varphi_{i(T)}(p_0, P)$, and a degenerate final stage.

The preceding argument demonstrates that a basic institution with $T$ proposal rounds and a degenerate final stage is equivalent to another basic institution with $T-1$ proposal rounds and a degenerate final stage, where the policy space has been appropriately reduced, and where the initial status quo has been appropriately transformed. The same argument implies that these institutions are in turn equivalent to another basic institution with $T-2$ proposal rounds and a degenerate final stage, where the policy space has been further reduced to $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$, and where the initial status quo has been further transformed.

Where does this argument ultimately lead? Recursive application of the same equivalence principle implies that the original institution is equivalent to a basic institution with *zero* proposal rounds and a degenerate final stage, where the policy space is

$$\Phi_{i(1)} \circ ... \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P),$$

and where the initial status quo has been appropriately transformed. Since this institution is completely degenerate, the transformed initial status quo is simply enacted into law.

According to the preceding argument, for *any* initial status quo $p_0 \in P$, the initial institution *must* generate an outcome in the set $\Phi_{i(1)} \circ ... \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. Notice that we can solve for this set through mechanical application of the $\Phi_i$ mappings. This allows us to completely characterize all possible outcomes of the legislative process, for any conceivable initial status quo, without fully specifying the equilibrium strategies.

For some of the arguments appearing later in this paper, it is also convenient to define a function $Q_t(p_{t-1})$ that maps the status quo $p_{t-1}$ in round $t$ to the eventual equilibrium outcome. The map is defined recursively as follows:

$$Q_T(p_{T-1}) \equiv \varphi_{i(T)}\,(p_{T-1}, P)$$

and, for $t < T$,

$$Q_t(p_{t-1}) \equiv \varphi_{i(t)}(Q_{t+1}(p_{t-1}), Q_{t+1}(P)).$$

This intuitive construction corresponds to backward induction. Consider the problem of individual $i(t)$ in round $t$ when the status quo is $p_{t-1}$. If proposal $p'$ passes in round $t$, the status quo for round $t+1$ is $p'$, and the eventual outcome is $Q_{t+1}(p')$. If no new proposal passes in round $t$, the status quo for round $t+1$ is $p_t = p_{t-1}$, and the eventual outcome is $Q_{t+1}(p_{t-1})$. Thus, $i(t)$'s problem is to choose the best policy in the set of continuation outcomes $Q_{t+1}(P)$ that can (weakly) defeat the continuation status quo $Q_{t+1}(p_{t-1})$ by majority rule. The solution is $\varphi_{i(t)}(Q_{t+1}(p_{t-1}), Q_{t+1}(P))$.

Note that $Q_t(P) = \Phi_{i(t)} \circ ... \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. Thus, $Q_t(P)$ denotes the set of policies that can emerge as final outcomes if one places no restrictions on the status quo for round $t$. Since $\Phi_l(R) \subseteq R$ for all $R \subseteq P$, every application of a $\Phi_l$ mapping shrinks the set of possible final outcomes. It follows that the sets $\{Q_t(P)\}_{t=1}^{T}$ are nested: $Q_1(P) \subseteq Q_2(P) \subseteq ... \subseteq Q_T(P)$.

### 3.3 Pareto efficiency

One can evaluate institutions with respect to the efficiency and distributional characteristics of the outcomes they generate. With respect to efficiency, we have the following simple result:

**Theorem 1:** *Consider an institution with a degenerate final stage and a policy set $P$ satisfying A1 and A2. Then the outcome, $p_T$, is Pareto efficient in $P$.*

**Proof:** We know that $p_T \in \Phi_{i(T)}(P)$. Consequently, we need only demonstrate that all points in $\Phi_{i(T)}(P)$ are Pareto efficient in $P$. Consider some $p \in \Phi_{i(T)}(P)$, and suppose contrary to the theorem that it is not Pareto efficient in $P$. In light of assumption A2, there is some $p^* \in P$ such that every individual strictly prefers $p^*$ to $p$. We know that there is some $p'$ such that $p =$

$\varphi_{i(T)}(p', P) = \arg\max_{q \in Z(p',P)} v_{i(T)}(q)$. But $p^* \in Z(p', P)$, and $v_{i(T)}(p^*) > v_{i(T)}(p)$, which is a contradiction. Q.E.D.

Theorem 1 assures us that the collective outcome will always lie on the Pareto frontier. Consequently, starting in the next section, our analysis focuses on distributional politics.

### 3.4 Selection of Condorcet winners

Bearing in mind the equivalence result of section 3.1, we will continue to focus on processes with degenerate final stages. In general, there is no reason to believe that the policy set $P$ will contain a Condorcet winner (defined as a policy that is majority preferred to all other policies). However, it is natural to wonder whether the collective choice process will select a Condorcet winner if one exists. As it turns out, this question is central to a number of the results proven in later sections.

Plainly, there are institutions of the form considered here that do not select Condorcet winners. As an example, consider an institution with a single proposal round. For any given initial status quo $p_0$, there is no particular reason to believe that the Condorcet winner, $p^c$, is the recognized individual's preferred outcome in $Z(p_0, P)$. Indeed, it is entirely possible that this individual prefers $p_0$ to $p^c$.

Despite the preceding observation, the group will select a Condorcet winner, assuming that one exists, provided that a sufficiently diversified set of individuals have opportunities to make proposals.

**Theorem 2:** *Consider an institution with a degenerate final stage, and a policy set satisfying A1 and A2. Suppose that there is a Condorcet winner $p^c$ in $P$. Then $p^c$ is the final outcome regardless of the initial status quo (i.e., $Q_1(P) = \{p^c\}$) whenever:*

*(1) $|J| \geq M$, or*

*(2) $p^c$ is the preferred policy in $P$ for some individual $l \in J$.*

**Proof:** Consider any $R \subseteq P$ with $p^c \in R$. For all $l$, we have $p^c \in \Phi_l(R)$ (since $p^c = \varphi_l(p^c, R)$). Moreover, for all $p \in \Phi_l(R)$, we have $v_l(p) \geq v_l(p^c)$ (since $p^c \in Z(p', R)$ for all $p' \in P$). Using the fact that $Q_t(P) = \Phi_{i(t)}(Q_{t+1}(P))$ (with $Q_{T+1}(P) \equiv P$) and applying induction, we therefore know that $v_l(p) \geq v_l(p^c)$ for all $l \in J$ and $p \in Q_1(P)$. By assumption A2, it follows that either

$p = p^c$ or $v_l(p) > v_l(p^c)$ for all $l \in J$. Conditions (1) and (2) both rule out the latter possibility. Q.E.D.

For environments with single-dimensional policy sets and single-peaked preferences, theorem 2 provides conditions under which the desires of the median voter prevail (just as in Downs' [1957] model of electoral competition). From lemma 1, we know that an analog of theorem 2 holds for basic institutions with non-degenerate final stages whenever there exists a Condorcet winner in $\Omega(P)$. This latter observation will prove useful in the next section.

# 4 Dictatorship results

To characterize the possible outcomes of the collective choice process with greater precision, one must place some restrictions on the set of feasible policies. In this section, we restrict attention to a particular class of policy sets that give rise to rich distributional politics. Models with similar payoff structures appear elsewhere in the theoretical literature concerning legislative policy making (see, e.g., Ferejohn [1974] or Ferejohn, Fiorina, and McKelvey [1987]). We extend our analysis to other types of policy sets in section 5.

For each individual, we assume that there is an associated "elementary policy." Let $E \equiv \{1, ..., N\}$ denote the set of all elementary policies. Each $l \in E$ produces highly concentrated benefits and diffuse costs. In particular, policy $l$ generates a net benefit $b_l > 0$ for individual $l$, and a cost $c_l > 0$ for every individual (including $l$). A policy $p$ is a collection of elementary policies. The set of feasible policies $P$ is the power set of $E$; that is, the set of all possible combinations of elementary policies. $P$ includes the empty set $\emptyset$, which represents inaction (nothing is implemented). Payoffs are additively separable:

$$v_l(p) = -\sum_{j \in p} c_j + \begin{cases} b_l & \text{if } l \in p \\ 0 & \text{otherwise.} \end{cases}$$

Henceforth, we will refer to $P$ as a CBDC policy set (for concentrated benefits, diffuse costs). We impose two additional assumptions:

**Assumption A3:** Total costs are increasing in the number of elementary policies. Specifically, $|p| < |p'| \Rightarrow \sum_{j \in p} c_j < \sum_{j \in p'} c_j$.

**Assumption A4:** A mutually beneficial policy (relative to $p = \emptyset$) exists for all coalitions consisting of $M$ or fewer individuals. In particular, for every policy $p$ with $|p| \leq M$, $b_l > \sum_{j \in p} c_j$ for all $l \in p$.

When all elementary policies are equally costly, Assumption A3 is trivially satisfied. Consequently, this assumption effectively restricts the degree to which costs can vary across elementary policies. For our main result, it is possible to relax this assumption considerably (see section 5).

Assumption A4 guarantees the existence of policies that are preferred to inaction by a majority of voters. It also guarantees that there exists such a policy for any bare-majority coalition. If there does not exist a policy that is mutually beneficial for all members of some majority coalition, then $p = \emptyset$ is a Condorcet winner. It follows that the group selects $\emptyset$ under the conditions identified in Theorem 2, as well as when $p_0 = \emptyset$. Ironically, the ability to assemble majoritarian coalitions is therefore essential for the emergence of the dictatorial outcomes derived below. Note finally that, under assumption A4, the universalistic policy $p = E$ need not maximize social surplus. Consider, for example, the case of $N = 5$ with $b_1 = ... = b_5 = 8$, $c_1 = c_2 = c_3 = 2$, and $c_4 = c_5 = 1$. Assumption A4 is clearly satisfied, but the surplus maximizing policy is $\{4, 5\}$.

We divide the analysis of CBDC policy sets into three subsections. The first considers deterministic institutions in which many individuals have opportunities to make proposals ("inclusive recognition orders"). We demonstrate that, as long a sufficient number of individuals are recognized at some point during deliberations, a dictatorial outcome emerges for every recognition order and every initial status quo. The second considers deterministic institutions in which relatively few individuals have opportunities to make proposals ("exclusive recognition orders"). Our analysis of these environments shows that the dictatorial policy occurs with high frequency when this outcome is not guaranteed. The third considers institutions in which the recognition order, the number of proposal rounds, or both may be random ("random recognition processes"). Our discussion subsumes the possibility that there is no finite bound on the number of proposal rounds. We show that a dictatorial outcome continues to emerge under reasonable, though somewhat restrictive assumptions about the manner in which uncertainty is resolved.

## 4.1 Inclusive recognition orders

Some collective choice processes plainly yield majoritarian outcomes. Consider, for example, an institution with a degenerate final stage and one proposal round ($T = 1$). Imagine that the initial status quo is inaction ($p_0 = \emptyset$). Then the outcome necessarily consists of $M$ elementary policies. Specifically, the policy includes the elementary policy $i(1)$ and the $M - 1$ least costly elementary policies other than $i(1)$.

Compare the institution discussed in the previous paragraph to one that is more inclusive. In particular, imagine that a large fraction of the individuals – perhaps all of them – have opportunities to make proposals (an inclusive recognition order). The latter institution certainly seems more egalitarian. Our next result shows that, in the presence of real-time agenda setting, greater inclusiveness can concentrate all political power in the hands of a single individual.

**Theorem 3:** *Consider an institution with a degenerate final stage, a CBDC policy set satisfying A1-A4, and $N \geq 5$ individuals. Provided that either $|J| > M$ or $i(T)$ proposes more than once, the unique outcome is the policy $p = \{i(T)\}$.*

Theorem 3 identifies conditions under which the last proposer, $i(T)$, is a dictator in the following sense: she obtains her most preferred outcome, $\{i(T)\}$, irrespective of the initial status quo, the order of recognition, or the costs and benefits associated with any particular elementary policy (provided that A1 through A4 are satisfied). It is important to emphasize the perversity of this outcome. When, for example, the initial status quo is the null policy $\emptyset$, all individuals other than $i(T)$ *strictly prefer* it to the final outcome. If the group simply failed to meet, everyone would be better off except $i(T)$. The group produces a result that is contrary to the interests of almost every member, even though no proposal can pass without majority support.

Theorem 3 has the following ironic implication: within the class of institutions considered in this paper, reforms that appear to be inclusive from a procedural perspective (by promoting "participatory democracy" or guaranteeing a "right to be heard") can have the unintended effect of concentrating political power. For example, a majoritarian outcome results when the group entertains only a single proposal, but dictatorship emerges when every individual is allowed to make a proposal.

### 4.1.1 An example

We illustrate the logic of theorem 3 through a simple example. Suppose that $N = 5$, and $c_1 < ... < c_5$. To stack things against our result, we assume that the last proposer is associated with the most costly elementary policy ($i(T) = 5$). The first step in solving for an equilibrium is to identify the final outcome for every status quo, $p_{T-1}$, that the last proposer might inherit. There are two possible cases: (1) $p_{T-1}$ does not include 5, and (2) $p_{T-1}$ includes 5.

Consider case (1). There are four possibilities. Possibility (a): $| p_{T-1}| = 4$ (i.e., $|p_{T-1}| = \{1, 2, 3, 4\}$). In this case $i(T)$ proposes $\{1, 2, 5\}$, which passes with the support of individuals 1, 2, and 5. Possibility (b): $| p_{T-1}| = 3$ . In this case, $i(T)$ drops the two most expensive policies and adds her own. For example, if $p_{T-1} = \{1, 2, 4\}$, she proposes $\{1, 5\}$, which passes with the support of individuals 1, 3, and 5. Possibility (c): $| p_{T-1}| = 2$ (e.g., $\{1, 2\}$). In this case $i(T)$ drops both of the elementary policies in $p_{T-1}$ and adds her own. This leads to the dictatorial outcome $\{5\}$. Possibility (d): $| p_{T-1}| = 1$. In this case $i(T)$ drops the elementary policy in $p_{T-1}$ and adds 3 others, including her own. For example, if $p_{T-1} = \{1\}$, she proposes $\{2, 3, 5\}$. Thus, in case (1), the only possible outcomes are of the form $\{5\}$, $\{5, x\}$, or $\{5, x, y\}$ (where $x$ and $y$ are elementary policies other than 5).

Now consider case (2). There are two possibilities. Possibility (a): $|p_{T-1}| \geq 3$. In this case, $i(T)$ drops at least the two most expensive policies other than her own. For example, if $p_{T-1} = \{1, 2, 3, 5\}$, she proposes $\{1, 5\}$, and this passes with the support of individuals 1, 4, and 5.[10] Possibility (b): $|p_{T-1}| < 3$. In this case, $i(T)$ proposes $\{5\}$, and this passes with the support of $i(T)$ along with all individuals whose elementary policies are excluded from $p_{T-1}$. For example, if $p_{T-1} = \{2, 5\}$, individuals 1, 3, 4, and 5 vote in favor of $\{5\}$.[11] Thus, in case (2), the only possible outcomes are also of the form $\{5\}$, $\{5, x\}$, and $\{5, x, y\}$.

From the preceding arguments, we know that, given the equilibrium behavior of the last proposer, the outcome must be $\{5\}$, $\{5, x\}$, or $\{5, x, y\}$. But this implies that $\{5\}$ is a Condorcet winner within the set of policies that can survive the final proposal round. In particular, $\{5\}$ majority defeats $\{5, x\}$ for all $x$ (only individual

[10] When $p_{T-1} = \{1, 2, 3, 4, 5\}$, $i(T)$ may propose either $\{1, 2, 5\}$ or $\{5\}$. The latter proposal will receive majority support as long as the policy $\{1, 2, 3, 4\}$ is not mutually beneficial for at least two of the included individuals.

[11] When $p_{T-1} = \{5\}$, all individuals are indifferent between voting for and against the proposal, but the outcome is identical in both cases.

$x$ is opposed), and $\{5\}$ majority defeats $\{5, x, y\}$ for all $x$, $y$ (only individuals $x$ and $y$ are opposed). Theorem 2 then delivers the desired conclusion.

To illustrate this final step, suppose that $T = 2$ and $i(1) = 4$ (the second-to-last proposer is associated with the second most costly elementary policy). A careful review of the preceding arguments reveals that the final outcome never includes elementary policy 4. Consequently, regardless of the initial status quo, individual 4 proposes either $\{5\}$ or (equivalently) something that the final proposer can successfully replace with $\{5\}$ (such as $\{1, 2, 5\}$), and this proposal passes. Note that individual 4 may take the initiative in advocating individual 5's most preferred outcome. Note also that proposals may pass with the opposition of those who (naively) appear to benefit. For example, if the initial status quo is inaction ($p_0 = \emptyset$) and individual 4 proposes $\{1, 2, 5\}$, the proposal passes with the support of individuals 3, 4, and 5, and with the opposition of individuals 1 and 2. This is because, if $\{1, 2, 5\}$ is victorious, $\{5\}$ is the ultimately outcome, but if $\{1, 2, 5\}$ is defeated, $\{1, 2, 5\}$ is the final outcome (since individual 5 proposes it again in the final round).

Real-time agenda setting plays a crucial role in the preceding argument, inasmuch as individual 5 must have the flexibility to make different proposals when she inherits different status quos. To illustrate, consider again the case where $T = 2$, $i(1) = 4$, and $p_0 = \emptyset$, and the equilibrium in which individual 4 proposes $\{5\}$. If the proposal passes, individual 5 proposes $\{5\}$, but if the proposal is defeated, individual 5 proposes $\{1, 2, 5\}$. To see that this flexibility is critical, consider an otherwise identical institution with advance agenda setting, where individual 5 must make the same proposal regardless of the prevailing status quo. Suppose that, as on the path of the equilibrium just considered, individual 4 proposes $\{5\}$ and individual 5 proposes $\{5\}$. If individual 4's proposal passes, the ultimate outcome is $\{5\}$. If individual 4's proposal is defeated, the ultimate outcome is $\emptyset$ (since $\emptyset$ majority defeats $\{5\}$). Thus, individuals 1, 2, 3, and 4 all vote against $\{5\}$ in round 1, and the ultimate outcome is $\emptyset$ rather than $\{5\}$. The same argument holds for any other round 1 proposal that would ultimately lead to the implementation of $\{5\}$. Thus, with advance agenda setting, there is no equilibrium that yields the dictatorial outcome.

#### 4.1.2 Intuition and proof

The proof of theorem 3 is based on three intuitive observations. First, if the final proposer inherits her most-preferred policy as the status quo for round $T$, then this policy is implemented (formally, $\{i(T)\} = \varphi_{i(T)}(\{i(T)\}, P)$). This implies:

**Property 1:** $\{i(T)\} \in \Phi_{i(T)}(P)$.

Second, regardless of the status quo prevailing prior to the final round, the outcome never includes the elementary policies of more than $M$ individuals. This is because the last proposer only needs to secure the approval of a minimal winning coalition. Formally:

**Property 2:** $p \in \Phi_{i(T)}(P) \Rightarrow |p| \leq M$ (proof in appendix).

Third, regardless of the status quo prevailing prior to the final round, the outcome always contains the elementary policy of the final proposer. This is because she can always tailor her proposal to the prevailing status quo. In some instances, she deletes some elementary policies and adds her own; in others, she adds a collection of elementary policies including her own. Formally:

**Property 3:** $p \in \Phi_{i(T)}(P) \Rightarrow i(T) \in p$ (proof in appendix).

Using properties 1 through 3, we argue that $\{i(T)\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$. Consider any other policy $p' \in \Phi_{i(T)}(P)$ other than $\{i(T)\}$. By properties 2 and 3, there are at least $M - 1$ individuals other than $i(T)$ whose associated elementary policies are excluded from $p'$. By property 3, all of these excluded individuals together with $i(T)$ (a majority) prefer $\{i(T)\}$ to $p'$. In general, the identity of the winning majority coalition depends on the choice of $p'$.

The desired conclusion now follows from lemma 1 and theorem 2. By lemma 1, the institution under consideration is equivalent to one in which there are $T - 1$ proposal rounds, and for which the policy space is $\Phi_{i(T)}(P)$ (one must also transform the initial status quo appropriately, but this is inconsequential). By theorem 2, the institution therefore selects $\{i(T)\}$ as long as either $i(T)$ is recognized twice, or at least $M$ distinct individuals are recognized in proposal rounds 1 through $T - 1$. If $|J| > M$, the latter condition is satisfied even if $i(T)$ is recognized only once.

### 4.1.3 Some remarks on the theorem

A few further remarks concerning theorem 3 are in order. First, aside from the requirement that $|J| > M$, we have placed no restrictions on the order of recognition. Some individuals may be recognized once or more than once, while others never have opportunities to make proposals. There is no need to cycle through those who are recognized in any particular order. Indeed, a single individual may be recognized in several consecutive rounds. It is natural to conjecture that consecutive proposals are redundant, but this is not the case. Somewhat surprisingly, an individual may be able to accomplish some objective with two consecutive proposals, but not with a single proposal. For example, with $T = 1$, the institution produces a policy with $M$ elementary components including $i(T)$. However, with $T > 1$ and $i(T-1) = i(T)$, the outcome is $\{i(T)\}$ (this follows by part (2) of theorem 2).

Second, using an alternative argument, one can extend the result to environments for which different elementary policies have the same costs (this violates assumption A2).[12] Since we consider this a knife-edge case, we omit the proof. We take up other extensions and generalizations in section 5.

Finally, the theorem does not hold for institutions with three individuals ($N = 3$). The proof breaks down when one tries to establish property 3. To illustrate, suppose that $T = 3$, $i(t) = t$, and $c_1 < c_2 < c_3$. Then the set of continuation outcomes for any status quo $p_T$ is given by

| $p_{T-1}$ | $Q_T(p_{T-1})$ |
|---|---|
| $\emptyset$ | $\{1,3\}$ |
| {1} | {2,3} |
| {2} | {1,3} |
| {3} | {3} |
| {1,2} | {1} |
| {1,3} | {3} |
| {2,3} | {3} |
| {1,2,3} | {1,3} |

.

Note that if $p_{T-1} = \{1,2\}$ the eventual outcome is $\{1\}$. Since 1 and 2 prefer $\{1\}$ to $\{3\}$, the latter is no longer a Condorcet winner in $Q_T(P)$. This undermines the dynamics that generate dictatorial outcomes. In this case, depending on the initial status quo, the outcome is either $\{1\}$ or $\{2,3\}$. Since most collective choice problems involve more than three decision makers in practice, we regard this as a technical curiosity.

[12] An alternative argument is required because $\varphi_i(p', P')$ may be set-valued. One must also make an assumption concerning the manner in which individuals resolve indifference.

### 4.2 Exclusive recognition orders

It is natural to question the general applicability of theorem 3. Several objections come to mind. First, the result requires individuals to know the recognition order as of round 1. For realistic institutions, there may be considerable uncertainty concerning who will be recognized two or three rounds in the future, let alone twenty or thirty rounds. A second related concern is that individuals must know the number of proposal rounds as of round 1. Though it is plausible to assume that there is a finite upper bound on the number of proposal rounds that can precede any time-dated policy, such as the passage of a budget for a given fiscal year, deliberations on any given proposal may vary randomly in length, creating variation in the realized number of rounds. A third concern is that the result appears to require highly sophisticated strategic reasoning. The familiar centipede game admits a single subgame perfect equilibrium, but this solution presupposes an ability to think through many layers of strategy. In practice, play of the centipede game fails to unravel as predicted by theory. Conceivably, our result may be vulnerable to the same criticism. Finally, the requirement that $|J| > M$ is particularly demanding for groups with large numbers of members.

In this section, we describe one potential avenue for addressing all of these criticisms simultaneously. Each of the concerns mentioned above relates in some way to the number of proposal rounds. When relatively few individuals have opportunities to make proposals (formally, $|J| \leq M$), one can show that there are *always* recognition orders and initial status quos for which $\{i(T)\}$ is not the outcome. In this sense, one cannot "improve" upon the requirement that $|J| > M$. However, it turns out that non-dictatorial outcomes are unusual: a high fraction of possible recognition orders generate $\{i(T)\}$ for all initial status quos even when $|J|$ is small relative to $M$. Consequently, we obtain a dictatorial or near-dictatorial outcome "most of the time" (in a sense made precise below), regardless of the group's size, as long as individuals can think ahead strategically only a small of number of steps, and as long as they properly anticipate the number of remaining rounds and the order of recognition once the end of deliberations draws near.

We begin our analysis of exclusive recognition orders by deriving several conditions under which the dictatorial outcome emerges even for small $|J|$. The statement of this theorem requires the following definitions: $H_K$ denotes the set of individuals associated with the $K$ most costly policies in $E \backslash i(T)$, and $i^*_K$ is the individual associated with the $K$-th most costly policy in $E \backslash i(T)$.

**Theorem 4:** *Consider an institution with a degenerate final stage, a CBDC policy set satisfying A1-A4, and $N \geq 5$ individuals.*

*(a) If $T \geq 2$ and $i(T-1) \neq i^*_{M-1}$, then the outcome is either $\{i(T)\}$ or $\{i(T-1), i(T)\}$.*

*(b) Under either of the following conditions, the unique outcome is the policy $p = \{i(T)\}$:*

*(b1) some member of $H_{M-2} \cup \{i(T)\}$ has the opportunity to make at least one proposal prior to round $T$.*

*(b2) $i(T-1) \neq i^*_{M-1}$ and $i(t) \neq i(T-1)$ for some $t < T-1$.*

Theorem 4 implies that institutions with short recognition orders can produce non-dictatorial outcomes only in relatively unlikely circumstances. We demonstrate this by deriving a lower bound on the fraction of recognition orders that generate the dictatorial outcome $\{i(T)\}$ for all initial status quos.

**Theorem 5:** *Consider an institution with $T > 1$ proposal rounds, a degenerate final stage, a CBDC policy set satisfying A1-A4, and $N \geq 5$ individuals. The fraction of recognition orders that generate the outcome $\{i(T)\}$ for all $p_0 \in P$ is not less than*

$$B(N,T) \equiv 1 - \frac{1}{2}\left(\frac{N-1}{N}\right)\left[\left(\frac{1}{N}\right)^{T-2} + \left(\frac{2}{N}\right)\left(\frac{M}{N}\right)^{T-2}\right].$$

If one imagines that a recognition order is selected at random in an initial stage, and that this selection process is governed by a uniform distribution over the set of all feasible recognition orders, then $B(N,T)$ provides a lower bound on the probability that the collective choice process yields $\{i(T)\}$. Figure 1 illustrates the manner in which this bound changes with the numbers of rounds and individuals. Notice that, regardless of whether $N$ is large or small, the bound approaches unity for relatively small values of $T$. Also notice that the bound is more sensitive to the number of proposal rounds than to the number of individuals. To understand why this is the case, consult part (b1) of theorem 4. If any member of $H_{M-2} \cup \{i(T)\}$ is recognized prior to round $T$, the outcome is $\{i(T)\}$. The probability of not recognizing a member of this group in any particular round is approximately $1/2$ for all $N$. This probability compounds rapidly with the number of rounds, thereby generating the observed convergence with $T$. Notice also that the bound is actually increasing in the number of individuals. This suggests that, contrary to the apparent implications of the requirement in theorem 3 that $|J| > M$, for fixed $T$ dictatorial outcomes are even more likely in large groups than in small ones.

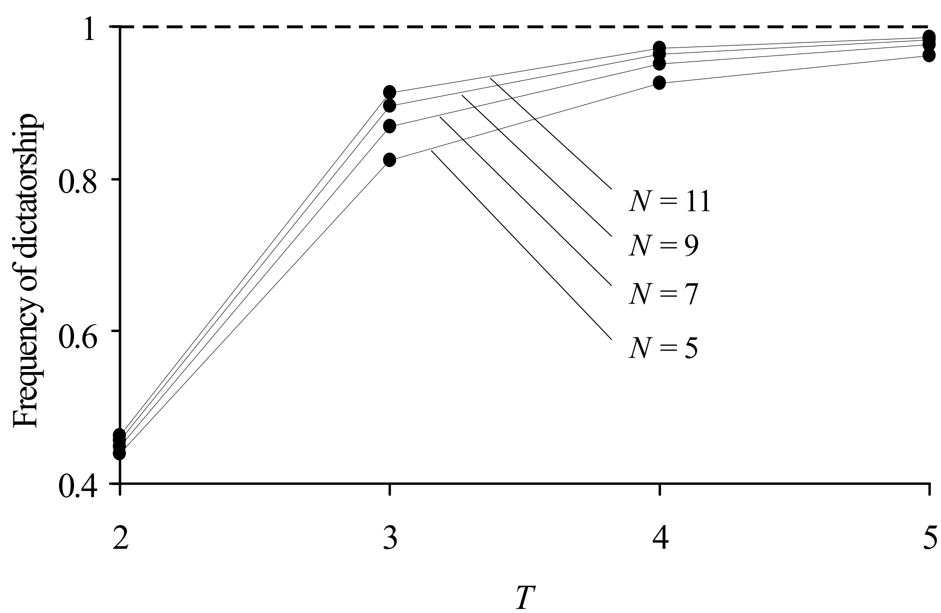


Figure 1: The function $B(N,T)$

The preceding result concerns the fraction of possible recognition orders that produce $\{i(T)\}$ for all initial status quos even when $|J| \leq M$. We now consider the conditions under which a particular initial status quo produces $\{i(T)\}$ regardless of the recognition order, again even though $|J| \leq M$.

**Theorem 6:** *Consider an institution with a degenerate final stage, a CBDC policy set satisfying A1-A4, and $N \geq 5$ individuals. An initial status quo $p_0 \in P$ leads to the outcome $\{i(T)\}$ provided that at least one of the following conditions is satisfied:*

*(i)* $|J| > 2$ *and either* $p_0 = \emptyset$ *or* $\sum_{j \in p_0} c_j > c_{i(T)}$,

*(ii)* $|p_0| \leq M - 1$ *and* $\sum_{j \in p_0} c_j > c_{i(T)}$,

*(iii)* $|J| > |Q_T(p_0)|$.

Part (i) tells us that, with three or more distinct proposers (a very weak condition indeed), $\{i(T)\}$ can be avoided only if the initial status quo consists of a single elementary policy that is less costly than $\{i(T)\}$. This is a small fraction of all feasible initial status quos; moreover, this fraction goes to zero as the number of individuals, $N$, becomes large. Consequently, if a status quo is selected at random in the initial stage, and if at least three individuals are recognized, a large group is almost certain to produce the dictatorial outcome $\{i(T)\}$. Part (ii) tells us that the dictatorial outcome also emerges when the initial status quo is more costly than $\{i(T)\}$ but contains fewer than than $M$ elementary policies, regardless of how many individuals are recognized. Part (iii) tells us that any initial status quo $p_0$ leads to the outcome $\{i(T)\}$ provided that the number of recognized individuals exceeds the number of elementary policies that would be implemented were $i(T)$ to inherit $p_0$ as the round $T$ status quo. Since $|Q_T(p_0)| < M$ for any initial status quo other than $p_0 = \emptyset$, $p_0 = E$, and $p_0 = \{j\}$ for $j$ with $c_j < c_{i(T)}$, part (iii) is typically weaker than $|J| > M$ (the requirement in theorem 3).

### 4.3 Random recognition processes

Next we turn our attention to a more explicit treatment of the possibility, mentioned at the outset of the previous section, that individuals may be uncertain about the recognition order, the number of proposal rounds, or both. We demonstrate that our central result depends not on the absence of uncertainty, but rather on the timing of the resolution of uncertainty from the perspective of the participants. Under the conditions identified below, uncertainty can persist as late as the beginning of the second-to-last proposal round without undermining our conclusions. Notably, our analysis subsumes cases for which there is no finite bound on the number of proposal rounds.

Imagine that, prior to the first proposal round, nature selects the pair $(T, i(\cdot))$, where $T$, the number of rounds, is chosen from the positive integers $\{1, 2, 3, ...\}$, and $i : \{1, ..., T\} \to \{1, ..., N\}$ is a recognition order. Nature might, for example, select $T = T'$ with probability $(1 - \lambda)^{T'-1}\lambda$ for $\lambda \in (0, 1)$, and it might select the proposer in each round through an independent draw from a uniform distribution over the set of participants. In that case, $\lambda$ represents a constant termination probability, there is no finite bound on the number of rounds, and each participant stands a one-in-$N$ chance of being named as the proposer in any given round. For our purposes, it is unnecessary to provide a general description of the class of random process governing the selection of $(T, i(\cdot))$.

Though nature technically selects the number of rounds and the recognition order in advance, some or all of this information may be revealed to the participants at later points in time. We identify two polar cases. At one extreme, participants do not learn the identity of the proposer in each round $t$ until the beginning of $t$, and $T$ is not revealed until after the final round. At the opposite extreme, $(T, i(\cdot))$ is revealed prior to the first round. We characterize intermediate cases as follows. If, for any given process, $(T, i(\cdot))$ is completely revealed (for the first time) by the end of round $T - K$, we say that the process is characterized by $K$ degrees of advance revelation. The first polar case corresponds to zero degrees of advance revelation ($K = 0$), while the second corresponds to $T$ degrees of advance revelation ($K = T$ for each realization of $T$).

Our basic framework subsumes the case where $(T, i(\cdot))$ is revealed prior to the first round ($T$ degrees of advance revelation). Theorem 5 allows us to extend this conclusion to intermediate cases: it tells us that, under plausible assumptions about the random process governing the selection of proposers (e.g., that all participants have an equal chance of being selected in any round), for any process with $K$ degrees of advance revelation, the probability of a dictatorial outcome is close to unity provided that $K$ is sufficiently large. One does not need to make strong, simplifying assumptions concerning the nature of the random process generating $(T, i(\cdot))$, nor to assume that the horizon is bounded.[13] Moreover, it is apparent from figure 1 that even small values of $K$ are sufficient to assure a dictatorial outcome with high probability.

To underscore this last point, consider the case of $K = 2$, which requires that, before the second-to-last proposal is made, participants know both $T$ (i.e. that there will be two more rounds) and the identity of the final proposer. Part (b1) of Theorem 4 tells us that the outcome is dictatorial ($\{i(T)\}$) if $i(T-1) \in H_{M-2} \cup \{i(T)\}$. Thus, if all individuals stand an equal chance of being selected as $i(T-1)$, the probability of achieving a dictatorial outcome is at least $\frac{M-1}{N} = \frac{N-1}{2N}$, which converges to $\frac{1}{2}$ for large groups. Part (a) of Theorem 4 tells us that the outcome

[13] As an example, imagine that, at the end of each period, there is a probability $\lambda$ that the players are told the process will end in $K$ rounds and all remaining proposers are identified; with probability $1 - \lambda$, no information concerning $T$ is revealed, and only the next proposer is identified. Note that, from the perspective of the participants, the potential number of rounds remains unbounded until the process enters the final $K$ rounds.

is either dictatorial or nearly dictatorial ($\{i(T)\}$ or $\{i(T-1), i(T)\}$) if $i(T-1) \in i^*_{M-1}$. Thus, if all individuals stand an equal chance of being selected as $i(T-1)$, the probability of achieving a dictatorial or near dictatorial outcome is at least $\frac{2(M-1)}{N} = \frac{N-1}{N}$, which converges to unity for large groups.

For some plausible processes, $K = 2$ (two degrees of advance revelation) is sufficient to guarantee a dictatorial outcome with high probability. To illustrate, imagine that there exists some set of individuals $\Theta \subseteq \{1, ...N\}$ with $|\Theta| \geq 3$ (e.g., elders, faction leaders, or possibly even the entire group) each of whom is always given one opportunity to make a final proposal before the process terminates. More specifically, $|\Theta|$ rounds prior to termination, participants are told that the process has entered a *final proposal phase* consisting of $|\Theta|$ rounds, and that, in each successive round, the proposer will be selected at random (with equal probabilities) at the beginning of the round from among those in $\Theta$ who have not yet made final proposals. We make no other assumptions about the random process generating $(T, i(\cdot))$, or about the nature of information revealed prior to round $T - |\Theta|$. Note that any process with a final proposal phase is characterized by two degrees of advance revelation: once the second-to-last proposer is identified, all participants can identify the last proposer by default. However, because the process also reveals some information about $i(T)$ and $i(T-1)$ in period $T-2$ (specifically, the identities of the last two proposers but not their order), we obtain a stronger result than for the general case of $K = 2$. Specifically, we have:

**Theorem 7:** *Consider an institution with a degenerate final stage, a CBDC policy set satisfying A1-A4, and $N \geq 5$ individuals. Imagine that $(T, i(\cdot))$ is determined randomly, but that the process is characterized by a final proposal phase. Then, with probability not less than $\frac{N-2}{N}$, the group adopts the elementary policy corresponding to whichever individual is randomly chosen to propose last, and nothing else.*

Observe that the probability that the last proposer is effectively a dictator converges to unity as the size of the group becomes large.

## 5 Alternative Policy Spaces

In section 4, our analysis focused on a particularly simple class of policy spaces. It is natural to wonder whether our central conclusions hold more generally. In this section, we consider three alternative policy spaces. The first generalizes the CBDC assumptions to cases for which there may be multiple elementary policies associated with each individual. The second considers cases for which multiple individuals are associated with the same elementary policy, and therefore have common interests. The third examines the canonical problem of dividing a fixed payoff.

### 5.1 Multiple projects

By adjusting the CBDC assumptions to allow for the possibility that each individual is associated with several elementary projects, each with distinct costs and benefits, we add considerable richness and flexibility to the policy space. Naturally, we can no longer uniquely associate individuals with elementary policies. Hence, we define $I = \{1, ..., N\}$ as the set of individuals, and $E$ as the set of elementary policies. An elementary policy $e$ is a pair, $(i, k)$ (denoting the $k$-th elementary policy associated with the $i$-th individual). Let $E_i$ be the subset of $E$ consisting of elementary policies associated with individual $i$. Under the assumptions stated below, $E_i$ is the outcome that $i$ would pick if $i$ were a dictator. Let $K_i \equiv |E_i|$. We will continue to use $P$ to denote the power set of $E$, and $p \in P$ to denote a policy (a subset of $E$). For any policy $p$, define $p_i = p \cap E_i$. That is, $p_i$ is the set of elementary policies associated with individual $i$ in the policy $p$.

As before, we assume that each elementary policy benefits one individual at a cost to all others. We write the benefit of the elementary policy $(i, k)$ to $i$ as $b_{ik}$, and we write the cost of this policy to each individual as $c_{ik}$. We continue to assume that the cost of any elementary policy is the same to all of the individuals. Accordingly, we define the total costs of any policy $p$ as $C(p) = \sum_{e \in p} c_e$. The proof of the theorem stated below actually uses only the fact that the ranking of the set $P$ by costs is the same for all individuals. Thus, the assumption can be weakened to some extent without altering the proof. We also continue to assume that costs and benefits are additive over elementary policies.

We impose assumptions A1 and A2 as before. Henceforth, $\overline{K}$ represents the largest number of elementary policies associated with any one individual, $\overline{c}$ represents the highest ($\underline{c}$ represents the lowest) cost associated with any elementary policy, and $\underline{b}$ represents the lowest benefit associated with any elementary policy. We replace A3 with the following:

**Assumption A5:** $\overline{K}\overline{c} < (M - 1)\underline{c}$.

Assumption A5 imposes an upper bound on the total costs of all elementary policies associated with any one individual. For the case of $K_i = 1$ for all $i$, this assumption weakens A3 in the sense that it requires each elementary policy to be less expensive than $M - 1$ other elementary policies, rather than two other elementary policies.[14] We also replace A4 with the following:

**Assumption A6:** $\underline{b} > \overline{c}\left(M - 1 + \overline{K}\right)$.

Assumption A6 implies that, if one adopts a collection of $M - 1 + \overline{K}$ elementary policies, all individuals associated with elementary policies in this set are better off. For the case of $\overline{K} = 1$, this coincides (approximately) with assumption A4.[15] For $\overline{K} > 1$, A6 is more demanding that A4 since it requires the benefit from any elementary policy to exceed the costs of more than $M$ other elementary policies.

We are now equipped to state the theorem.

**Theorem 8:** *Consider an institution with a degenerate final stage, a CBDC policy set (possibly with multiple elementary policies for each individual) satisfying assumptions A1, A2, A5, and A6, and $N \geq 5$ individuals. Provided that either $|J| > M$ or $i(T)$ proposes more than once, the unique outcome is the policy $p = E_i$.*

Notice that theorem 3 is a special case of theorem 8 (subject to the technical qualifications noted in the previous two footnotes).[16] It states that, if the recognition order is sufficiently inclusive, then all elementary policies associated with the final proposer, and only these policies, are adopted. Thus, the final proposer emerges as an effective dictator in a very strong sense.

## 5.2 Quasi-distributional politics

Next we extend the CBDC framework to subsume cases in which members of subgroups share common interests. Policies are then "quasi-distributional," in the sense

[14] By writing the assumption in terms of bounds rather than in terms of the costs of particular collections of policies, we have actually imposed some incidental restrictions that are not implied by A3. One can state a version of A5 in terms of collections of policies and thereby generalize A3, but the notation is less compact.

[15] The assumptions are not identical, as A6 is stated in terms of bounds, while A4 is stated in terms of the costs and benefits for collections of policies. One can state a more cumbersome version of A6 in the latter form that specializes exactly to A4 when $K_i = 1$ for all $i$.

[16] The proof is more complicated, however, because it is difficult to show that any policy $q \in \Phi_{i(T)}(P)$ contains $E_{i(T)}$ (this was straightforward for the case $|E_i| = 1$ considered in theorem 3).

that they affect the distribution of payoffs among subgroups, rather than across individuals. For this class of environments, we can no longer uniquely associate elementary policies with individuals. As in section 5.1, we define $I = \{1, ..., N\}$ as the set of individuals. The set of individuals is partitioned into groups $I_s$, $s = 1, ..., N^G$, where $N^G \leq N$. Let $N_s = |I_s|$ (the number of individuals in group $s$). The assignment of individuals into groups is described by a function $r : I \rightarrow \{1, ..., N^G\}$, so that $l \in I_{r(l)}$.

Let $N^*$ denote the median value of $N_{r(l)}$ over all individuals. Note that this is not the median group size. For example, if $N^G = 5$ and the group sizes are 4, 3, 2, 1, and 1, then the median group size is 2, but $N^* = 3$.

There is one elementary policy for each group. Thus, the set of elementary policies is given by $E = \{1, ..., N^G\}$. Note that we can also use $E$ to denote the set of groups. When the elementary policy for group $s \in E$ is implemented, every individual bears a cost $c_s$. Each individual $l \in I_s$ receives a benefit $b_s$. As before, payoffs are additive. Note that there is a common payoff function, $u_s(p)$, for all members of the group $s$, so $v_l(p) = u_{r(l)}(p)$. The policy set $P$ is once again the power set of $E$. Note that we can use $p \in P$ either to represent a policy or a collection of groups.

We continue to impose assumptions A1 and A2. However, we modify assumptions A3 and A4. In particular, we replace A3 with the following:

**Assumption A7:** For any $p \in P$ with $\sum_{j \in p} N_j \geq M - N^*$, we have $\sum_{j \in p} c_j > \max_{s \in E} c_s$.

One can think of assumption A7, like A3, as imposing an upper bound on the costs of individual elementary policies. For the case where each group consists of a single individual, it implies that each elementary policy costs less than the combination of $M - 1$ other elementary policies. Clearly, this relaxes A3.

Alternatively, one can also think of A7 as imposing a limit on the sizes of the groups. It always requires $N^* < M - 1$.[17] Under one set of plausible conditions, it implies $N^* < M/2$.[18] These conditions are not, however, necessary for the result

[17] If $N^* \geq M - 1$, then A7 requires every elementary polilcy to cost strictly more than $\max_{s \in E} c_s$, which is impossible.

[18] If one assumes that total costs are increasing in the number of individuals whose elementary policies are included in $p$ (that is, $\sum_{j \in p} N_j < \sum_{j \in p'} N_j \Rightarrow \sum_{j \in p} c_j < \sum_{j \in p'} c_j$, which is a natural generalization of A3), then A7 reduces to the statement that $\overline{N} + N^* < M$ (where $\overline{N}$ denotes the

stated below. For example, when $N_{r(i(T))} = N^* \geq M$, $\{r(i(T))\}$ is a Condorcet winner in $P$. Consequently, the same conclusion as in theorem 9 below follows directly from theorem 2.

To state our next assumption, we need the following definition.

**Definition:** A collection of groups $L \subseteq E$ is *decisive* if $\sum_{s \in L} N_s \geq M$. Moreover, $L$ is *minimally decisive* if, in addition, $\sum_{s \in L} N_s - N_k < M$ for all $k \in L$. Let $\Lambda$ denote the set of all minimally decisive collections of groups.

We replace A4 with the following:

**Assumption A8:** A mutually beneficial policy (relative to $p = \emptyset$) exists for all minimally decisive collections of groups. That is, for all $p \in \Lambda$, we have $b_s > \sum_{j \in p} c_j$ for all $s \in p$.

Note that, for the case where each group consists of a single individual, assumptions A4 and A8 are equivalent.

Finally, for any recognition order, we define $J' = \cup_{j \in J} I_{r(j)}$. This represents the set of individuals whose interests coincide with someone who has at least one opportunity to make a proposal.

Now we are equipped to state the theorem. It is easy to verify that A7 is never satisfied for $N = 3$. Consequently, the stated assumptions subsume our usual requirement that $N \geq 5$.

**Theorem 9:** *Consider an institution with a degenerate final stage and a quasi-distributional CBDC policy set satisfying A1, A2, A7, and A8. Assume that $N_{r(i(T))} \geq N^*$. Provided that either $|J'| \geq M + |I_{r(i(T))}|$ or some member of $I_{r(i(T))}$ proposes prior to round $T$, the outcome is the policy $p = \{r(i(T))\}$.*

Note that the requirement in Theorem 9 is $|J'| \geq M + |I_{r(i(T))}|$, rather than $|J| > M$. In other words, instead of requiring that at least $M + 1$ individuals have opportunities to make proposals, we require only that at least $M$ individuals other than those in $I_{r(i(T))}$ are "represented by" those with opportunities to make proposals. When groups are large, the latter condition is easily satified even if only a small number of agents make proposals.

size of the largest group). Note that the latter inequality implies $N^* < M/2$. It holds whenever the size of the largest group is less than $N/4$, which in turn requires $N^G \geq 5$.

Note also that the conclusion stated in theorem 9 depends on the identity of the last proposer. In particular, this individual must belong to a group of size $N^*$ or larger. By definition, more than half of all individuals belong to such a group. Thus, if the last proposer is chosen at random (with equal probabilities), then theorem 9 tells us that one obtains a dictatorial outcome with probability greater than one-half.

It is useful to identify the analytic problem that arises when $N_{r(i(T))} < N^*$. The proof of theorem 3 makes use of the fact that $i(T)$ always contrives to make herself pivotal. This may not be possible in a quasi-distributional setting when $i(T)$'s group is small. Consider an example in which there are four groups of sizes 2, 2, 2, and 1, so that $N = 7$, $M = 4$, and $N^* = 2$. Suppose $N_{r(i(T))} = 1$. To build a majority coalition, $i(T)$ must have the support of at least four other members (instead of the three needed before). Accordingly, if $p_{T-1} = E$, the outcome must include the elementary policies associated with two groups other than $I_{r(i(T))}$. But then $\varphi_{i(T)}(p_{T-1}, P)$ majority-defeats $\{r(i(T))\}$, so $\{r(i(T))\}$ is not a Condorcet winner in $\Phi_{i(T)}(P)$.

There is, however, reason to believe that the result might hold with greater generality, even when $N_{r(i(T))} < N^*$. In the previous example, $i(T)$ in essence is required to find coalitions of five individuals (including $i(T)$) to support any proposal, rather than the bare majority of four proposals. Thus, the situation facing $i(T)$ is quite similar to that arising with a supermajority requirement. We treat supermajority requirements in the sequel to this paper (Bernheim, Rangel, and Rayo [2002]), and demonstrate that the dictatorship result is surprisingly robust.

Despite the foregoing, note that when the groups are of equal sizes, all groups have size $N^*$, so it is no longer necessary to impose a condition on the identity of the last proposer. Thus, theorem 9 has the following immediate corollary (where $M^G \equiv (N^G + 1)/2$ and $J^G$ denotes the set of groups for which at least one member makes a proposal):

**Corollary:** *Consider an institution with a degenerate final stage and a quasi-distributional CBDC policy set satisfying A1 and A2. Suppose that*

(i) $N_j = N^*$ *for all* $j \in E$,

(ii) *For any* $p \in P$ *with* $|p| \geq M^G - 1$, *we have* $\sum_{j \in p} c_j > \max_{s \in E} c_s$, *and*

(iii) *For any* $p \in P$ *with* $|p| \leq M^G$, *we have* $b_s > \sum_{j \in p} c_j$ *for all* $s \in p$.

*Then, provided that either* $\left|J^G\right| > M^G$ *or some member of* $I_{r(i(T))}$ *proposes prior to round* $T$, *the outcome is the policy* $p = \{r(i(T))\}$.

Condition (i) simply says that all groups are of the same size. Under condition (i), A7 implies (ii), and A8 implies (iii). When each group contains exactly one individual, condition (ii) generalizes A3, and (iii) is equivalent to A4. Condition (ii) is never satisfied when $N^G = 3$; hence the corollary requires $N^G \geq 5$ (the analog of $N \geq 5$). Note also that $|J| > M$ implies $\left|J^G\right| > M^G$.

Both theorem 9 and its corollary imply that the policy outcome is $\{r(i(T))\}$ whenever *any* member of $I_{r(i(T))}$, and not just $i(T)$, proposes prior to round $T$. If we interpret groups as political parties, this requirement seems innocuous. Thus, if the last proposer is a member of a sufficiently large political party, that party ordinarily dictates the policy outcome.

### 5.3 Splitting a fixed payoff

Finally, we consider the canonical problem of dividing a fixed payoff ($P = \Delta^{N-1}$, the unit simplex in $\mathbb{R}^N$, and $v_i(p) = p_i$). This policy space violates assumption A1 and, more importantly, A2 (the generic no-indifference condition).

The violation of A2 undermines the uniqueness of continuation equilibria, and thereby complicates the analysis considerably. Nevertheless, provided that one adopts a reasonable and consistent rule for resolving this indifference, the outcome is approximately dictatorial.

To understand the issues raised by the possibility of indifference, it is useful to start by defining a *weak Condorcet winner* within a set $R$ as a policy $q \in R$ such that, for all $q' \in R$, a majority of individuals weakly prefer $q$ to $q'$. Notice that, in general, nothing assures the uniqueness of a weak Condorcet winner.

Let $p^j$ denote the policy for which $j$ receives a payoff of unity ($p^j_j = 1$) and every other party receives a payoff of zero ($p^j_l = 0$ for $l \neq j$). This is the alternative that $j$ would select if $j$ were a dictator. It is straightforward to show that $p^{i(T)}$ is a weak Condorcet winner in the set $\Phi_{i(T)}(\Delta^{N-1})$.[19] Somewhat surprisingly, it

[19] Technically, $\Phi_{i(T)}(\Delta^{N-1})$ is not well-defined unless, as below, one resolves indifference in a way that is consistent with the existence of $\varphi_{i(T)}(p, \Delta^{N-1})$ for all $p \in \Delta^{N-1}$.

is also possible to show that this outcome is the unique weak Condorcet winner in $\Phi_{i(T)}(\Delta^{N-1})$.[20]

Were it possible to prove an analog of theorem 2 for unique weak Condorcet winners, then the implications of theorem 3 would generalize immediately to the problem of dividing a fixed payoff. When the policy set includes a unique weak Condorcet winner $p^{wc}$, and when the recognition order is sufficiently inclusive, there does indeed exist an equilibrium that selects $p^{wc}$. The problem with indifference is that, by appropriately contriving the resolution of indifference at various stages of the game, one can in many instances achieve other outcomes.

From our perspective, the most reasonable equilibria in such circumstances are the ones that select $p^{wc}$. To sustain other outcomes, one must assume that individuals who will receive zero payoffs in all continuation paths, and who therefore have absolutely nothing at stake, resolve their indifference when making proposals and casting votes by supporting the course that inflicts the most damage on individual $i(T)$. It is difficult to sustain these outcomes once one rules out malevolence by imposing a consistent rule for resolving indifference.

The violation of A1 also introduces some technical problems related to continuity and openness. We avoid these issues by assuming that the policy space is a discretized version of $\Delta^{N-1}$. Specifically, select some positive integer $m$, and let $\varepsilon = \frac{1}{m}$. Define

$$\Delta_{\varepsilon}^{N-1} \equiv \left\{ p \in \mathbb{R}^N \;\mid\; p \geq 0,\; \sum_{l=1}^{N} p_l = 1, \text{ and } p_l = n\varepsilon \text{ for some } n \in \{0, 1, ..., m\} \right\}.$$

For our next result, we assume that individuals vote in favor of a proposal only if they expect to be strictly better off should the proposal pass. We also rule out complex history dependent punishments by focusing on Markov-perfect equilibria, which can be described by outcome functions $(Q_t)_{t=1}^{T} : \Delta_{\varepsilon}^{N-1} \to \Delta_{\varepsilon}^{N-1}$ (a sequence of functions that, for each round $t$, map the status quo to a final outcome). We demonstrate that, with these restrictions, and with at least three proposers (rather than five as in our previous results), the final proposer receives virtually all of the surplus. This holds for every possible initial status quo, including equal division. In such cases, an approximately dictatorial outcome emerges even though every individual except $i(T)$ would be strictly better off if the group took no action,

[20]Indeed, for any other element of this set, $p'$, there is some other element, $p''$, such that a majority of individuals strictly prefers $p''$ to $p'$.

and even though every proposal requires the approval of a majority to pass. The outcome is approximately dictatorial in the following sense: as $\varepsilon$ approaches zero, $i(T)$'s equilibrium payoff converges to unity.

**Theorem 10:** *Consider an institution with a degenerate final stage, a policy set* $\Delta_{\varepsilon}^{N-1}$, and $N \geq 3$ *individuals. Consider any Markov-perfect equilibrium outcome functions* $(Q_t)_{t=1}^{T}$ *under which each individual* $l$ *votes in favor of* $p_t^m$ *in round* $t$ *if and only if* $(Q_{t+1}(p_t^m))_l > (Q_{t+1}(p_{t-1}))_l$. *Provided that either* $|J| > M$ or $i(T)$ *proposes more than once, we have* $(p_T)_{i(T)} \geq 1 - N\varepsilon$ *for all* $p_0 \in P$.

A natural alternative assumption is that individuals resolve their indifference in favor of the current proposal, rather than against it. This case is considerably more complex. However, one can demonstrate that the outcome satisfies the following two properties (proof omitted): (i) virtually all surplus is divided between $i(T)$, $i(T-1)$, and $i(T-2)$, and (ii) if $\varepsilon < \frac{\delta}{N}$ for some sufficiently small $\delta$, then as $N$ goes to infinity, the surplus received by $i(T-1)$ goes to zero at the rate $\frac{1}{N}$, and the surplus received by $i(T-2)$ goes to zero at the rate $\frac{1}{N^2}$. Thus, in large groups, the last proposer again receives essentially all of the surplus. One can extend these results to the non-discretized simplex by invoking suitable equilibrium refinements.

# 6 Summary and Conclusions

In this paper, we have explored the effect of real-time agenda setting on democratic policy making. Our analysis reveals a surprisingly robust tendency for a natural and simple class of democratic institutions to produce high concentrations of political power. In particular, for some familiar classes of policy spaces with rich distributional politics, the last proposer is effectively a dictator whenever a sufficient number of individuals have opportunities to make proposals, as well as under a variety of related conditions. Thus, under reasonably general assumptions, control of the final proposal with real-time agenda setting confers as much political power as control of the entire agenda. Moreover, this outcome is more likely to arise when more individuals have opportunities to make proposals. Ironically, the last proposer need not have dictatorial powers unless a sufficient number of individuals take part in setting the agenda. Accordingly, within the class of institutions considered in this paper, reforms that appear to be inclusive from a procedural perspective (by promoting "participatory democracy" or guaranteeing a "right to be heard") can have the unintended effect of concentrating political power. We have also demonstrated that institutions belonging to the class considered here yield Pareto efficient outcomes and select Condorcet winners when they exist.

In Bernheim, Rangel, and Rayo [2002], we examine the sensitivity of our central conclusions to variations in institutional rules. Some apparently minor procedural details matter a great deal, while seemingly important rules are actually of little consequence. Supermajority requirements do little to overcome the dictatorial power of the final proposer. Endogenizing the order of recognition has no effect on the high concentration of political power when a chair chooses the order in advance of deliberations, or when the chair makes these decisions round by round but is aligned with a particular member of the group. In the latter case, even a chair with "universalistic" objectives (one that wishes to implement as many elementary policies as possible) may find it impossible to manipulate the order of recognition so as to enact a policy that benefits more than two individuals.

When the rules of the institution permit the group to terminate deliberations before the final round, the power of the last proposer may evaporate. However, the particular outcome depends on the details of the termination rule. For the least restrictive rule (one that allows individuals to bundle policy proposals with motions to preclude reconsideration), political power is simply transferred from the final proposer to the first proposer (and perhaps to one other individual) in a significant fraction of environments. When individuals are not permitted to bundle policy proposals with motions to preclude reconsideration, one can obtain almost anything from inaction to a universalistic outcome, depending on the initial status quo.

If each proposal is subject to amendment before being put to a vote against the prevailing status quo, the power of the last proposer evaporates, but in some instances the group nevertheless selects policies that benefit small minorities (even a single individual) at the expense of large majorities. A rule precluding the reconsideration of elementary policies once they are passed leads to outcomes that benefit groups no larger than minimal majorities. Ironically, when such a rule is combined with a supermajority requirement, the final outcome benefits a minority of members at the expense of a majority, and the number of individuals benefiting from the final outcome *shrinks* with the size of the required supermajority. Limitations on the introduction of new business near the conclusion of deliberations promote inaction.

# Appendix

**Proof of Lemma 1:** For this proof, we use the notation introduced in section 3.2, except that we write $Q_t(p; P, \Omega)$ rather than simply $Q_t(p)$ to make explicit the dependence of this function (suppressed in the notation of the text) on both the policy space $P$ and the final stage $\Omega$. Also, we use $\mathbb{I}$ to denote the identity mapping. Fix $P, \Omega$, and $p \in P$. We proceed by induction. Using the definition of $Q_T$ we obtain:

$$Q_T(p; P, \Omega) = \varphi_{i(T)}\left(\Omega(p), \Omega(P)\right) = \varphi_{i(T)}\left(\mathbb{I} \circ \Omega(p), \mathbb{I} \circ \Omega(P)\right) = Q_T\left(\Omega(p); \Omega(P), \mathbb{I}\right),$$

which proves the claim for $t = T$.

Now suppose the claim is true for $t+1$, which together with the definition of $Q_t$ implies:

$$Q_t(p; P, \Omega) = \varphi_{i(t)}\left(Q_{t+1}(p; P, \Omega), Q_{t+1}(P; P, \Omega)\right) =$$
$$\varphi_{i(t)}\left(Q_{t+1}\left(\Omega\left(p\right); \Omega\left(P\right), \mathbb{I}\right), Q_{t+1}\left(\Omega\left(P\right); \Omega\left(P\right), \mathbb{I}\right)\right) = Q_t(\Omega(p); \Omega(P), \mathbb{I}),$$

establishing the result. Q.E.D.

**Proof of Theorem 3:** Aside from properties 2 and 3, theorem 3 is proven in the text.

To prove property 2, consider $p \in \Phi_{i(T)}(P)$ and any $q \in P$ such that $p \in \varphi_{i(T)}(q, P)$. Suppose that $|p| > M$. Choose any set $S$ with $i(T) \in S$ and $|S| = M$, such that $v_l(p) \geq v_l(q)$ for all $l \in S$ (existence of $S$ is guaranteed because $p \in \varphi_{i(T)}(q, P)$, but it may not be unique). $S$ represents a minimal decisive coalition including the last proposer, all the members of which prefer $p$ to $q$. Consider some new policy $p'$ formed by deleting from $p$ all elementary policies associated with individuals not in $S$ (formally, $p' = p \cap S$). Note that $v_l(p') > v_l(p)$ for all $l \in S$ (since total costs are lower). But then $v_l(p') > v_l(q)$ for all $l \in S$, which implies $p' \in Z(q, P)$, contradicting the fact that $p = \varphi_{i(T)}(q, P)$.

To prove property 3, it is useful to distinguish between three mutually exclusive and exhaustive cases, defined by the characteristics of $p_{T-1}$. For each case, we describe a $p'$ containing the elementary policy $i(T)$ with $|p'| \leq M$, and we prove that $p' \in Z(p_{T-1}, P)$. Assumption A4 guarantees that $v_{i(T)}(p') > 0$. Since any policy $p$ excluding the elementary policy $i(T)$ provides individual $i(T)$ with a non-positive payoff, this implies that $i(T)$'s best choice must result in the implementation of some policy $p''$ containing the elementary policy $i(T)$, as desired.

Case 1: $i(T) \in p_{T-1}$. If $|p_{T-1}| \leq M$, take $p' = p_{T-1}$. If $|p_{T-1}| > M$, suppose that $i(T)$ proposes the policy $p'$ obtained by dropping $M-1$ elementary policies from $p_{T-1}$ other than $i(T)$. The proposed policy improves the payoff to any individual associated with an elementary policy that is not dropped. Since this group forms a majority, $p' \in Z(p_{T-1}, P)$.

Case 2: $i(T) \notin p_{T-1}$ and $|p_{T-1}| < M$. Consider any $p'$ such that and $i(T) \in p'$, $|p'| = M$, and $p' \cap p_{T-1} = \emptyset$. By A4, all individuals associated with elementary policies in $p'$ strictly prefer $p'$ to $p$. Since this group forms a majority, $p' \in Z(p_{T-1}, P)$.

Case 3: $i(T) \notin p_{T-1}$ and $|p_{T-1}| \geq M$. Consider any $p'$ such that $i(T) \in p'$, $|p'| = |p_{T-1}| - (M-2)$, and $p' \backslash \{i(T)\} \subset p_{T-1}$ (in other words, drop $M-1$ elementary policies from $p_{T-1}$ and add $i(T)$). By A3 and $N \geq 5$ (which implies $M \geq 3$), all individuals are better off except those in $p_{T-1} \backslash p'$. But $|p_{T-1} \backslash p'| = M - 1$, so $p' \in Z(p_{T-1}, P)$. Q.E.D.

**Proof of Theorem 4:** The following argument makes use of properties 1-3 from section 4.1.2. We begin with a lemma. Note that the four cases stated in the lemma are mutually exclusive and exhaustive.

**Lemma 4.1:** *Consider any* $q \in P$.

(i) *If* $\sum_{k \in q} c_k > c_{i(T)}$, *then* $\varphi_{i(T)}(q, P) \cap \left(H_{M-2} \cup \{i^*_{M-1}\}\right) = \emptyset$,

(ii) *If* $q = \{j\}$ *with* $c_j < c_{i(T)}$ *and* $j \notin H_{M-2}$, *then* $\varphi_{i(T)}(q, P) = E \backslash \left(H_{M-2} \cup \{j\}\right)$,

(iii) *If* $q = \{j\}$ *with* $c_j < c_{i(T)}$ *and* $j \in H_{M-2}$, *then* $\varphi_{i(T)}(q, P) = E \backslash \left(H_{M-2} \cup \{i^*_{M-1}\}\right)$, and

(iv) *If* $q = \emptyset$, *then* $\varphi_{i(T)}(q, P) = E \backslash \left(H_{M-2} \cup \{i^*_{M-1}\}\right)$.

**Proof:** For the proof of each part, we use $p = \varphi_{i(T)}(q, P)$ for notational simplicity.

(i) We begin by showing that there exists $p' \in Z(q, P)$ such that $i(T) \in p'$ and $\sum_{k \in p'} c_k < \sum_{k \in q} c_k$. If $i(T) \in q$, this is trivial (simply drop any elementary policy from $q$). If $i(T) \notin q$, the construct $p'$ as follows: if $|q| > 1$, drop any two elementary policies from $q$ and add $i(T)$; if $|q| = 1$, then $p' = \{i(T)\}$.

Clearly, $\sum_{k\in p} c_k \leq \sum_{k\in p'} c_k$ (otherwise $i(T)$ would propose $p'$ rather than $p$). Thus, $\sum_{k\in p} c_k < \sum_{k\in q} c_k$.

Next we argue that $j \notin q$ and $j \neq i(T)$ implies $j \notin p$. Suppose not. Consider $p'' = p\backslash\{j\}$. All individuals but $j$ prefer $p''$ to $p$, and $j$ prefers $p''$ to $q$ (since $j$'s elementary policy is excluded from both $p''$ and $q$ and since the cost of $p''$ is lower). Thus, all those who prefer $p$ to $q$ also prefer $p''$ to $q$, so $p'' \in Z(q, P)$. But then $i(T)$ would propose $p''$ rather than $p$, a contradiction.

Now suppose that, contrary to part (i) of the lemma, there exists some $j \in p \cap \left(i^*_{M-1} \cup H_{M-2}\right)$. Let $S = \{i \in E \mid v_i(p) > v_i(q)\}$. Then $|S| = M$ (if $|S| > M$, then $p\backslash\{j\} \in Z(q, P)$, and $i(T)$ prefers $p\backslash\{j\}$ to $p$, so $p \notin \varphi_{i(T)}(q, P)$). Since $p$ is less costly than $q$, we must have $E\backslash S \subseteq q$ and $(E\backslash S) \cap p = \emptyset$ (otherwise an individual in $E\backslash S$ would prefer $p$ to $q$). Since $|E\backslash S| = M-1$, and since $i(T) \notin E\backslash S$, there is some $j' \in E\backslash S$ with $c_{j'} < c_j$. Consider $\widehat{p}$ formed by deleting $j$ from $p$ and inserting $j'$. Clearly, $\sum_{k\in\widehat{p}} c_k < \sum_{k\in p} c_k < \sum_{k\in q} c_k$. Thus, all members of $S\backslash\{j\}$ strictly prefer $\widehat{p}$ to $p$ and $p$ to $q$, while $j'$ clearly prefers $\widehat{p}$ to $q$. Since $|(S\backslash\{j\}) \cup \{j'\}| = M$, we have $\widehat{p} \in Z(q, P)$. But since $i(T)$ strictly prefers $\widehat{p}$ to $p$, this contradicts $p = \varphi_{i(T)}(q, P)$.

(ii) Since $i(T) \in p$, we know that $j$ strictly prefers $q$ to $p$. It follows that $j \notin p$ (if not, then $p\backslash\{j\} \in Z(q, P)$, and $i(T)$ strictly prefers $p\backslash\{j\}$ to $p$, a contradiction). Moreover, since $i(T) \in p$ and $c_{i(T)} > c_j$, we know that individual $i$ prefers $p$ to $q$ only if $i \in p$. Thus, $|p| \geq M$. Clearly, $E\backslash\left(H_{M-2} \cup \{j\}\right) \in Z(q, P)$. Moreover, with $j \notin H_{M-2}$, this is the lowest cost policy containing $i(T)$ and at least $M-1$ other elementary components other than $j$. Thus, it is $i(T)$'s best choice.

(iii) The proof is identical to that of part (ii), except that $E\backslash\left(H_{M-2} \cup \{i^*_{M-1}\}\right)$ takes the place of $E\backslash\left(H_{M-2} \cup \{j\}\right)$.

(iv) Since $i(T) \in p$ and $c_{i(T)} > 0$, we know that individual $i$ prefers $p$ to $q$ only if $i \in p$. Thus, $|p| \geq M$. Clearly, $E\backslash\left(H_{M-2} \cup \{i^*_{M-1}\}\right) \in Z(q, P)$. Moreover, it is the lowest cost policy containing $i(T)$ and at least $M-1$ other elementary components. Thus, it is $i(T)$'s best choice. Q.E.D.

**Proof of part (b1):** Lemma 4.1 and $i(T) \in p$ imply that all members of $H_{M-2}$ prefer $\{i(T)\}$ to all other elements of $\Phi_{i(T)}(P)$. The same is obviously true for individual $i(T)$. Given that $\{i(T)\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$ (see the proof of theorem 3), part (b1) of theorem 4 follows from theorem 2, part (2).

**Proof of part (a):** If $i(T-1) \in H_{M-2} \cup \{i(T)\}$, we know the outcome is $\{i(T)\}$ by part (b1). So, throughout the rest of the proof of part (a), we will suppose that

$i(T-1) \in E\backslash\left(\{i(T), i^*_{M-1}\} \cup H_{M-2}\right) \equiv L_{M-1}$ (note that $L_{M-1}$ is the set of $M-1$ least costly elementary policies other than $i(T)$).

We claim that $\{i(T-1), i(T)\} \in \Phi_{i(T)}(P)$. In particular, consider $q = \{i(T), i(T-1), i^*_{M-1}\} \cup H_{M-2}$. Note that $\{i(T-1), i(T)\} \in Z(q, P)$ (it is strictly preferred to $q$ by members of $E\backslash\left(\{i^*_{M-1}\} \cup H_{M-2}\right)$). The only elements of $P$ that $i(T)$ prefers to $\{i(T-1), i(T)\}$ are $\{i(T)\}$ and $\{i(T), j\}$ where $c_j < c_{i(T-1)}$. Note that all members of $\{i(T-1), i^*_{M-1}\} \cup H_{M-2}$ strictly prefer $q$ to both $\{i(T)\}$ and $\{i(T), j\}$ (since $i(T-1) \in L_{M-1}$ and $c_j < c_{i(T-1)}$ implies $j \notin \{i(T-1), i^*_{M-1}\} \cup H_{M-2}$). Consequently, $\{i(T)\}, \{i(T), j\} \notin Z(q, P)$. But then $\{i(T-1), i(T)\} = \varphi_{i(T)}(q, P)$, as required.

Now we establish that $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{\{i(T)\}, \{i(T-1), i(T)\}\}$. By theorem 3, we know that $\{i(T)\} \in \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. Since $i(T) \in p$ for all $p \in \Phi_{i(T)}(P)$ and since $\{i(T-1), i(T)\} \in \Phi_{i(T)}(P)$, we have $\{i(T-1), i(T)\} = \varphi_{i(T-1)}(\{i(T-1), i(T)\}, \Phi_{i(T)}(P)) \in \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. We claim that $p \notin \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$ for all other $p \in \Phi_{i(T)}(P)$. Suppose not. Consider any $q \in \Phi_{i(T)}(P)$ such that $p \in \varphi_{i(T-1)}(q, \Phi_{i(T)}(P))$ for $p \notin \{\{i(T)\}, \{i(T-1), i(T)\}\}$. There are two cases to consider.

(i) $i(T-1) \notin p$. Since $\{i(T)\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$, we know that $\{i(T)\} \in Z(q, \Phi_{i(T)}(P))$. But since $i(T) \in p$, individual $i(T-1)$ strictly prefers $\{i(T)\}$ to $p$. Thus, $p \notin \varphi_{i(T-1)}(q, \Phi_{i(T)})$, a contradiction.

(ii) $i(T-1) \in p$. Since $i(T) \in p$, we know that $|p| \geq 3$. Since $p \in Z(q, \Phi_{i(T)}(P))$ and $|p| \leq M$, we must also have $|q| \geq 3$ (otherwise all members of $(E\backslash p) \cup \{i(T)\}$, a majority, would prefer $q$ to $p$). Everyone in $(E\backslash q) \cup \{i(T)\}$ must then prefer $\{i(T-1), i(T)\}$ to $q$. Since $|q| \leq M$ and $i(T) \in q$, this is a majority, so $\{i(T-1), i(T)\} \in Z\left(q, \Phi_{i(T)}(P)\right)$. But since $i(T-1)$ prefers $\{i(T-1), i(T)\}$ to $p$, we have a contradiction. This completes the proof of part (a).

**Proof of part (b2):** From part (a), we know that $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{\{i(T)\}, \{i(T-1), i(T)\}\}$. All individuals other than $i(T-1)$ strictly prefer $\{i(T)\}$ to $\{i(T-1), i(T)\}$. Consequently, for any $j \neq i(T-1)$, $\Phi_j(\{\{i(T-1), i(T)\}, \{i(T)\}\}) = \{i(T)\}$. The desired result follows directly. Q.E.D.

**Proof of Theorem 5:** The proof consists of two steps.

**Step 1:** Characterization of $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. There are four cases to consider.

(a) $i(T-1) \in H_{M-2} \cup \{i(T)\}$. Then, by theorem 4, part (b1), we know that $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$.

(b) $i(T-1) \in L_{M-1}$ (defined, as in the proof of theorem 4, as the $M-1$ lowest cost elementary policies other than $i(T)$). From the proof of theorem 4, part (a), we know that $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{\{i(T)\}, \{i(T-1), i(T)\}\}$. Note that, in this case, $\Phi_j \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$ for any $j \neq i(T-1)$.

(c) $i(T-1) = i^*_{M-1}$ and $i(T)$ is not the lowest cost elementary policy. We claim that, if $p$ is an element of $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$, then either $p = \{i(T)\}$, or $p$ is of the form $E \backslash (H_{M-2} \cup \{j\})$ for $j \in L_{M-1}$. Suppose $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$ contains some other policy $p$. Consider any $q \in \Phi_{i(T)}(P)$ such that $p = \varphi_{i(T)}(q, \Phi_{i(T)}(P))$. From the proof of theorem 3, we know that $\{i(T)\} \in Z(q, \Phi_{i(T)}(P))$. By lemma 4.1, we know that $i^*_{M-1} \notin p$. Thus, $i^*_{M-1}$ strictly prefers $\{i(T)\}$ to $p$, which is a contradiction. Note that, in this case, $\Phi_j \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$ for any $j \in H_{M-2} \cup \{i(T)\}$.

(d) $i(T-1) = i^*_{M-1}$ and $i(T)$ is the lowest cost elementary policy. Then by lemma 4.1 parts (i) and (iv), $i^*_{M-1} \notin p$ for all $p \in \Phi_{i(T)}(P)$. But then, by theorem 2 part (2), $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$.

**Step 2.** Computation of the function $B(N,T)$. There are two cases to consider.

Case 1: $i(T)$ is not the lowest cost elementary policy. In that case, according to step 1, the process can yield an outcome other than $\{i(T)\}$ only if either (a) $i(t) = i(T-1) \in L_{M-1}$ for all $t < T-1$, or (b) $i(T-1) = i^*_{M-1}$ and $i(t) \notin H_{M-2} \cup \{i(T)\}$ for all $t < T-1$. Case 1(a) occurs for the following fraction of recognition orders:

$$\left(\frac{N-1}{N}\right)\left(\frac{M-1}{N}\right)\left(\frac{1}{N}\right)^{T-2}.$$

Case 1(b) occurs in the fraction of recognition orders:

$$\left(\frac{N-1}{N}\right)\left(\frac{1}{N}\right)\left(\frac{M}{N}\right)^{T-2}.$$

Case 2: $i(T)$ is the lowest cost elementary policy. In this case, according to step 1, the process can yield an outcome other than $\{i(T)\}$ only if $i(t) = i(T-1) \in L_{M-1}$ for all $t < T-1$, which occurs in the following fraction of orders:

$$\left(\frac{1}{N}\right)\left(\frac{M-1}{N}\right)\left(\frac{1}{N}\right)^{T-2}.$$

Combining these expressions, we find that the fraction of orders producing $\{i(T)\}$ is at least:

$$1 - \left(\frac{M-1}{N}\right)\left(\frac{1}{N}\right)^{T-2} - \left(\frac{N-1}{N}\right)\left(\frac{1}{N}\right)\left(\frac{M}{N}\right)^{T-2}.$$

We replace $M-1$ with $\frac{N-1}{2}$ and factor to obtain the formula for $B(N,T)$. Q.E.D.

**Proof of Theorem 6:** We begin the proof with a lemma.

**Lemma 6.1:** *For any $p \in P$ and $t < T$, the total cost of $Q_t(p)$ is not greater than the total cost of $Q_T(p)$.*

**Proof:** Note first that, for any $p \in P$, if $Q_t(p) \neq Q_{t+1}(p)$, then a majority of individuals must strictly prefer $Q_t(p)$ to $Q_{t+1}(p)$.

Next we argue that, for all $t = 1, ..., T-1$, the total cost of $Q_t(p)$ is not greater than the total cost of $Q_{t+1}(p)$ (the lemma follows directly). Suppose on the contrary the that total cost of $Q_t(p)$ is greater than the total cost of $Q_{t+1}(p)$ for some such $t$. Then all members of $E \backslash Q_t(p)$ strictly prefer $Q_{t+1}(p)$ to $Q_t(p)$. Individual $i(T)$ also prefers $Q_{t+1}(p)$ to $Q_t(p)$ (since both policies include $i(T)$). But $|\{i(T)\} \cup (E \backslash Q_t(p))| \geq M$ (since $i(T) \in Q_t(p)$ and $|Q_t(p)| \leq M$), which contradicts the fact that $Q_t(p)$ must be majority-preferred to $Q_{t+1}(p)$. Q.E.D.

Now we prove the theorem.

Part (i): Consider once again the four mutually exclusive and exhaustive cases discussed in the proof of theorem 5, step 1. For cases (a) and (d), $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$, so $Q_1(p_0) = \{i(T)\}$. For case (b), $\Phi_j \circ \Phi_{i(T-1)} \circ \Phi_{i(T)}(P) = \{i(T)\}$ for any $j \neq i(T-1)$, so as long as $|J| > 2$ (which assures $i(t) \neq i(T-1)$ for some $t < T-1$), $Q_1(p_0) = \{i(T)\}$.

Now consider case (c). We claim that $Q_T(p_0)$ is no more costly than the policy $L_{M-1} \cup \{i(T)\}$. By assumption, either $p_0 = 0$ or $\sum_{j \in p_0} c_j > c_{i(T)}$. First consider the subcase where $p_0 = \emptyset$. By lemma 4.1 part (iv), $Q_T(p_0) = E \backslash \left(H_{M-2} \cup \{i^*_{M-1}\}\right) = L_{M-1} \cup \{i(T)\}$, as required. Next consider the subcase where $\sum_{j \in p_0} c_j > c_{i(T)}$. Note that $(p_0 \cap L_{M-1}) \cup \{i(T)\} \in Z(p_0, P)$ (since it omits from $p_0$ no more than $M-1$ elementary policies). Thus, $Q_T(p_0)$ must be no more costly than $L_{M-1} \cup \{i(T)\}$ (or $i(T)$ would propose $(p_0 \cap L_{M-1}) \cup \{i(T)\}$ instead).

In light of the preceding claim, lemma 6.1 tells us that $Q_1(p_0)$ is no more costly than $L_{M-1} \cup \{i(T)\}$. We know from the proof of theorem 5 that if $p$ is an element of $\Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$, then either $p = \{i(T)\}$, or $p$ is of the form $E \backslash (H_{M-2} \cup \{j\})$ for $j \in L_{M-1}$. Also recall that $Q_1(p_0) \in \Phi_{i(T-1)} \circ \Phi_{i(T)}(P)$. But $E \backslash (H_{M-2} \cup \{j\})$ for $j \in L_{M-1}$ is more costly than $L_{M-1} \cup \{i(T)\}$ (since the latter policy is the same as $E \backslash \left(H_{M-2} \cup \{i^*_{M-1}\}\right)$. Thus, $Q_t(p_0) = \{i(T)\}$, as desired.

Part (ii): If $|p_0| \leq M-1$ and $\sum_{j\in p_0} c_j > c_{i(T)}$, then $\{i(T)\} \in Z(p_0, P)$, so $Q_T(p_0) = \{i(T)\}$. Since $\{i(T)\}$ is strictly less costly than all other policies in $\Phi_{i(T)}(P)$ (given that all such policies contain $i(T)$), lemma 6.1 implies immediately that $Q_1(p_0) = \{i(T)\}$.

Part (iii): From the argument given in the proof of theorem 2 and the fact that $\{i(T)\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$, we know that $v_l\left(Q_1(p_0)\right) \geq v_l(\{i(T)\})$ for all $l \in J\backslash\{i(T)\}$. By lemma 6.1, $|Q_1(p_0)| \leq |Q_T(p_0)| < |J|$. Consequently, since $i(T) \in Q_1(p_0)$, there exists $l' \in J\backslash\{i(T)\}$ such that $l' \notin Q_1(p_0)$. But then $v_{l'}\left(Q_1(p_0)\right) \geq v_{l'}(\{i(T)\})$ implies $Q_1(p_0) = \{i(T)\}$. Q.E.D.

**Proof of Theorem 7:** Consider the selection of a proposal in round $T-2$. Let $i^A$ and $i^B$ denote the two members of $\Theta$ who make proposals after $T-2$. Suppose in addition that $i^*_{M-1} \notin \{i^A, i^B\}$ (which occurs with probability $\frac{N-2}{N}$). According to part (a) of theorem 4, all individuals know that, if $i^k$ is chosen to propose last ($k = A, B$), the final outcome will be either $\{i^k\}$ or $\{i^A, i^B\}$. From the perspective of round $T-2$, all individuals other than $i^A$ and $i^B$ (including $i(T-2)$) would therefore strictly prefer an outcome that produces $i^A$ when $i^A$ is chosen to propose last, and $i^B$ when $B$ is chosen to propose last, to any other outcome that is achievable given the continuation equilibria for rounds $T-1$ and $T$. Suppose that $i(T-2)$ proposes $\{i^A, i^B\}$, and that the proposal passes. In round $T-1$, all individuals other than $i^A$ and $i^B$ know that if they vote against $p_t^m$ (regardless of what it is), the ultimate outcome will be $i^k$ (with $p_{T-1} = \{i^A, i^B\}$, $i^k$ will propose $\{i^k\}$ in the final round, and this will pass). Consequently, they will vote for $p_t^m$ only if it also yields $\{i^k\}$ as the final outcome. Thus, passing the proposal $\{i^A, i^B\}$ in round $T-2$ produces $i^A$ when $i^A$ is chosen to propose last, and $i^B$ when $i^B$ is chosen to propose last. If $i(T-2)$'s equilibrium proposal led to any other outcome, $i(T-2)$ would have an incentive to deviate to the proposal $\{i^A, i^B\}$, and all individuals other than $i^A$ and $i^B$ would have an incentive to support it. Thus, as long as $i^*_{M-1} \notin \{i^A, i^B\}$, the process implements the elementary policy associated with the last proposer, and nothing else. Q.E.D.

**Proof of Theorem 8:** We begin with some notation. For any $i \in I$ and $q \in P$, let $G(q)$ denote some set of $M-1$ individuals, not including $i(T)$, satisfying the following condition: $j \in G(q)$ implies $v_j\left(\varphi_{i(T)}(q, P)\right) \geq v_j(q)$. That is, $G(q)$ represents a group of $M-1$ individuals other than $i(T)$ who (weakly) prefer $\varphi_{i(T)}(q, P)$

to $q$ (by A2, equality will hold only if $q = \varphi_{i(T)}(q, P)$). By the definition of $\varphi_{i(T)}(\cdot)$, one can always identify such a group, though it need not be unique. When it is not unique, $G(q)$ represents some arbitrary selection from the set of groups satisfying this condition. Let $\Gamma(q)$ denote the set of $M-1$ individuals which is the complement of $G(q) \cup \{i(T)\}$ in $I$. The support of individuals in $\Gamma(q)$ is not required for the passage of $\varphi_i(q, P)$ (although in any given instance some members of this group may nevertheless prefer this outcome to $q$). Thus, one can think of $\Gamma(q)$ as the "minority." Finally, we define $\varphi^{-1}_{i(T)}(p, P)$ as the set of policies $q$ such that $p = \varphi_{i(T)}(q, P)$. Note that this inverse exists for all $p \in \Phi_{i(T)}(P)$.

**Step 1:** As in the proof of theorem 3, we proceed by demonstrating that $\Phi_{i(T)}(P)$ has the following three properties.

**Property 1:** $E_{i(T)} \in \Phi_{i(T)}(P)$. As before, this is straightforward, since $E_{i(T)} = \varphi_{i(T)}(E_{i(T)}, P)$.

**Property 2:** For any $p \in \Phi_{i(T)}(P)$ and $q \in \varphi^{-1}_{i(T)}(p, P)$, we have $|p_j| = 0$ for all $j \in \Gamma(q)$.

Suppose not. Consider any policy $q \in \varphi^{-1}_{i(T)}(p, P)$. Define $p' = p \cap \left(\cup_{j \in G(q) \cup \{i(T)\}} E_j\right)$ (in other words, delete all elementary policies associated with individuals in $\Gamma(q)$ from $p$). Plainly, all $j \in G(q) \cup \{i(T)\}$ strictly prefer $p'$ to $p$ and weakly prefer $p$ to $q$, so $p' \in Z(q, P)$. But since $i(T)$ prefers $p'$ to $p$, this contradicts the fact that $p = \varphi_{i(T)}(q, P)$.

**Property 3:** For any $p \in \Phi_{i(T)}(P)$ with $p \neq E_{i(T)}$, we have $C(p) > C(E_{i(T)})$.

By assumption A5, this is obviously the case for any $p$ with $|p| \geq M-1$. So for the remainder of this proof, we focus on the case of $p$ with $|p| < M-1$. Consider any $q \in \varphi^{-1}_{i(T)}(p, P)$. Note for future reference that $|p_j| = 0$ for some $j \in G(q)$ (since $|p| < M-1$).

We prove property 3 through a series of three claims.

**Claim 1:** $C(q) \geq C(p)$. Assume on the contrary that $C(p) > C(q)$. For all $j \in G(q)$ with $|p_j| = 0$, we have $v_j(p) = -C(p) < -C(q) \leq v_j(q)$, which contradicts $j \in G(q)$.

**Claim 2:** If $C(q) = C(p)$, then $p = E_{i(T)}$. By assumption A2, $C(q) = C(p)$ implies $q = p$. Assume contrary to the claim that $p \neq E_{i(T)}$. There are two cases to consider.

Case 1: $|p_j| = 0$ for all $j \neq i(T)$. Consider any $p'$ such that $\left|p'_{i(T)}\right| = K_{i(T)}$, $\left|p'_j\right| = 1$ for $j \in G(q)$, and $\left|p'_j\right| = 0$ for $j \notin G(q) \cup \{i(T)\}$. By assumption A6, all

$j \in G(q) \cup \{i(T)\}$ strictly prefer $p'$ to $p$. Since $p = q$, we have $p' \in Z(q, P)$. Since $i(T)$ strictly prefers $p$ to $p'$, we have $p \neq \varphi_{i(T)}(q, P)$, a contradiction.

Case 2: $|p_{j^*}| > 0$ for some $j^* \neq i(T)$. Consider $p'$ constructed as follows: $p' = p \backslash E_{j^*}$. All individuals but $j^*$ strictly prefer $p'$ to $p$. Since $p = q$, $p' \in Z(q, P)$. Since $i(T)$ strictly prefers $p'$ to $p$, we have $p \neq \varphi_{i(T)}(q, P)$, a contradiction.

**Claim 3:** If $C(q) > C(p)$ and $p \neq E_{i(T)}$, then $C(p) > C(E_{i(T)})$. We prove claim 3 through a series of three steps. Throughout, we assume that $C(q) > C(p)$ and $p \neq E_{i(T)}$.

**3.1:** $\exists j^* \in G(q)$ such that $|p_{j^*}| > 0$. Suppose not. Then, by property 2, $|p_j| = 0$ for all $j \neq i(T)$. Arguing exactly as in the proof of case 1 for claim 2, this can only be the case if $p = E_{i(T)}$, a contradiction.

**3.2:** $|q_j| > 0$ for all $j \in \Gamma(q)$. Suppose on the contrary that there exists some $j \in \Gamma(q)$ such that $|q_j| = 0$. Then, since $C(q) > C(p)$, we have $v_j(p) > v_j(q)$. In that case, $p$ is preferred to $q$ by all members of $G(q) \cup \{i(T), j\}$, which constitutes a supermajority. Consider $p'$ defined as follows: $p' = p \backslash E_{j^*}$ (where $j^*$ was identified in step 3.1). Note that $p'$ is strictly preferred to $p$ by all individuals in $G(q) \cup \{i(T), j\}$ except for $j^*$. Since $p$ is preferred to $q$ by all members of this same group, a strict majority prefers $p'$ to $q$. Thus, $p' \in Z(q, P)$. Since $i(T)$ strictly prefers $p'$ to $p$, we have $p \neq \varphi_{i(T)}(q)$, a contradiction.

**3.3:** $p_{i(T)} = E_{i(T)}$ (from which it follows immediately that $C(p) > C(E_{i(T)})$, as desired). Suppose not. Divide the set $G(q)$ into the following two subsets:

$$A = \{j \in G(q) \mid C(p_j) < C(q_j) - \overline{c}\} \text{ and } B = G(q) \backslash A.$$

Now construct $p'$ as follows:

$$\begin{aligned} p'_{i(T)} &= E_{i(T)}, \\ p'_j &= 0 = p_j \text{ for } j \in \Gamma(q), \\ p'_j &\supseteq p_j \text{ and } \left|p'_j\right| = |p_j| + 1 \text{ for } j \in A, \\ p'_j &= q_j \text{ for } j \in B. \end{aligned}$$

Note that $C(p'_j) \leq C(p_j) + \overline{c}$ for all $j \in G(q)$. Thus, $C(p') \leq C(p) + \left(\overline{K} + M - 1\right)\overline{c}$.

Note also that $C(p'_j) \leq C(q_j)$ for $j \in G(q)$. Thus,

$$\begin{aligned} C(p') &= C(p'_{i(T)}) + C\left(\cup_{j \in G(q)} p'_j\right) \\ &\leq \overline{K}\overline{c} + C\left(\cup_{j \in G(q)} q_j\right) \\ &\leq \overline{K}\overline{c} + C\left(\cup_{j \in G(q)} q_j\right) + \left[C\left(\cup_{j \in \Gamma(q)} q_j\right) - (M-1)\underline{c}\right] + C(q_{i(T)}) \\ &= C(q) - \left[(M-1)\underline{c} - \overline{K}\overline{c}\right] \\ &< C(q) \end{aligned}$$

(where we have used step 3.2 for the second inequality and A5 for the final inequality). Thus, $C(p') < C(q)$.

Consider any $j \in A \cup \{i(T)\}$. Note that

$$\begin{aligned} v_j(p') - v_j(p) &\geq \underline{b} - \left(C(p') - C(p)\right) \\ &\geq \underline{b} - \left(\overline{K} + M - 1\right)\overline{c} \\ &> 0 \end{aligned}$$

by assumption A6. Thus, all such $j$ strictly prefer $p'$ to $p$. Since $A \cup \{i(T)\} \subseteq G(q) \cup \{i(T)\}$, these same individuals strictly prefer $p$ to $q$, and therefore strictly prefer $p'$ to $q$.

Now consider any $j \in B$. Since $p'_j = q_j$, we have $v_j(p') - v_j(q) = C(q) - C(p') > 0$. Thus, all such $j$ strictly prefer $p'$ to $q$.

From the preceding, it follows that all $j \in G(q) \cup \{i(T)\}$ strictly prefer $p'$ to $q$. Thus, $p' \in Z(q, P)$. But since $i(T)$ strictly prefers $p'$ to $p$, this implies that $p \neq \varphi_{i(T)}(q, P)$, which is a contradiction.

**Step 2:** Now we argue that $E_{i(T)}$ is a Condorcet winner in $\Phi_{i(T)}(P)$. By property 1, we know that $E_{i(T)} \in \Phi_{i(T)}(P)$. Consider any $p \in \Phi_{i(T)}(P)$ other than $E_{i(T)}$. Let $q$ be any policy in $\varphi^{-1}_{i(T)}(p, P)$. By properties 2 and 3, we know that, for all $j \in \Gamma(q)$, we have $v_j(E_{i(T)}) - v_j(p) = C(p) - C(E_{i(T)}) > 0$. Thus, all such $j$ strictly prefer $E_{i(T)}$ to $p$. Obviously, $i(T)$ also strictly prefers $E_{i(T)}$ to $p$. Since $|\Gamma(q)| = M - 1$, a majority of individuals strictly prefer $E_{i(T)}$ to $p$, as required.

**Step 3:** To complete the proof of the theorem, we apply lemma 1 and theorem 2, exactly as in the proof of theorem 3. Q.E.D.

**Proof of Theorem 9:** Consider any $p \in \Phi_{i(T)}(P)$. Throughout, we use $q$ to denote some policy for which $p = \varphi_{i(T)}(q, P)$ (if there is more than one such policy, we select one arbitrarily).

The proof of this theorem requires the following two preliminary results.

**Lemma 9.1:** *Suppose $N_s \geq N^*$. Consider any decisive set $L \subseteq E$ with $s \in L$. Then there is a minimally decisive set $L' \subseteq L$ with $s \in L'$.*

**Proof:** Let $S \equiv |L|$. Let the function $\mu : \{1, ..., S\} \to L$ be such that, for all $l \in \{1, ..., S\}$, we have (i) for all $l, l' \in \{1, ..., S\}$, $\mu(l) \neq \mu(l')$, (ii) for all $l \in \{2, ..., S\}$, $N_{\mu(l)} \geq N_{\mu(l-1)}$, and (iii) if $N_{\mu(l)} = N_s$ for some $l$ with $\mu(l) \neq s$, then $l < \mu^{-1}(s)$. In other words, $\mu^{-1}(\cdot)$ indexes the elements of $L$ in order of increasing size (with $s$ placed after other groups of equal size). Since $L$ is decisive, we know that $\sum_{l=1}^{S} N_{\mu(l)} \geq M$. Moreover, $N_{\mu(S)} < M$ (otherwise we would have $N^* = N_{\mu(S)} \geq M$, which would violate A7). Accordingly, there exists some integer $z \in \{1, ..., S\}$ such that $\sum_{l=z}^{S} N_{\mu(l)} \geq M$ and $\sum_{l=z+1}^{S} N_{\mu(l)} < M$. Let $L' = \{\mu(z), \mu(z+1), ..., \mu(S)\}$. In light of (ii), $\sum_{l=z+1}^{S} N_{\mu(l)} < M$ implies that $\sum_{j \in L'} N_j - N_k < M$ for all $k \in L'$. Thus, $L'$ is minimally decisive. Let $L'' = \{l \in E \mid N_l > N^*\}$. In light of (iii) and the fact that $N_s \geq N^*$, we have $\sum_{l=\mu^{-1}(s)+1}^{S} N_{\mu(l)} \leq \sum_{j \in L''} N_j < M$. Thus, $z \leq \mu^{-1}(s)$, which implies $s \in L'$. Q.E.D.

**Lemma 9.2:** *There exists $p' \in Z(q, P)$ such that $r(i(T)) \in p'$ and $p' \subseteq L$ for some $L \in \Lambda$.*

**Proof:** There are two cases to consider.

**Case 1:** $r(i(T)) \in q$.

If $q$ is not decisive, take $p' = \{r(i(T))\}$. Notice that $p'$ is preferred to $q$ (weakly if $q = p'$) for the decisive set $E \backslash q$, so $p' \in Z(q, P)$. Moreover, we know that $\{r(i(T))\} \subseteq L$ for some $L \in \Lambda$ (this follows directly from lemma 9.1, since one can always find a decisive set containing $r(i(T))$).

If $q$ is minimally decisive, simply take $p' = q$.

If $q$ is decisive but not minimally decisive, then, by lemma 9.1, there exists some minimally decisive set $p' \subseteq q$ such that $r(i(T)) \in p'$. Since $\sum_{j \in p'} c_j < \sum_{j \in q} c_j$, all $j \in p'$ strictly prefer $p'$ to $q$, so $p' \in Z(q, P)$.

**Case 2:** $r(i(T)) \notin q$.

First suppose that $\sum_{j \in q} N_j \geq M$. By deleting elements of $q$ starting with those associated with the smallest groups and moving to the largest, one can find some $q' \subset q$ such that $M > \sum_{j \in q'} N_j \geq M - N^*$. Construct $q'' = (q \backslash q') \cup \{r(i(T))\}$. By A7, $\sum_{j \in q''} c_j < \sum_{j \in q} c_j$. Thus, all individuals in any group $l \in E \backslash q'$ strictly prefer

$q''$ to $q$. By construction, $E\backslash q'$ is decisive. By lemma 9.1, we know there exists some $L \subseteq E\backslash q'$ with $r(i(T)) \in L$ and $L \in \Lambda$. Let $p' = L \cap q''$. Individuals within groups belonging to $L$ (weakly) prefer $p'$ to $q''$, and strictly prefer $q''$ to $q$. Thus, $p' = Z(q, P)$, as required.

Now imagine that $\sum_{j \in q} N_j < M$. Note that the set $E\backslash q$ is decisive. Thus, by lemma 9.1, there is a minimally decisive set $p' \subseteq E\backslash q$ with $r(i(T)) \in p'$. By A8, all individuals belonging to groups within $p'$ strictly prefer $p'$ to $q$. Thus, $p' \in Z(q, P)$. Q.E.D.

Now we return to the proof of the theorem. Much as in the proof of theorem 3, we proceed by establishing three properties of $\Phi_{i(T)}(P)$.

**Property 1:** $\{r(i(T))\} \in \Phi_{i(T)}(P)$.

Since $\{r(i(T))\}$ is the favorite policy of individual $i(T)$, we have $\{r(i(T))\} = \varphi_{i(T)}(\{r(i(T))\}, P)$.

**Property 2:** $p \in \Phi_{i(T)}(P)$ implies $p \subseteq L$ for some $L \in \Lambda$.

Assume not. Let $\Psi$ denote the set of groups that (weakly) prefer $p$ to $q$. Clearly, $\Psi$ is decisive, and $\Psi$ contains $r(i(T))$. From lemma 9.1, we know that there exists some $L \in \Lambda$ with $L \subseteq \Psi$ and $r(\{i(T)\}) \in L$. Consider the policy $p' = p \cap L$. Since it is not the case that $p \subseteq L$, we have $p' \subset p$, so $\sum_{l \in p'} c_l < \sum_{l \in p} c_l$. Consequently, all members of any group $j \in p'$ (including $r(i(T))$) strictly prefer $p'$ to $p$. Since these individuals also (weakly) prefer $p$ to $q$, they strictly prefer $p'$ to $q$, which implies $p' \in Z(q, P)$. But since $i(T)$ strictly prefers $p'$ to $p$, this contradicts $p \in \varphi_{i(T)}(q, P)$.

**Property 3:** $p \in \Phi_{i(T)}(P)$ implies $r(i(T)) \in p$.

Suppose that the property is false, i.e. that there exists $p \in \Phi_{i(T)}(P)$ with $r(i(T)) \notin p$. Then $v_{i(T)}(p) \leq 0$. Lemma 9.2 establishes the existence of some $p' \in Z(q, P)$ such that $r(i(T)) \in p'$ and $p' \subseteq L$ for some $L \in \Lambda$. Clearly, $v_{i(T)}(p') \geq v_{i(T)}(L) > 0$, where the second inequality follows from A8. Thus, $p \notin \varphi_{i(T)}(q, P)$, which is a contradiction.

Now we prove the theorem. In particular, we claim that $\{r(i(T))\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$. By property 1, we know that $\{r(i(T))\} \in \Phi_{i(T)}(P)$. Consider any other $p \in \Phi_{i(T)}(P)$. By property 3, $\sum_{j \in p} c_j > c_{r(i(T))}$. Thus, all individuals within groups $j \in E\backslash p$ strictly prefer $\{r(i(T))\}$ to $p$, as do individuals within $r(i(T))$. Note that the set of individuals who strictly prefer $\{r(i(T))\}$ to $p$ can be written as $E\backslash (p\backslash\{r(i(T))\})$. Since, by property 2, $p \subseteq L$ for some $L \in \Lambda$, we know that $p\backslash\{r(i(T))\}$ is not decisive. But then $E\backslash (p\backslash\{r(i(T))\})$ is decisive, as required.

Since $\{r(i(T))\}$ is a Condorcet winner in $\Phi_{i(T)}(P)$, we know (from the argument used in the proof of Theorem 2) that, for all $p \in Q_1(P)$, we have $v_i(p) \geq v_i(\{r(i(T))\})$ for all $i \in J'' \equiv \{j \mid j = i(t) \text{ for some } t = 1, ..., T-1\}$, and hence for all $i \in \cup_{j \in J''} I_{r(j)}$. If $J'' \cap I_{r(i(T))}$ is non-empty, then the fact that $v_i(p) \geq v_i(\{r(i(T))\})$ for $i \in J'' \cap I_{r(i(T))}$ requires $p = \{r(i(T))\}$. If $|J'| \geq M + \left|I_{r(i(T))}\right|$, then $\left|\cup_{j \in J''} I_{r(j)}\right| \geq M$, in which case the fact that $v_i(p) \geq v_i(\{r(i(T))\})$ for all $i \in \cup_{j \in J''} I_{r(j)}$ implies $p = \{r(i(T))\}$ (otherwise $\{r(i(T))\}$ would not be a Condorcet winner). Q.E.D.

**Proof of Theorem 10:** The theorem is trivial when $\varepsilon \geq 1/N$. Consequently, assume $\varepsilon < 1/N$. We begin with two lemmas.

**Lemma 10.1:** *Consider any $p_{T-1} \in \Delta_\varepsilon^{N-1}$. Under either of the following conditions, there exists some set of individuals $S$ with $i(T) \notin S$ and $|S| = M - 1$ such that $(Q_T(p_{T-1}))_l = 0$ if and only if $l \in S$.*

(i) $Q_T(p_{T-1}) \neq p_{T-1}$,

(ii) $(Q_T(p_{T-1}))_{i(T)} < 1 - N\varepsilon$.

**Proof:** (i) Since $Q_T(p_{T-1}) \neq p_{T-1}$, there exists a set $S''$ with $i(T) \notin S''$ and $|S''| = M - 1$ such that $(Q_T(p_{T-1}))_l > (p_{T-1})_l \geq 0$ for $l \in S''$. We claim that $(Q_T(p_{T-1}))_l = 0$ for all $l \in E \backslash (S'' \cup \{i(T)\})$. Suppose not. Consider $p'$ constructed as follows: $p'_l = 0$ for $l \in E \backslash (S'' \cup \{i(T)\})$, $p'_l = (Q_T(p_{T-1}))_l$ for $l \in S''$, and $p'_{i(T)} = (Q_T(p_{T-1}))_{i(T)} + \sum_{i \in E \backslash S''} (Q_T(p_{T-1}))_i$ (in other words, divert all surplus from members of $E \backslash (S'' \cup \{i(T)\})$ to $i(T)$). Plainly, $p'_l > (p_{T-1})_l$ for $l \in S'' \cup \{i(T)\}$. Since $|S'' \cup \{i(T)\}| = M$, the policy $p'$ would pass if proposed in round $T$; since $p'_{i(T)} > (Q_T(p_{T-1}))_{i(T)}$, individual $i(T)$ would therefore have an incentive to propose it. But this contradicts the hypothesis that $Q_T(p_{T-1})$ is $i(T)$'s optimal proposal. Since $(Q_T(p_{T-1}))_l = 0$ for $l \in E \backslash (S'' \cup \{i(T)\})$ and $|E \backslash (S'' \cup \{i(T)\})| = M - 1$, we must have $(Q_T(p_{T-1}))_{i(T)} > (p_{T-1})_{i(T)} \geq 0$ (or $Q_T(p_{T-1})$ would not pass). Taking $S = E \backslash (S'' \cup \{i(T)\})$ delivers the desired conclusion.

(ii) In light of part (i), we prove this by showing that $Q_T(p_{T-1}) \neq p_{T-1}$. Suppose on the contrary that $Q_T(p_{T-1}) = p_{T-1}$. Since $(p_{T-1})_{i(T)} < 1 - N\varepsilon$, we have $\sum_{j \neq i(T)} (p_{T-1})_j \geq (N+1)\varepsilon$, which implies the existence of a set $S'$ with $i(T) \notin S'$

and $|S'| = \frac{N-1}{2} = M-1$ such that $\sum_{j\in S'} (p_{T-1})_j \geq \left(\frac{N+1}{2}\right)\varepsilon = M\varepsilon$. It is therefore possible to construct a policy $p'$ with the property that $p'_j \geq (p_{T-1})_j + \varepsilon$ for all $j \notin S'$. All members of $E\backslash S'$, including $i(T)$, strictly prefer $p'$ to $p_{T-1}$. Since $|E\backslash S'| = M$, the policy $p'$ would pass if proposed in round $T$, and $i(T)$ would therefore have an incentive to propose it. But this contradicts the hypothesis that $Q_T(p_{T-1}) = p_{T-1}$. Q.E.D.

**Lemma 10.2:** *Consider any set $A$ with $i(T) \notin A$ and $|A| = M-1$. Define the policy $x^A$ as follows: $x^A_{i(T)} = 1-(M-1)\varepsilon$; for $l \in A$, $x^A_l = \varepsilon$; for $l \notin A\cup\{i(T)\}$, $x^A_l = 0$. Then $Q_t(x^A) = x^A$ for all $t = 1, ..., T$.*

**Proof:** First consider the case of $t = T$. If $Q_T(x^A) \neq x^A$, then there must be $M$ individuals with $\left(Q_T(x^A)\right)_l > x^A_l$. But this can only be the case if $\left(Q_T(x^A)\right)_{i(T)} < x^A_{i(T)}$, which implies that $i(T)$ would not propose $Q_T(x^A)$, a contradiction.

Now suppose that $Q_{t+1}(x^A) = x^A$. Imagine that, contrary to the claim, $Q_t(x^A) \neq x^A$. Then there must be $M$ individuals with $\left(Q_t(x^A)\right)_l > \left(Q_{t+1}(x^A)\right)_l = x^A_l$. Once again, this can only be the case if $\left(Q_t(x^A)\right)_{i(T)} < x^A_{i(T)}$. But then we know that there is some set $B$ with $|B| = M$ and $i(T) \notin B$ such that $\left(Q_t(x^A)\right)_l > \left(Q_{t+1}(x^A)\right)_l = x^A_l$. Since $|A| = M-1$, there is at least on individual, $j^*$, in both $A$ and $B$.

Consider some equilibrium from round $t$ onward, given $p_{t-1} = x^A$, for which the outcome is $Q_t(x^A)$. Let $p^A_{T-1}$ denote the status quo at the outset of round $T$ on the equilibrium path. Plainly, $Q_T(p^A_{T-1}) = Q_t(x^A)$. There are two possibilities.

(i) $p^A_{T-1} = Q_T(p^A_{T-1})$. Consider any set $D = B\backslash\{j\}$ for some $j \in B\backslash\{j^*\}$. Since $\left(Q_t(x^A)\right)_l \geq \varepsilon$ for $l \in D\backslash\{j^*\}$ and $\left(Q_t(x^A)\right)_{j^*} \geq 2\varepsilon$, we have $\sum_{l\in D}\left(Q_t(x^A)\right)_l \geq M\epsilon$. Consequently, we can construct a policy $p'$ with $p'_l = \left(Q_t(x^A)\right)_l + \varepsilon$ for $l \in E\backslash D$. Since $p^A_{T-1} = Q_t(x^A)$ and $|E\backslash D| = M$, if $p'$ is proposed in round $T$, it will pass. Since $i(T)$ prefers $p'$ to $Q_t(x^A)$, $i(T)$ will propose $p'$, which contradicts the hypothesis that the outcome is $Q_t(x^A)$.

(ii) $p^A_{T-1} \neq Q_T(p^A_{T-1})$. Then, by lemma 10.1 part (i), there exists a set $S$ with $i(T) \notin S$ and $|S| = M-1$ such that $(Q_T(p_{T-1}))_l = 0$ for all $l \in S$. But this contradicts the existence of the set $B$. Q.E.D.

Now suppose, contrary to the theorem, that there exists $p_0$ such that $(Q_1(p_0))_{i(T)} < 1 - N\varepsilon$. Consider some equilibrium resulting in the outcome $Q_1(p_0)$, and suppose that, for each $t = 1, .., T$, the policy $\widehat{p}_{t-1}$ is the status quo at the outset of each round

$t$ on the equilibrium path (by construction $\widehat{p}_0 = p_0$). Plainly, $Q_t(\widehat{p}_{t-1}) = Q_1(p_0)$.

We claim that, for all $t \in 1, ..., T-1$, we have $(Q_{t+1}(\widehat{p}_{t-1}))_{i(T)} < 1 - N\varepsilon$. Suppose on the contrary that there is some $t'$ for which $(Q_{t'+1}(\widehat{p}_{t'-1}))_{i(T)} \geq 1 - N\varepsilon$. Then we know that, in equilibrium, $\widehat{p}_{t'} \neq \widehat{p}_{t'-1}$ is proposed in round $t'$ and it passes (so that the outcome is $Q_{t'+1}(\widehat{p}_{t'})$ rather than $Q_{t'+1}(\widehat{p}_{t'-1})$). But, by lemma 10.1 part (ii), there exists a set of individuals $S$ with $i(T) \notin S$ and $|S| = M - 1$ such that $(Q_{t'+1}(\widehat{p}_{t'}))_l = (Q_T(\widehat{p}_{T-1}))_l = 0$ for all $l \in S$. Moreover, $(Q_{t'+1}(\widehat{p}_{t'}))_{i(T)} < (Q_{t'+1}(\widehat{p}_{t'-1}))_{i(T)}$. Thus, under the assumption that those who are indifferent vote against proposals, at least $M$ individuals (those in $S \cup \{i(T)\}$) vote against $\widehat{p}_{t'}$, which contradicts the hypothesis that $\widehat{p}_{t'}$ passes.

Since $(Q_{t+1}(\widehat{p}_{t-1}))_{i(T)} < 1 - N\varepsilon$, we know from lemma 10.1 part (ii) that, for every round $t$, there exists some set of individuals $S(t)$ with $i(T) \notin S(t)$ and $|S(t)| = M - 1$ such that $(Q_{t+1}(\widehat{p}_{t-1}))_l = 0$ if and only if $l \in S(t)$. Imagine that $i(t)$ proposes $x^{S(t)}$ (as described in lemma 10.2, with $A = S(t)$). If the proposal passes, we know by lemma 10.2 that the outcome is $x^{S(t)}$. Since $x_l^{S(t)} > (Q_{t+1}(\widehat{p}_{t-1}))_l$ for all $l \in S(t)$ as well as for $l = i(T)$ (given that $(Q_{t+1}(\widehat{p}_{t-1}))_{i(T)} < 1 - N\varepsilon$), we know that the proposal passes (given that $|S(t) \cup \{i(T)\}| = M$). Consider the case where $i(T)$ proposes more than once. In any round $t < T$ for which $i(t) = i(T)$, individual $i(T)$ would gain by deviating to the proposal $x^{S(t)}$, contradicting the assumption of equilibrium. Now consider the case where $|J| > M$. We know from lemma 10.1 part (ii) that there is some $t'' \in \{1, ..., T-1\}$ such that $(Q_{t''+1}(\widehat{p}_{t''}))_{i(t'')} = 0$, which implies $(Q_{t''+1}(\widehat{p}_{t''-1}))_{i(t'')} = 0$ (otherwise $i(t'')$ would propose $\widehat{p}_{t''-1}$), and hence $i(t'') \in S(t'')$. But then individual $i(t'')$ would gain by deviating to the proposal $x^{S(t'')}$, contradicting the assumption of equilibrium. Q.E.D.

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