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A GENUINE RANK-DEPENDENT GENERALIZATION OF THE VON NEUMANN-MORGENSTERN EXPECTED UTILITY THEOREM

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This paper uses "revealed probability trade-offs" to provide a natural foundation for probability weighting in the famous von Neumann and Morgenstern axiomatic set-up for expected utility. In particular, it shows that a rank-dependent preference functional is obtained in this set-up when the independence axiom is weakened to stochastic dominance and a probability trade-off consistency condition. In contrast with the existing axiomatizations of rank-dependent utility, the resulting axioms allow for complete flexibility regarding the outcome space. Consequently, a parameter-free test/elicitation of rank-dependent utility becomes possible. The probability-oriented approach of this paper also provides theoretical foundations for probabilistic attitudes towards risk. It is shown that the preference conditions that characterize the shape of the probability weighting function can be derived from simple probability trade-off conditions.

KEYWORDS: Probability weighting, expected utility, rank-dependent utility, probability trade-offs, probabilistic risk attitude.

1. INTRODUCTION AND MOTIVATION

SOON AFTER THE PUBLICATION of the fundamental reference of expected utility theory by von Neumann and Morgenstern (1944), its descriptive validity was challenged by experimental observations suggesting that individuals use decision weights instead of probabilities (Preston and Baratta (1948)). Various descriptive models of individual decision under risk that incorporate the transformation of single-outcome probabilities have been proposed in the 1970's (Handa (1977), Karmarkar (1978), Kahneman and Tversky (1979)). Subsequently, Quiggin (1982) avoided the violations of stochastic dominance implied by the single-outcome probability transformation approach, by transforming cumulative probabilities. The transformation of cumulative probabilities into decision weights underlies both rank-dependent utility and Tversky and Kahneman's cumulative prospect theory (Tversky and Kahneman (1992)). Indeed, there have been many axiomatic derivations of rank-dependent utility (Wakker (1994) describes their main characteristics).

All the axiomatic approaches of rank-dependent utility (RDU) have assumed richness of the outcome space. For instance, it is a continuum in Quiggin (1982),

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Segal (1989, 1990), Chew (1989), Chateauneuf (1999), Wakker (1994), and the references therein, and a solvable space in Nakamura (1995). Consequently, the resulting rank-dependent generalizations of expected utility (EU) cannot be applied to the simple and elegant von Neumann-Morgenstern axiomatic set-up which does not impose any restriction on the outcome set. In this sense, it is desirable to provide a rank-dependent generalization of EU that preserves the von Neumann-Morgenstern setting.

This paper shows that if the independence axiom is replaced by a condition called probability trade-off consistency and stochastic dominance, we obtain a rank-dependent generalization of the von Neumann-Morgenstern EU theorem. What is done here is very similar to what Gilboa (1987) did to obtain Choquet expected utility in the Savagean axiomatic set-up. The first contribution of the paper therefore consists in providing a genuine generalization of the von Neumann-Morgenstern EU theorem with complete flexibility regarding the outcome space. This flexibility results in transparent axioms for RDU and, hence, the possibility of a parameter-free testing of this theory. It straightforwardly extends the application of RDU to domains with general outcomes that need not constitute continuums (health states, cars, houses, etc.).

The second contribution of the paper is that its techniques are immediately directed towards the nonlinear processing of probabilities, i.e., the new dimension of risk attitude that RDU adds to expected utility. The approaches of Wakker (1994) and others were primarily directed towards outcomes and nonlinear probabilities were only derived indirectly.[2] The techniques used in this paper immediately lead to the derivation of the nonlinear probability weighting function and to natural characterizations of its basic properties. These techniques also allow for quantitative measurements of probability weighting.

The paper is organized as follows. Section 2 presents the basic definitions and notation used in the paper. Section 3 presents the idea of probability trade-offs, with Subsection 3.1 introducing probability trade-offs in the probability triangle through the Allais Paradox and Subsection 3.2 generalizing this concept to lotteries having more than three outcomes. The main proposition establishes a straightforward link between probability trade-offs and probability weighting.

Section 4 shows that, under the usual conditions of a continuous weak order and stochastic dominance, a probability trade-off consistency condition is necessary and sufficient to obtain a rank-dependent extension of the von Neumann-Morgenstern EU theorem. Subsection 4.1 provides the intuition behind probability trade-off consistency and formulates the condition in terms of the preference relation; it also shows that this condition is necessary for RDU. Then, it is shown that when the independence axiom is replaced by stochastic dominance and probability trade-off consistency, we obtain a genuine rank-dependent generalization of the von Neumann-Morgenstern EU theorem

[2] The approach of Yaari (1987) is also directed towards outcomes (and needs a continuum of outcomes), but it is less general than the other approaches because it does not derive utility from preferences.

(Theorem 9, Proposition 10). Subsection 4.2 gives some experimental evidence regarding probability trade-off consistency.

Section 5 uses the probability-oriented approach of the paper to analyze probabilistic risk attitude and the shape of the probability weighting function. First, it is shown how, under RDU, the probability weighting functions of different decision makers can be compared independently of their utility functions (Theorem 12, Corollary 14). Then, some general preference conditions concerning the shape of the probability weighting function are presented and the inverse S-shaped probability weighting function is formalized in terms of probability trade-offs.

2. PRELIMINARIES

Let $\mathbb{P}^s$ denote the set of *simple* probability measures (i.e., probability measures with a finite support), called *lotteries*, over an arbitrary set of outcomes $\mathscr{C}$ that may be finite or infinite. By $\succcurlyeq$ we denote the preference relation of a decision maker on $\mathbb{P}^s$, with $\sim, \succ$ defined as usual. $\succcurlyeq$ is a *weak order* if it is complete and transitive. The preference relation $\succcurlyeq$ is naturally extended to the set of outcomes $\mathscr{C}$ through the degenerate lotteries in $\mathbb{P}^s$.

Let $X = \{x_1, \dots, x_n\} \subset \mathscr{C}$ be the generic symbol for a finite set of outcomes and $\mathbb{P}_X$ the set of probability distributions over X. Consequently, $\mathbb{P}^s$ is the union of all sets $\mathbb{P}_X$, i.e., $\mathbb{P}^s = \bigcup_{X \subset \mathscr{C}} \mathbb{P}_X$. Throughout this paper, the notation $\mathbb{P}_X$ assumes that $x_n \succcurlyeq \cdots \succcurlyeq x_1$, i.e., $X = \{x_1, \dots, x_n\}$ is a *rank-ordered* subset of $\mathscr{C}$. $\mathbb{P}_X$ may be written as a subset of the Cartesian product of unit intervals

$$(1) \qquad \mathbb{P}_X = \left\{(p_1, \dots, p_n) \in [0,1]^n : \sum_{i=1}^{n} p_i = 1\right\}.$$

It is a *mixture space*, i.e., $\forall P, Q \in \mathbb{P}_X, \forall \alpha \in [0,1] : \alpha P + (1-\alpha) Q \in \mathbb{P}_X$.

The binary relation $\succcurlyeq$ satisfies *stochastic dominance* on $\mathbb{P}^s$ if for all $P, Q \in \mathbb{P}^s, P \succ Q$ whenever $P \neq Q$ and $\forall x \in \mathscr{C}, P(\{y \in \mathscr{C} : y \succcurlyeq x\}) \geq Q(\{y \in \mathscr{C} : y \succcurlyeq x\})$. Following Fishburn (1970), the preference relation $\succcurlyeq$ is *Jensen-continuous* if for all lotteries $P, Q, R \in \mathbb{P}^s$, if $P \succ Q$ then there exist $\lambda, \mu \in (0,1)$ such that $\lambda P + (1-\lambda) R \succ Q$ and $P \succ \mu Q + (1-\mu) R$. The binary relation $\succcurlyeq$ satisfies *vNM-independence* if for all lotteries $P, Q, R \in \mathbb{P}^s$ and all $\lambda \in (0,1)$, $P \succcurlyeq Q$ if and only if $\lambda P + (1-\lambda) R \succcurlyeq \lambda Q + (1-\lambda) R$.

For this paper it is convenient to work with distribution functions instead of lotteries. Therefore, the set $\mathbb{P}_X$ is transformed into the following set, called a *rank-ordered Cartesian product*,

$$(2) \qquad \mathbb{P}_X^* = \left\{\left(p_2^*, \dots, p_n^*\right) \in [0,1]^{n-1} : p_2^* \geq \cdots \geq p_n^*\right\}$$

through a function $(\cdot)^*\colon \mathbb{P}_X \longrightarrow \mathbb{P}_X^*$ such that $(P)^* = P^* = \left(p_2^*, \dots, p_n^*\right)$, where $p_i^* = \sum_{j=i}^{n} p_j, i = 2, \dots, n$. In other words, $(\cdot)^*$ transforms each lottery P into a function P^* assigning to each x_i the probability of receiving x_i or any outcome rank-ordered above in X.[3] Like $\mathbb{P}_X, \mathbb{P}_X^*$ is a mixture space. Furthermore, $\mathbb{P}_X$ and

[3] If $x_n \succ \cdots \succ x_1$, then P^* is the decumulative distribution corresponding to P, assigning to each x_i the probability of receiving x_i or any better outcome in X.

$\mathbb{P}_X^*$ are isomorphic, i.e., $(\cdot)^*$ is one-to-one and $\forall P, Q \in \mathbb{P}^s, \forall \alpha \in [0,1] : (\alpha P + (1-\alpha)Q)^* = \alpha P^* + (1-\alpha)Q^*$. The preference relation $\succcurlyeq$ on $\mathbb{P}_X$ is naturally extended to the set $\mathbb{P}_X^*$ by means of the equivalence

$$(3) \qquad \forall P, Q \in \mathbb{P}_X, \quad P \succcurlyeq Q \Longleftrightarrow P^* \succcurlyeq Q^*.$$

This equivalence allows straightforward extension of properties of $\succcurlyeq$ on $\mathbb{P}_X$ such as weak ordering, Jensen-continuity, or vNM-independence to the corresponding rank-ordered Cartesian product $\mathbb{P}_X^*$. $\succcurlyeq$ satisfies *monotonicity* on $\mathbb{P}_X^*$, i.e., for all $P^*, Q^* \in \mathbb{P}_X^*$, $P^* \succ Q^*$ whenever $P^* \neq Q^*$ and $p_i^* \geq q_i^*$ for all $i \in \{2, \ldots, n\}$, if and only if it satisfies stochastic dominance on $\mathbb{P}_X$.

A function V: $\mathbb{P}^s \longrightarrow \mathbb{R}$ *represents* $\succcurlyeq$ if $P \succcurlyeq Q \Longleftrightarrow V(P) \geq V(Q)$ for all P, $Q \in \mathbb{P}^s$. The function corresponding to RDU is denoted by V_{RDU}.

DEFINITION 1: *Rank-dependent utility* (RDU) holds if there exists a strictly increasing continuous probability weighting function w: $[0,1] \to [0,1]$ with $w(0) = 0$ and $w(1) = 1$ and a strictly increasing utility function u: $\mathscr{C} \to \mathbb{R}$ such that $\succcurlyeq$ is represented by V_{RDU} as follows:

$$(4) \qquad \forall P \in \mathbb{P}_X, \quad V_{RDU}(P) = u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})] w\big(p_i^*\big),$$

for all finite sets $X \subset \mathscr{C}$.

3. DERIVED TRADE-OFFS FOR PROBABILITIES

This section adapts the idea of derived trade-offs for outcomes used in Wakker (1994) to the dual context where outcomes are given and probabilities are variable. The objective of this adaptation is to obtain a direct derivation of probability weighting from preferences. More specifically, the preference relation $\succcurlyeq$ on $\mathbb{P}_X^*$ is used to compare the impact for the decision maker of replacing probability β by probability α with the impact of replacing probability δ by probability γ. Formally, this is done through the comparison of pairs of probabilities representing probability replacements. A replacement of probability β by probability α is called a *probability trade-off* and is denoted by $[\alpha; \beta]$. The following subsection resorts to the famous Allais example to introduce the idea of revealed probability trade-offs. These results are then generalized to general $\mathbb{P}_X^*$'s in Subsection 3.2.

3.1. *Probability Trade-offs in the Rank-ordered Probability Triangle*

Suppose that there exists a weak order $\succcurlyeq$ on $\mathbb{P}_{\{x_1, x_2, x_3\}}$ with $x_3 \succ x_2 \succ x_1$. This lottery domain can be represented by means of the probability triangle, i.e., the unit triangle in the $p_1 p_3$ space. Similar to $P \in \mathbb{P}_{\{x_1, x_2, x_3\}}$ being represented by the point (p_1, p_3) in the probability triangle, P^* can be represented by a pair of numbers, say p_2^* and p_3^*, in the rank-ordered probability triangle of summits (0, 0), (1, 0), and (1, 1) as shown in Figure 1. The rank-ordered probability triangle

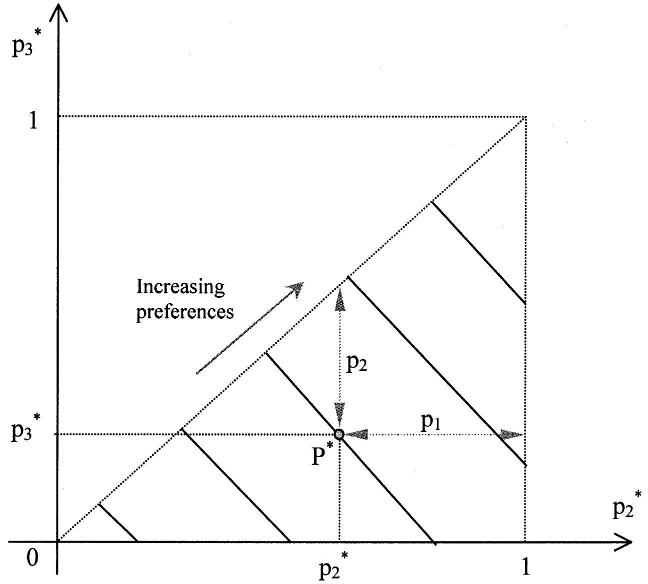

FIGURE 1.—Rank-ordered probability triangle.

(in the $p_2^*p_3^*$ space) is actually the mirror image of the probability triangle (in the p_1p_3 space). A lottery P stochastically dominates a lottery Q if the point $P^* = (p_2^*, p_3^*)$ is located northeast of the point $Q^* = (q_2^*, q_3^*)$. When EU holds, the indifference curves are parallel straight lines.

The intuition behind derived probability trade-offs can be demonstrated through the famous Allais Paradox. Table I gives the two choice problems proposed in the Allais example. By taking $x_1 = 0, x_2 = 1M, x_3 = 5M$, where M is one million dollars, lotteries P, Q, R, S can be represented by P^*, Q^*, R^*, S^* respectively in the rank-ordered probability triangle. In the Allais experience most subjects prefer P to Q and S to R and therefore violate EU. This inconsistency with the standard model of decision under risk may be seen as revealing the

TABLE I

ALLAIS PARADOX $(x_1 = 0, x_2 = 1M, x_3 = 5M)$[a]

	Alternatives in $\mathbb{P}_X$	Alternatives in $\mathbb{P}_X^*$
Problem 1	$P = (0, 1, 0)$	$P^* = (1, 0)$
	$Q = \left(\frac{1}{100}, \frac{89}{100}, \frac{10}{100}\right)$	$Q^* = \left(\frac{99}{100}, \frac{10}{100}\right)$
Problem 2	$R = \left(\frac{89}{100}, \frac{11}{100}, 0\right)$	$R^* = \left(\frac{11}{100}, 0\right)$
	$S = \left(\frac{90}{100}, 0, \frac{10}{100}\right)$	$S^* = \left(\frac{10}{100}, \frac{10}{100}\right)$

[a] M denotes \$1,000,000.

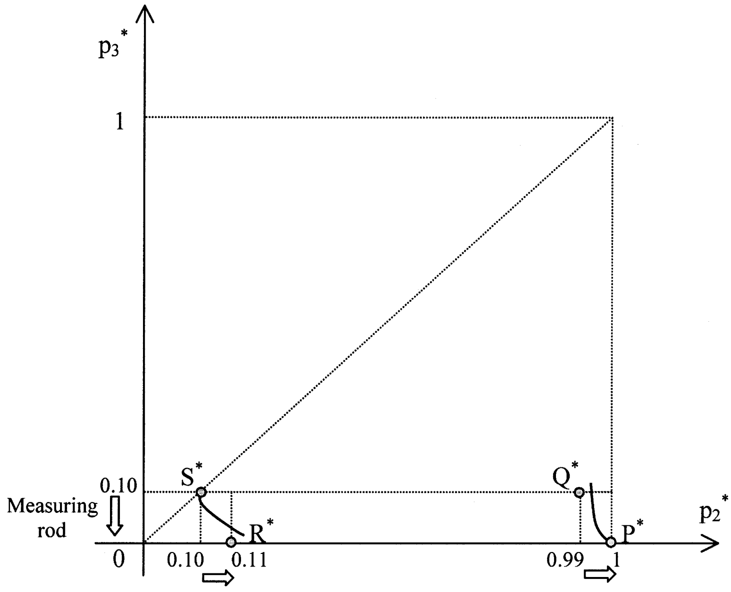

FIGURE 2.—Revealed probability trade-offs from the Allais paradox.

subjects' tendency to assign a less important subjective impact to the replacement of probability .10 by probability .11 than to the replacement of probability .99 by probability 1. In other words, the subjects seem to assign different impacts to the probability trade-offs [.11; .10] and [1; .99].

Using Figure 2, this claim can be elaborated as follows. A preference for P over Q implies that the advantage of the probability trade-off [1; .99] on the p_2^*-axis (receiving at least $1M$ with probability 1 is better than only with probability .99) outweighs the disadvantage of the probability trade-off [0; .10] on the p_3^*-axis (receiving at least $5M$ with probability 0 is worse than with probability .10). Similarly, a preference for S over R implies that the disadvantage of the probability trade-off [0; .10] on the p_3^* -axis outweighs the advantage of the probability trade-off [.11; .10] on the p_2^*-axis. Therefore, using the probability trade-off [0; .10] on the p_3^*-axis as a measuring rod, the choice pattern PS implies that the probability trade-off [1; .99] has more impact than the probability trade-off [.11; .10]. Subsection 3.2 (Proposition 4) shows that, under EU, the probability trade-offs [1; .99] and [.11; .10] must have the same impact (because of the linearity in the probabilities of the EU preference functional). Observing indifferences $P \sim Q$ and $R \sim S$ instead of the choice pattern PS reveals (through a similar reasoning) that the probability trade-offs [1; .99] and [.11; .10] have the same impact.

Under RDU, the choice pattern PS implies that $V_{RDU}(P) > V_{RDU}(Q)$ and $V_{RDU}(S) > V_{RDU}(R)$. Therefore Definition 1 (see Proposition 4) implies that $w(1) - w(.99) > w(.11) - w(.10)$. Similarly, the indifferences $P \sim Q$ and $R \sim S$ imply $w(1) - w(.99) = w(.11) - w(.10)$. This suggests a straightforward link

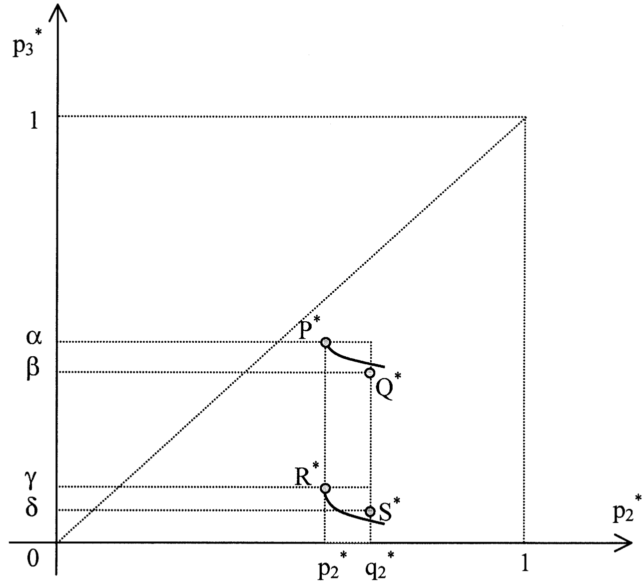

FIGURE 3.—Probability trade-offs (on the p_3^*-axis).

between probability trade-offs on the one hand and probability weighting on the other hand.

Similarly, we can compare probability trade-offs on the p_3^*-axis by using a measuring rod on the p_2^*-axis. Let P, Q, R, S be four lotteries satisfying $p_2^* = r_2^*$ and $q_2^* = s_2^*$ as shown in Figure 3. Suppose that P and S are preferred to Q and R respectively. The choice pattern PS implies that the advantage of the probability trade-off $[\alpha; \beta]$ outweighs the disadvantage of the probability trade-off $[p_2^*; q_2^*]$ on the p_2^*-axis which outweighs the advantage of the probability trade-off $[\gamma; \delta]$. Consequently, the probability trade-off $[\alpha; \beta]$ has more impact than the probability trade-off $[\gamma; \delta]$.

The idea of revealed probability trade-offs in the rank-ordered triangle consists of using a measuring rod on one axis to compare probability trade-offs on the other axis. Ideally, the ordering of revealed probability trade-offs over the unit interval should be independent of the measuring rod, the rank-ordered set $\{x_1, x_2, x_3\}$ and the axis used to obtain them. As we will see, this independence holds if RDU is assumed. The approach of this subsection is generalized to $\mathbb{P}_X^*$ for any finite rank-ordered set $X \subset \mathscr{C}$ in the following subsection.

3.2. *Probability Trade-offs on General $\mathbb{P}_X^*$'s*

Formally, obtaining revealed orderings of probability trade-offs from the preference relation $\succcurlyeq$ on $\mathbb{P}_X^*$ results in constructing *quarternary relations* $\succcurlyeq^t$, $\succ^t$ and $\backsim^t$, meaning "has equal or more impact than," "has more impact than," and "has the same impact as," respectively. These quarternary relations are derived from $\succcurlyeq$ and in that sense are not primitives. Wakker (1989, 1994) used similar relations to compare trade-offs for outcomes. In the following definitions, (λ, P^*_{-i}) is written for P^* with p^*_i replaced by λ. This notation implies that $p^*_{i-1} \geq \lambda \geq p^*_{i+1}$ for $i \geq 3$ and $\lambda \geq p^*_{i+1}$ for $i = 2$.

DEFINITION 2: For probabilities $\alpha, \beta, \gamma, \delta$ we write $[\alpha; \beta] \succcurlyeq^t [\gamma; \delta]$ if

$$\begin{cases} (\alpha, P^*_{-i}) \succcurlyeq (\beta, Q^*_{-i}) \\ (\gamma, P^*_{-i}) \preccurlyeq (\delta, Q^*_{-i}), \end{cases} \tag{5}$$

for some rank-ordered set $X = \{x_1, \ldots, x_n\}, i \in \{2, \ldots, n\}$ such that $x_i \succ x_{i-1}$ and $P^*, Q^* \in \mathbb{P}^*_X$.

Note that by holding p^*_k and q^*_k fixed for $k \neq i$, we implicitly use them as a measuring rod. The other quarternary relations $\succ^t$ and $\backsim^t$ are defined similarly.

DEFINITION 3: For probabilities $\alpha, \beta, \gamma, \delta$ we write

$[\alpha; \beta] \succ^t [\gamma; \delta]$ if we have $\prec$ instead of $\preccurlyeq$ in Definition 2;

$[\alpha; \beta] \sim^t [\gamma; \delta]$ if we have $\backsim$ instead of $\succcurlyeq$ and $\preccurlyeq$ in Definition 2.

The following proposition (and its elementary proof) further clarifies the meaning of the quarternary relations $\succcurlyeq^t, \succ^t, \sim^t$ and their link with probability weighting under RDU.

PROPOSITION 4: *Under RDU, we have*
(i) $[\alpha; \beta] \succcurlyeq^t [\gamma; \delta] \Longrightarrow w(\alpha) - w(\beta) \geq w(\gamma) - w(\delta)$,
(ii) $[\alpha; \beta] \succ^t [\gamma; \delta] \Longrightarrow w(\alpha) - w(\beta) > w(\gamma) - w(\delta)$,
(iii) $[\alpha; \beta] \sim^t [\gamma; \delta] \Longrightarrow w(\alpha) - w(\beta) = w(\gamma) - w(\delta)$.

The proof of implication (i) may clarify: suppose we have $(\alpha, P^*_{-i}) \succcurlyeq (\beta, Q^*_{-i})$ and $(\gamma, P^*_{-i}) \preccurlyeq (\delta, Q^*_{-i})$ for some $X = \{x_1, \ldots, x_n\}, i \in \{2, \ldots, n\}, x_i \succ x_{i-1}$ and P^*, Q^* in $\mathbb{P}^*_X$, i.e., $[\alpha; \beta] \succcurlyeq^t [\gamma; \delta]$. Substituting RDU for these preferences gives

$$A + [u(x_i) - u(x_{i-1})]w(\alpha) \geq B + [u(x_i) - u(x_{i-1})]w(\beta);$$

$$A + [u(x_i) - u(x_{i-1})]w(\gamma) \leq B + [u(x_i) - u(x_{i-1})]w(\delta),$$

where $A = \sum_{k \neq i}[u(x_k) - u(x_{k-1})]w(p^*_k)$ and $B = \sum_{k \neq i}[u(x_k) - u(x_{k-1})]w(q^*_k)$. Rewriting and rearranging these two inequalities implies $w(\alpha) - w(\beta) \geq w(\gamma) - w(\delta)$. The implications (ii) and (iii) are similarly obtained.

As a corollary, we obtain the following proposition. It is a mere consequence of the absence of probability weighting under EU.

PROPOSITION 5: *Under EU, we have*
(i) $[\alpha;\beta] \succcurlyeq^t [\gamma;\delta] \Longrightarrow \alpha-\beta \geq \gamma-\delta$,
(ii) $[\alpha;\beta] \succ^t [\gamma;\delta] \Longrightarrow \alpha-\beta > \gamma-\delta$,
(iii) $[\alpha;\beta] \backsim^t [\gamma;\delta] \Longrightarrow \alpha-\beta = \gamma-\delta$.

The following section describes the main necessary condition for the existence of a probability weighting function. It is shown that this condition rules out inconsistent probability trade-offs.

4. TRADE-OFF CONSISTENCY AND PROBABILITY WEIGHTING

4.1. *Probability Weighting Derived from Probability Trade-offs*

As noted in Subsection 3.1, derived trade-offs are useful when they are independent of the measuring rod, the rank-ordered X, and the axis used to obtain them. If this independence condition is not satisfied, inconsistent orderings of probability trade-offs such as $[\alpha;\beta] \succ^t [\gamma;\delta]$ and $[\gamma;\delta] \succcurlyeq^t [\alpha;\beta]$ can be obtained. In this subsection, trade-off consistency is used in a manner dual to Wakker (1989, 1994).

Figure 4 gives two examples in the rank-ordered probability triangle where the derived probability trade-offs depend on the measuring rod used. In the first example, the measuring rod m implies $[\alpha;\beta] \backsim^t [\gamma;\delta]$ whereas the measuring rod m' implies $[\alpha;\beta] \succ^t [\gamma;\delta]$. This illustrates within-axis contradictory orderings

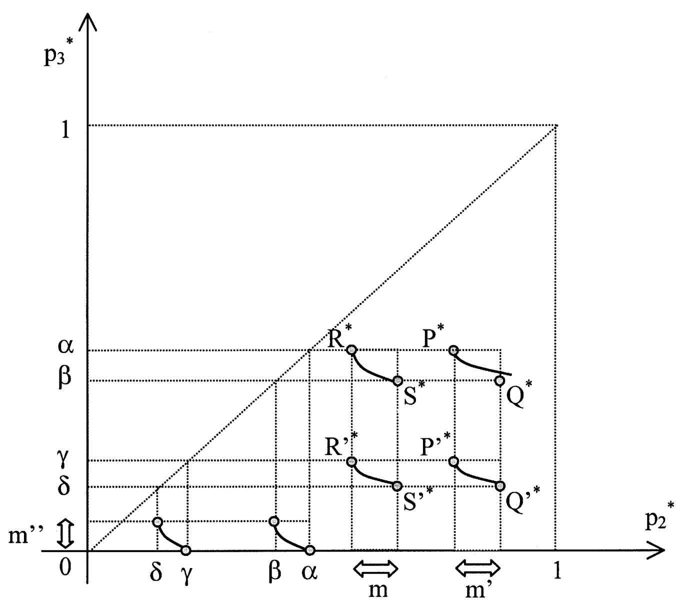

FIGURE 4.—Contradictory orderings of probability trade-offs.

of probability trade-offs. The second example illustrates across-axis contradictory orderings of probability trade-offs. It shows that the measuring rod m'' implies $[\alpha;\beta] \backsim^t [\gamma;\delta]$ whereas the measuring rod m' implies $[\alpha;\beta] \succ^t [\gamma;\delta]$.

The described inconsistencies contradict the existence of a probability weighting function w. Thus, a *probability trade-off consistency* condition is needed to avoid these kinds of contradictions.

DEFINITION 6: We say that $\succcurlyeq$ satisfies *probability trade-off consistency* if there do not exist probabilities $\alpha, \beta, \gamma, \delta$ such that both $[\alpha;\beta] \succcurlyeq^t [\gamma;\delta]$ and $[\gamma;\delta] \succ^t [\alpha;\beta]$.

Following Definitions 2 and 3, the ingredients X, i, and P^*, Q^* used to obtain $[\alpha;\beta] \succcurlyeq^t [\gamma;\delta]$ are not necessarily identical to those used to obtain $[\gamma;\delta] \succ^t [\alpha;\beta]$. This remark allows for an easy reformulation of probability trade-off consistency in terms of preferences.

LEMMA 7: *The binary relation $\succcurlyeq$ satisfies probability trade-off consistency if and only if, for all rank-ordered sets $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$, $i \in \{2, \ldots, n\}$ such that $x_i \succ x_{i-1}$, $j \in \{2, \ldots, m\}$ such that $y_j \succ y_{j-1}$, $P^*, Q^* \in \mathbb{P}_X^*$ and $R^*, S^* \in \mathbb{P}_Y^*$*

$$(6) \qquad \begin{cases} \begin{cases} (\alpha, P^*_{-i}) \preccurlyeq (\beta, Q^*_{-i}) \\ (\gamma, P^*_{-i}) \succcurlyeq (\delta, Q^*_{-i}) \end{cases} \\ (\alpha, R^*_{-j}) \succcurlyeq (\beta, S^*_{-j}) \end{cases}$$

$$(7) \qquad \Longrightarrow (\gamma, R^*_{-j}) \succcurlyeq (\delta, S^*_{-j}).$$

A violation of the implication (6) $\Longrightarrow$ (7), i.e., $\exists X, Y, i, j, P^*, Q^*$ in P_X^* and R^*, S^* in P_Y^* satisfying (6) and *not* (7), straightforwardly implies $[\gamma;\delta] \succcurlyeq^t [\alpha;\beta]$ and $[\alpha;\beta] \succ^t [\gamma;\delta]$ and thus results in contradictory orderings of probability trade-offs. The following proposition shows that probability trade-off consistency is satisfied when RDU holds. It is an immediate consequence of Definition 6 and Proposition 4.

PROPOSITION 8: *RDU implies probability trade-off consistency.*

Theorem 9 shows that, under usual conditions of stochastic dominance and Jensen-continuity of the weak order $\succcurlyeq$, probability trade-off consistency is not only necessary, but also sufficient, for RDU.

THEOREM 9: *Let $\succcurlyeq$ be a preference relation on $\mathbb{P}^s$. Then, RDU holds on $\mathbb{P}^s$ if and only if the following conditions are satisfied*:

A1. *Weak ordering*;

A2. *Stochastic dominance*;

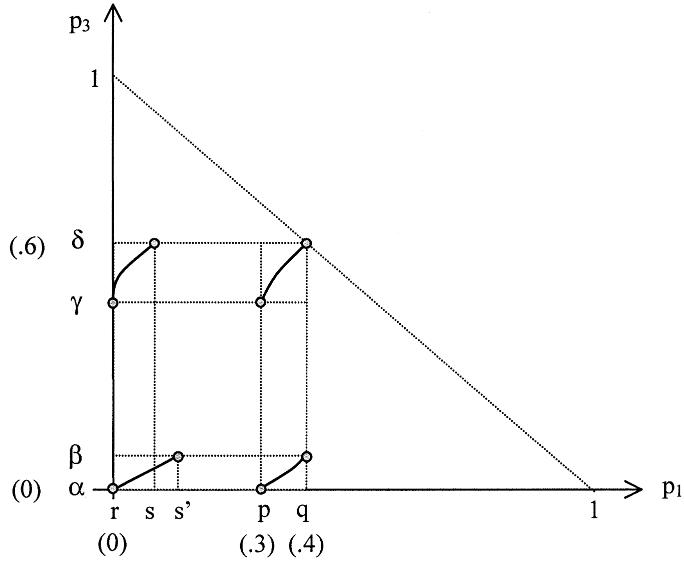

FIGURE 5.—Test of probability trade-off consistency.

A3. *Jensen-continuity*;
A4. *Probability trade-off consistency.*

The proof of Theorem 9 is given in Appendix A. The following proposition shows that the conditions in this theorem are immediate weakenings of the von Neumann-Morgenstern axioms. In particular, vNM-independence implies probability trade-off consistency.

PROPOSITION 10: *Let* $\succcurlyeq$ *be a vNM-independent weak order on* $\mathbb{P}^s$. *Then*
(i) $\succcurlyeq$ *satisfies stochastic dominance.*
(ii) $[\alpha; \beta] \succcurlyeq^t [\gamma; \delta] \Longrightarrow \alpha - \beta \geq \gamma - \delta$.
(iii) $[\alpha; \beta] \succ^t [\gamma; \delta] \Longrightarrow \alpha - \beta > \gamma - \delta$.
(iv) $[\alpha; \beta] \backsim^t [\gamma; \delta] \Longrightarrow \alpha - \beta = \gamma - \delta$.
(v) $\succcurlyeq$ *satisfies probability trade-off consistency.*

The proof of Proposition 10 is given in Appendix B. This proposition implies (in combination with Theorem 9) that if $\succcurlyeq$ is a preference relation on $\mathbb{P}^s$, then EU holds on $\mathbb{P}^s$ if and only if $\succcurlyeq$ is a Jensen-continuous weak order satisfying vNM-independence. Therefore, Theorem 9 is a genuine rank-dependent generalization of the von Neumann-Morgenstern EU theorem.

TABLE II

	Consistent with	
Cases	EU	RDU
$s \neq s'$	no	no
$s = s'$ and $\beta - \alpha = \delta - \gamma$	yes	yes
$s = s'$ and $\beta - \alpha \neq \delta - \gamma$	no	yes

4.2. *Some Empirical Evidence*

When the richness of the set of outcomes is used to derive probability weighting (e.g., Wakker (1994); Chateauneuf (1999)), no parameter-free test of RDU is possible in the probability triangle, the domain commonly used to test generalizations of EU theory. It can be shown that a nontrivial test of the comonotonic independence axiom[4] (a necessary condition for RDU) requires at least four distinct outcomes (see Wakker, Erev, and Weber (1994, Section 6, Observation 3)). By considering trade-offs between probabilities, this paper makes it possible to test RDU in the probability triangle.

Abdellaoui and Munier (1998b) tested probability trade-off consistency as given in Lemma 7 with the restrictions $i = j$ and $\backsim$ instead of $\succcurlyeq$.[5] This test needed the construction of four indifferences in the unit triangle:

(8) $$(p, \alpha) \backsim (q, \beta), (p, \gamma) \backsim (q, \delta),$$

(9) $$(r, \alpha) \backsim (s', \beta), (r, \gamma) \backsim (s, \delta),$$

where β, γ, s, s' were assessed by means of a bisection choice procedure. Probabilities p, q, r, α, and δ were fixed beforehand.[6]

Under RDU, indifferences (8) imply $w(\beta) - w(\alpha) = w(\delta) - w(\gamma)$. Indifferences (9) then allow us to check if the decision maker's probability trade-offs are consistent. If $s \neq s'$, the revealed probability trade-offs are contradictory. Table II presents the three possible cases.

The hypothesis $\beta - \alpha = \delta - \gamma$ was clearly rejected in favor of $\beta - \alpha < \delta - \gamma$ (paired t test, $t = 5.14$ and $p < .001$) for a sample of 36 subjects. This means that a probability increment from 0 has more impact than an equal probability increment away from 0. Furthermore, a paired t test showed that the (consistency) hypothesis $s = s'$ is not rejected by the data ($t = .54$, $p = .6$). Consequently, these results are consistent with RDU. The results also corroborate the idea that the boundary effects, observed in many experiments that use the probability triangle

[4] This axiom asserts that preference between lotteries will be unaffected by substitution of common consequences as long as these substitutions have no effect on the rank order of the outcomes in either lottery.

[5] Another experiment testing probability trade-off consistency through strict preferences and without the restriction $i = j$ is in preparation.

[6] This is a test of within-axis probability trade-off consistency. The same basic principle can be used to test across-axis probability trade-off consistency.

as a domain of choice, can be accommodated by probability weighting (see, e.g., Wu and Gonzalez (1996), Abdellaoui and Munier (1998a)).

5. PROBABILISTIC RISK ATTITUDE AND THE SHAPE OF THE PROBABILITY WEIGHTING FUNCTION

Probability weighting in RDU introduces a new component of the attitude towards risk: the *probabilistic risk attitude*. This poses the problem of isolating the effect of the curvature of the probability weighting function on the individual attitude towards risk from the effect induced by the curvature of the utility function. Chew, Karni, and Safra (1987) used the condition that a mean-preserving increase in risk is never favorable, to characterize the joint concavity of the utility function and convexity of the probability weighting function. Subsequently, Wakker (1994) proposed a utility-oriented approach that filters out the effect of the utility function to isolate the probabilistic risk attitude by means of midpoint-outcomes.[7] The probability-oriented axiomatic set-up proposed in this paper allows for a quantitative measurement of the probability weighting function, in addition to the qualitative testing of its shape.

The probability-oriented approach in this paper makes it possible to compare the attitudes towards probabilistic risk of two decision makers even if they have different utility functions. In other words, the comparison of probability trade-offs automatically filters out the effect of the utility function.

Consider two lotteries $P=(1-p,0,p)$ and $Q=(1-q,0,q)$ in a probability triangle. We want to see how for some probabilistic mixture $\lambda P+(1-\lambda)Q$, the first decision maker compares her increased probabilistic risk with the midpoint of the RDU values of the lotteries, i.e., $[V_{RDU}(P)+V_{RDU}(Q)]/2$, and how the second decision maker does this. Under RDU and with $u(x_1)=0$ and $u(x_3)=1$, the comparison of $V_{RDU}(\lambda P+(1-\lambda)Q)$ with $[V_{RDU}(P)+V_{RDU}(Q)]/2$ turns out to be equivalent to the comparison of $w(\lambda p+(1-\lambda)q)$ with $[w(p)+w(q)]/2$. This gives the following definition.

DEFINITION 11: An individual with a preference relation $\succcurlyeq_2$ is *more averse* to probabilistic risk than another individual with a preference relation $\succcurlyeq_1$ if for all probabilities α, β, γ satisfying $\alpha > \beta > \gamma$,

$$\begin{cases} [\alpha;\beta] \succcurlyeq_1^t [\beta;\gamma] \\ [\alpha;\beta] \prec_2^t [\beta;\gamma] \end{cases} \tag{10}$$

is excluded. We say that $\succcurlyeq_2$ is *more prone* to probabilistic risk if in (10) we interchange the roles of $\succcurlyeq_1$ and $\succcurlyeq_2$. $\succcurlyeq_2$ is *equally averse* to probabilistic risk if it is both more averse and more prone to probabilistic risk.

[7] A midpoint outcome between outcomes x and y is an outcome z such that $u(z)=(u(x)+u(y))/2$.

For EU, the classical results concerning comparisons of risk aversion have been established by Pratt (1964) and Arrow (1965) in terms of the shape of the utility function. For RDU, the next theorem states the relations more averse and more prone to probabilistic risk in terms of the shape of the probability weighting function. Therefore, it may be considered as the analogue of Pratt-Arrow for probability weighting.

THEOREM 12: *Suppose that RDU holds for* $\succcurlyeq_i, w_i, u_i, i = 1, 2$. *Then the following two statements are equivalent*:

(i) $w_2 = \theta \circ w_1$ *for a continuous, convex* (*respectively concave*), *strictly increasing* θ: $[0, 1] \to [0, 1]$;

(ii) $\succcurlyeq_2$ *is more averse* (*respectively prone*) *to probabilistic risk than* $\succcurlyeq_1$.

The proof of Theorem 12 is given in Appendix B. As a consequence of this theorem, probabilistic risk aversion, proneness, and neutrality can be defined as follows.

DEFINITION 13: $\succcurlyeq$ exhibits *probabilistic risk aversion* if $[\alpha; \beta] \succ^t [\alpha + \varepsilon; \beta + \varepsilon]$ is excluded for all probabilities considered with $\alpha > \beta, \varepsilon > 0$.

$\succcurlyeq$ exhibits *probabilistic risk proneness* if $[\alpha + \varepsilon; \beta + \varepsilon] \succ^t [\alpha; \beta]$ is excluded for all probabilities as above.

$\succcurlyeq$ exhibits *probabilistic risk neutrality* if it exhibits both risk aversion and risk proneness.

The following corollary (of Propositions 4 and 5) explicitly establishes that probabilistic risk aversion (respectively proneness) corresponds to convexity (respectively concavity) of w, and that probabilistic risk neutrality corresponds to the linearity of w.

COROLLARY 14: *Under RDU,* w *is convex* (*concave*, *linear*) *if and only if* $\succcurlyeq$ *exhibits probabilistic risk aversion* (*proneness*, *both aversion and proneness*).

Under RDU, violations of EU such as the common consequence effect or the common ratio effect (e.g., Allais (1953), Kahneman and Tversky (1979)) reflect violations of probabilistic risk neutrality. Furthermore, there is ample evidence that the weighting function in RDU is inverse S-shaped, concave over the first third of the unit interval and convex elsewhere (Tversky and Kahneman (1992); Camerer and Ho (1994); Wu and Gonzalez (1996); Gonzalez ansd Wu (1999); Abdellaoui (2000); Bleichrodt and Pinto (2000)). Preference conditions providing information about probability weighting without resorting to parametric specifications of the RDU model were proposed by Segal (1987), Tversky and Wakker (1995), Wu and Gonzalez (1996, 1998), Wakker (2001).

The definition of the quarternary relation $\succ^t$ allows us to translate the exclusion of some probability trade-offs into an implication in terms of $\succcurlyeq$. This is stipulated by the following lemma.

LEMMA 15: *Suppose that* $\succcurlyeq$ *is a weak order. Then, for all lotteries* P, Q, *for all probabilities* α, β *such that* $\alpha > \beta$, *and all positive and feasible* ε*'s*:

(i) *if* $[\alpha; \beta] \succ^t [\alpha+\varepsilon; \beta+\varepsilon]$ *is excluded, then* $\forall i \in \{2, \ldots, n\}$,

$$(\alpha, P^*_{-i}) \backsim (\beta, Q^*_{-i}) \Longrightarrow (\alpha+\varepsilon, P^*_{-i}) \succcurlyeq (\beta+\varepsilon, Q^*_{-i}); \tag{11}$$

(ii) *if* $[\alpha+\varepsilon; \beta+\varepsilon] \succ^t [\alpha; \beta]$ *is excluded, then* $\forall i \in \{2, \ldots, n\}$,

$$(\alpha, P^*_{-i}) \backsim (\beta, Q^*_{-i}) \Longrightarrow (\alpha+\varepsilon, P^*_{-i}) \preccurlyeq (\beta+\varepsilon, Q^*_{-i}). \tag{12}$$

PROOF: Immediate. *Q.E.D.*

Such implications give us (direct) preference conditions concerning the shape of the probability weighting function similar to those proposed by Tversky and Wakker (1995) and Wu and Gonzalez (1998).

The following theorem gives local conditions for the shape of the probability weighting function.[8] Its proof is given in Appendix B.

THEOREM 16: *Under RDU, the following two propositions are equivalent*:

(i) w *is concave* (*resp. convex*) *in the range* $]\underline{p}, \bar{p}[$;

(ii) *for all* $\alpha, \beta, \varepsilon$ *such that* $\bar{p} > \alpha+\varepsilon > \alpha > \beta \geq \underline{p}$, $[\alpha+\varepsilon; \beta+\varepsilon] \succ^t [\alpha; \beta]$ (*resp.* $[\alpha; \beta] \succ^t [\alpha+\varepsilon; \beta+\varepsilon]$) *is excluded.*

Tversky and Wakker (1995) propose two concepts to characterize inverse S-shaped probability weighting functions: *lower subadditivity* and *upper subadditivity*. Lower subadditivity means that a lower (probability) interval $[0, \beta]$ has more impact than a middle interval $[\alpha, \alpha+\beta]$. Upper subadditivity means that an upper interval $[1-\beta, 1]$ has more impact than a middle interval $[\alpha, \alpha+\beta]$. These ideas can be expressed in terms of probability trade-offs as follows.

DEFINITION 17: For constants $\varepsilon \geq 0$ and $\varepsilon' \geq 0$, the probability weighting function satisfies *subadditivity* (lower and upper) with respect to $\varepsilon, \varepsilon'$ if

$$(\alpha+\beta \leq 1-\varepsilon) \Longrightarrow \text{not } ([\alpha+\beta; \beta] \succcurlyeq^t [\beta; 0]) \quad \text{and} \tag{13}$$

$$(\alpha \geq \varepsilon') \Longrightarrow \text{not } ([\alpha+\beta; \beta] \succcurlyeq^t [1; 1-\beta]). \tag{14}$$

Implications (13) and (14) correspond to lower subadditivity and upper subadditivity respectively. They are straightforward statements of the preference conditions proposed by Tversky and Wakker (1995, Section 4).

[8] Results restricted to subdomains were also provided by Prelec (1998) and Wakker (2001).

6. CONCLUSION

This paper has introduced a new tool for analyzing probability weighting through derived probability trade-offs. It proposes a parsimonious axiomatization of RDU that preserves the standard setting of von Neumann-Morgenstern. Moreover, it provides a simple theoretical foundation that characterizes the prominent features of probability weighting studied in the literature. Derived probability trade-offs also simplify experimental investigations of probability weighting.

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APPENDIX A: PROOF OF THEOREM 9

PRELIMINARIES: Let $\mathscr{A}$ be a Cartesian product $\prod_{i=1}^{m}[0,1]$ and $\mathscr{B}$ a subset of $\mathscr{A}$. A function $V\colon \mathscr{B} \longrightarrow \mathbb{R}$ representing $\succcurlyeq$ on $\mathscr{B}$ is *additive* if $\forall r=(r_1,\dots,r_m)\in\mathscr{B}, V(r)=\sum_{i=1}^{m}V_i(r_i)$ for some functions $V_1,\dots,V_m$ to $\mathbb{R}$. If an additive function represents $\succcurlyeq$, then the V_i's are *additive value functions*. A function V is *cardinal* if it is unique up to a positive affine transformation. Additive value functions are *jointly cardinal* if they can be replaced by $W_1,\dots,W_m$ if and only if there exist real numbers $\beta_1,\dots,\beta_m$, and a positive $\alpha>0$, such that $\forall i=1,\dots,m: W_i=\beta_i+\alpha V_i$.

Note that formula (4) in Definition 1 may be rewritten as follows

$$\forall P\in\mathbb{P}_X, \qquad V_{RDU}(P)=u(x_1)+V(P^*)$$

where $V=\sum_{i=2}^{n}V_i$ is an additive representing function on $\mathbb{P}_X^*$. Here, $V_{RDU}(\cdot)$ is a positive affine transformation of $V(\cdot)$.

Euclidean continuity is used to derive an additive representation of $\succcurlyeq$ on $\mathbb{P}_X^*$. The following lemma gives sufficient conditions for Euclidean continuity on $\mathbb{P}_X^*$.

LEMMA 18: *Let $\succcurlyeq$ be a Jensen-continuous weak order satisfying stochastic dominance on $\mathbb{P}^s$. Then $\succcurlyeq$ satisfies Euclidean continuity on $\mathbb{P}_X^*$ for all finite sets $X\subset\mathscr{C}$.*

PROOF: Recall that $\succcurlyeq$ on $\mathbb{P}_X$ is extended to $\mathbb{P}_X^*$ through equivalence (3), that $(\cdot)^*$ is one-to-one and that stochastic dominance on $\mathbb{P}_X$ implies monotonicity on $\mathbb{P}_X^*$. In order to show that $\succcurlyeq$ satisfies Euclidean continuity on $\mathbb{P}_X^*$, it has to be proved that both sets

$$\{P^*\in\mathbb{P}_X^*: P^*\succ Q^*\} \quad\text{and}\quad \{P^*\in\mathbb{P}_X^*: Q^*\succ P^*\}$$

are open in $\mathbb{P}_X^*$ for each $Q^*\in\mathbb{P}_X^*$.

Let us prove that $A=\{P^*\in\mathbb{P}_X^*: Q^*\succ P^*\}$ is open; the proof for the other set is similar. Take any $P^*\in A$, hence $Q^*\succ P^*$. Let $\overline{P}$ be the best lottery in $\mathbb{P}_X$, i.e., $\overline{P}^*=(1,\dots,1)$ (x_n is the preferred outcome in X). $\overline{P}^*\succcurlyeq R^*$ for all $R^*\in\mathbb{P}_X^*$ by monotonicity. Because of Jensen-continuity, there exists $\lambda\in(0,1)$ such that $Q^*\succ\lambda\overline{P}^*+(1-\lambda)P^*$. Applying monotonicity again, we have for each $\lambda'\in[0,\lambda)$

$$Q^*\succ\lambda\overline{P}^*+(1-\lambda)P^*\succcurlyeq\lambda'\overline{P}^*+(1-\lambda')P^*\succcurlyeq\lambda' R^*+(1-\lambda')P^*.$$

Hence, if $P^* \in A$, then there exists an open neighborhood

$$\{S^* \in \mathbb{P}_X^* : S^* = \lambda' R^* + (1-\lambda')P^* \text{ for some } \lambda' \in [0, \lambda) \text{ and } R^* \in \mathbb{P}_X^*\}$$

of P^* in A. This implies that $\{P^* \in \mathbb{P}_X^* : Q^* \succ P^*\}$ is an open subset of $\mathbb{P}_X^*$. *Q.E.D.*

PROOF OF THEOREM 9: When RDU holds, the derivation of the preference conditions A1–A4 is rather straightforward.

Next suppose that A1–A4 hold. It must be shown that RDU holds. The proof is broken into two parts.

Part I: This part of the proof uses the generalized results on additive representation theory as developed in Wakker (1993, Theorem 3.2 and Proposition 3.5) and other more specific results from Wakker (1989).

The preference conditions A1–A4 and Lemma 18 imply that the binary relation $\succcurlyeq$ on $\mathbb{P}_X^*$ is a continuous monotonic weak order that satisfies generalized triple cancellation, i.e., the implication (6) $\Longrightarrow$ (7) in Lemma 7 with $i = j$. $\mathbb{P}_X^*$ is rank-ordered (i.e., $p_2^* \geq \cdots \geq p_n^*$) through $\geq$. Continuity of $\geq$ on [0, 1] is evident.

Suppose that the set X does not contain indifferent outcomes. By Theorem 3.2 in Wakker (1993), there exists an extended additive representation on $\mathbb{P}_X^*$ for arbitrary, finite ($n \geq 3$) and rank-ordered sets $X \subset \mathscr{C}$. Consequently, $\succcurlyeq$ is represented on $\mathbb{P}_X^*$ by

$$V(p_2^*, \ldots, p_n^*) = V_2(p_2^*) + \cdots + V_n(p_n^*)$$

where the V_i's are jointly cardinal and possibly V_2 assigns the value $-\infty$ to probability 0 and/or V_n assigns the value ∞ to probability 1. All other V_i values are real. It can be shown that probability trade-off consistency implies that the different V_i's locally order differences the same way when they are finite and hence can be taken proportional to each other (Wakker (1989, Lemma VI.8.2)). It follows that V_2 and V_n are well behaved in the upper and lower bound of the unit interval respectively by proportionality (Wakker (1993, Proposition 3.5)). Hence, we get the following new representation:

$$V(p_2^*, \ldots, p_n^*) = \mu_2 w(p_2^*) + \cdots + \mu_n w(p_n^*)$$

where w is an increasing and continuous function defined on [0,1] that is unique up to a positive affine transformation and the μ_i's are positive and sum up to one.

Suppose that X may contain indifferent elements. In this case we get a representation as before, however with $\mu_i = V_i = 0$ for each i with $x_i \backsim x_{i-1}$ because such a coordinate i does not affect preference (see Wakker (1989, Lemma VI.8.2)).

Part II: We define the utility function $X = \{x_1, \ldots, x_n\}$ by

$$\begin{cases} u(x_1) = 0 \\ u(x_i) = \sum_{k=2}^{i} \mu_k. \end{cases}$$

Therefore u represents $\succcurlyeq$ on X and we finally get

$$\text{(15)} \qquad \forall P^* \in \mathbb{P}_X^*, \quad V(P^*) = u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})] w(p_i^*).$$

By setting $w(0) = 0$ and $w(1) = 1$, w becomes a probability weighting function. Consequently, formula (15) gives a RDU representation of $\succcurlyeq$ on $\mathbb{P}_X^*$.

If u is replaced by a positive affine transformation, the resulting preference functional is still consistent with a RDU representation of $\succcurlyeq$ on $\mathbb{P}_X$ (Definition 1).

Conversely, let

$$\forall P^* \in \mathbb{P}_X^*, \quad \widehat{V}(P^*) = \hat{u}(x_1) + \sum_{i=2}^{n} [\hat{u}(x_i) - \hat{u}(x_{i-1})] \hat{w}(p_i^*)$$

be another RDU representation of $\succcurlyeq$ on $\mathbb{P}_X$. Then $[\hat{u}(x_2)-\hat{u}(x_1)]\hat{w},\ldots,[\hat{u}(x_n)-\hat{u}(x_{n-1})]\hat{w}$ are additive value functions for $\succcurlyeq$ on $\mathbb{P}_X^*$ too. By joint cardinality $b_2,\ldots,b_n$ and $a>0$ exist such that

$$(16)\qquad [\hat{u}(x_i)-\hat{u}(x_{i-1})]\hat{w}=a[u(x_i)-u(x_{i-1})]w+b_i \quad \forall i\in\{2,\ldots n\}.$$

This means that for each i, $\hat{w}$ is a positive affine transformation of w. Because $w(0)=\hat{w}(0)=0$ and $w(1)=\hat{w}(1)=1$, $\hat{w}$ and w must be identical and $b_i=0\ \forall i\in\{2,\ldots,n\}$. Consequently, formula (16) becomes

$$\hat{u}(x_i)-\hat{u}(x_{i-1})=a[u(x_i)-u(x_{i-1})],$$

showing that $\hat{u}$ is a positive affine transformation of u. The uniqueness of the utility function up to a positive affine transformation is established.

The RDU formula (15) might however suggest that the u and w values depend on the chosen rank-ordered set $X=\{x_1,\ldots,x_n\}$. To show that this is not true, consider three outcomes c, c', and c'' in $\mathscr{C}$ such that $c''\succ c'\succ c$. Consider also two finite rank-ordered sets $Y,Z\subset\mathscr{C}$ containing c, c', and c''. The RDU representations of $\succcurlyeq$ on $\mathbb{P}_Y$ and $\mathbb{P}_Z$ imply the existence of probability weighting functions w_Y and w_Z defined on the unit interval and functions u_Y and u_Z defined on Y and Z respectively. Suppose that $u_Y(c)=u_Z(c)=0$ and $u_Y(c'')=u_Z(c'')=1$. Now consider the rank-ordered set corresponding to $Y\cap Z\subset\mathscr{C}$. This set contains at least the rank-ordered set $\{c,c',c''\}$. Because of uniqueness, the two RDU models coincide on $\mathbb{P}_{Y\cap Z}$. Consequently, we can define a utility function u on $\mathscr{C}$ as follows. For $x\in\mathscr{C}$, take any X containing x and define $u(x)=u_X(x)$. Therefore the probability weighting function and the utility function are independent of Y and Z. Note that every pair of lotteries in $\mathbb{P}^s$ can be considered as contained in a set $\mathbb{P}_X$ with X containing c, c' and c''. Therefore, every preference is now represented by the RDU as constructed. In short, the desired RDU representation and uniqueness results have been obtained on $\mathbb{P}^s$. *Q.E.D.*

APPENDIX B

PROOF OF PROPOSITION 10: (i) This is well known.

(ii) Suppose we have $(\alpha,P^*_{-i})\succcurlyeq(\beta,Q^*_{-i})$ and $(\gamma,P^*_{-i})\preccurlyeq(\delta,Q^*_{-i})$, i.e., $[\alpha;\beta]\succcurlyeq^t[\gamma;\delta]$. By vNM-independence

$$\frac{1}{2}(\alpha,P^*_{-i})+\frac{1}{2}(\delta,Q^*_{-i})\succcurlyeq\frac{1}{2}(\beta,Q^*_{-i})+\frac{1}{2}(\delta,Q^*_{-i}),$$
$$\frac{1}{2}(\beta,Q^*_{-i})+\frac{1}{2}(\delta,Q^*_{-i})\succcurlyeq\frac{1}{2}(\beta,Q^*_{-i})+\frac{1}{2}(\gamma,P^*_{-i}).$$

Transitivity of $\succcurlyeq$ implies

$$\frac{1}{2}(\alpha,P^*_{-i})+\frac{1}{2}(\delta,Q^*_{-i})\succcurlyeq\frac{1}{2}(\beta,Q^*_{-i})+\frac{1}{2}(\gamma,P^*_{-i}).$$

Because of monotonicity, we finally get $\alpha+\delta\geq\beta+\gamma$ and therefore $\alpha-\beta\geq\gamma-\delta$.

The proof is similar for implications (iii) and (iv).

(v) follows from (ii) and (iii). *Q.E.D.*

PROOF OF THEOREM 12: Suppose that (ii) is false. This implies that (10) is not excluded. Suppose that (10) holds. Then with $\beta=\lambda\alpha+(1-\lambda)\gamma$ and $\lambda\in]0,1[$ we have

$$\begin{cases} w_1(\lambda\alpha+(1-\lambda)\gamma)\leq[w_1(\alpha)+w_1(\gamma)]/2,\\ w_2(\lambda\alpha+(1-\lambda)\gamma)>[w_2(\alpha)+w_2(\gamma)]/2.\end{cases}$$

This precludes that $w_2=\theta\circ w_1$ for a convex and increasing θ on $w_1([0,1])$. Consequently, a violation of (ii) implies a violation of (i), which establishes the implication (i)$\Longrightarrow$(ii). The proof of the implication (ii)$\Longrightarrow$(i) is given in Wakker (1994, Appendix 1, pp. 32–33). *Q.E.D.*

PROOF OF THEOREM 16: The implication (i)$\Longrightarrow$(ii) is evident. Suppose now that for all $\alpha, \beta, \varepsilon$ such that $\bar{p} > \alpha+\varepsilon > \alpha > \beta > \underline{p}$, $[\alpha+\varepsilon; \beta+\varepsilon] \succ^{\iota} [\alpha; \beta]$ is excluded. Suppose for an arbitrary i that $(\alpha, P^*_{-i}) \backsim (\beta, Q^*_{-i})$. Then by Lemma 15 (page 731), we must also have $(\alpha+\varepsilon, P^*_{-i}) \preccurlyeq (\beta+\varepsilon, Q^*_{-i})$. The preference functional $V_{RDU}(\cdot)$ straightforwardly implies the inequality $w(\alpha+\varepsilon) - w(\beta+\varepsilon) \leq w(\alpha) - w(\beta)$, which means that w is concave on the range $]\underline{p}, \bar{p}[$ (see Wu and Gonzalez (1998, Appendices A and B)).

The proof for convexity is similar. *Q.E.D.*

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