EC9AA Term 3: Lectures on Economic Inequality

Debraj Ray, University of Warwick, Summer 2024

- Slides 4: Measuring Upward Mobility

Introduction

Mobility centrally important in current debates:

- In the United States and Europe

  Chetty et al (2017), Alesina et al (2018), Manduca et al (2020)

- Connection to growth, inequality, aspirations etc.

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- The concept refers to:

- the ease of transition between various social categories;

- income, wealth, location, political persuasions …

What the Term Might Mean

- **Non-Directional:**
  - Pure movement: off-diagonals in transition matrix. Atkinson (1981), Bartholomew (1982), Conlisk (1974), Dardanoni (1993), Hart (1976), Prais (1955), Shorrocks (1978a,b) …

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+ all combinations of these ...

A Large But Still Incomplete List

Name	Measure	Directional	Non-directional	Absolute	Relative
King (1983)	$M_\kappa = 1 - \exp\left[-\frac{\gamma}{n}\sum\frac{\lvert z_i - y_i\rvert}{\mu_y}\right]$		✓		✓
Shorrocks index (1978)	$M_S = \frac{n-\text{Tr}(P)}{n-1}$		✓		✓
Variability of the eigenvalues	$\sigma(\gamma_i)$		✓		✓
Bartholomew (1982)	$M_B = \frac{1}{n-1}\sum_i\sum_j \pi_i p_{ij} \mid i-j \mid$		✓		✓
IG Income Elasticity (IGE)	$\beta = \frac{\text{Cov}(S_{it}, S_{it-1})}{\text{Var}(S_{it-1})}$		✓	✓	
Correlation coefficient (CE)	$\rho_S = \frac{\text{Cov}(S_{it}, S_{it-1})}{\sqrt{\text{Var}(S_{it})}\sqrt{\text{Var}(S_{it-1})}}$		✓	✓	
Slope rank-rank	$\rho_{PR} = \text{Corr}(P_i, R_i)$		✓		✓
IG rank association (IRA)	$\beta = \frac{\text{Cov}(p^y_{it}, p^X_{it})}{\text{Var}(p^X_{it})}$		✓		✓
Mitra & Ok (1998)	$\text{MO}_\alpha(\mathbf{x},\mathbf{y}) = \frac{1}{n}\gamma\left(\sum_i \lvert y_i - x_i\rvert^\alpha\right)^{1/\alpha}$		✓	✓	
Gini symmetric index of mobility	$GS = \frac{\sum_i (y_i - x_i)(F_{xi} - F_{yi})}{\sum_i (y_i - 1)F_{yi} + \sum_i (x_i - 1)F_{xi}}$		✓	✓	
Great Gatsby curve	Corr(Gini, IGE)		✓	✓	
Bhattacharya (2011)	$\nu = Pr(F_1(Y_1) - F_0(Y_0) > \tau \mid s_1 \le F_0(Y_0) \le s_2, X = x)$	✓			✓
Absolute upward mobility (1)	$p_{25} = \mathbb{E}(Y \mid X \le 25)$	✓			✓
Absolute upward mobility (2)	$A = \Phi\left(\frac{\mu_c - \mu_p}{\sqrt{\sigma_p^2 + \sigma_c^2 + 2\rho\sigma_p\sigma_c}}\right)$	✓			✓
Chetty et al (2017)	$\text{AM}(\mathbf{x},\mathbf{y}) = \frac{1}{n}\sum_i (1_{y_i] \ge x_i})$	✓		✓	
Rising up-up	$P_{20to100} = \mathbb{E}[Y = 100 \mid X = 20]$	✓			✓
Bottom half mobility	$\mu_0^{50} = \mathbb{E}(y \mid x \in [0, 50])$	✓			✓
Fields & Ok (1999)	$\text{FO}(\mathbf{x},\mathbf{y}) = \frac{1}{n}\sum_i (\ln(y_i) - \ln(x_i))$	✓		✓	
Card (2018)	$\mathbb{E}(y > 50 \mid x \in [45, 70])$	✓		✓	
Pro-poor growth	$G = \sum_{k=1}^5 w_k g_k$	✓		✓	

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Data demands

- Existing measures rely heavily on panel data (more discussion later).
- This has held back empirical work, especially on developing countries.

Upward Mobility

- We propose a measure of **upward mobility** that is:
  - Directional: rewards growth and punishes decline;
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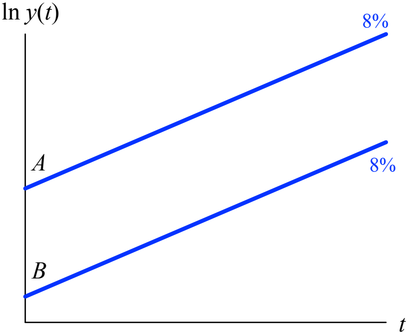

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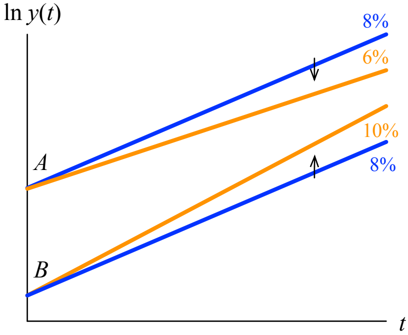

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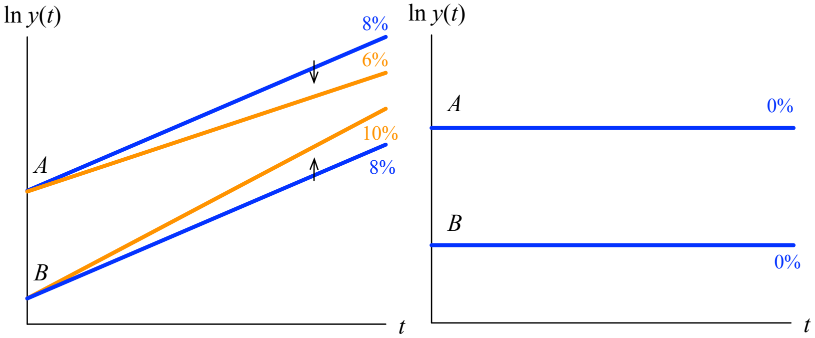

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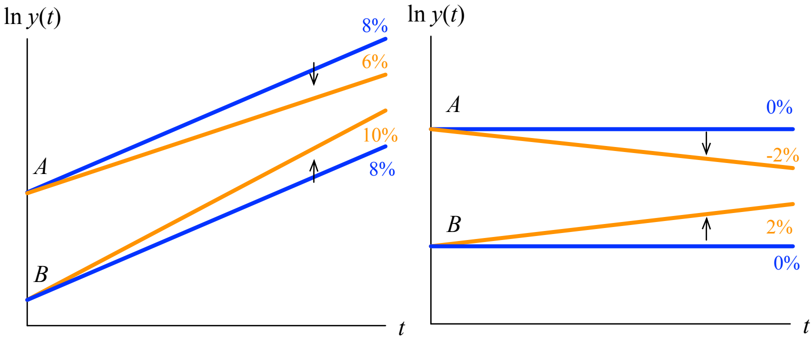

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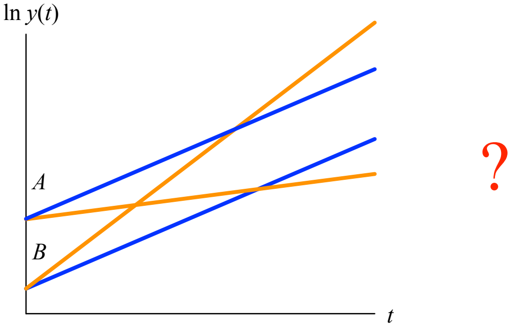

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  - An "instantaneous" measure or $\underbrace{\textbf{upward mobility kernel}}_{\textbf{intermediate step}}$ that is:
    - directional and progressive.
  - A $\underbrace{\textbf{mobility measure on trajectories}}_{\textbf{what we're after}}$ that is:
    - based on the collection of instantaneous kernels.

Instantaneous Upward Mobility

- **Central variable**: y, "income."
  - state variable for individual well-being.
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  - state variable for individual well-being.
  - e.g., "permanent income" or a proxy, such as consumption
- **Data**: For each person:
  - $y_i > 0$ baseline income
  - $g_i = \dot{y}_i / y_i$ instantaneous growth rate.
  - $\mathbf{z} =$ the full collection $\{z_i\}_{i=1}^n$, where $z_i = (y_i, g_i)$.

Instantaneous Upward Mobility

- **Upward mobility kernel**: $M(\mathbf{z})$, where $\mathbf{z} = \{z_i\}_{i=1}^{n}$, and $z_i = (y_i, g_i)$.
  - Anonymous, continuous.

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  - Anonymous, continuous.
  - Zero-growth normalization:

    $g_i = 0$ all $i \mapsto M(\mathbf{z}) = 0$.
  - Consistency under population mergers.

Details

Core Axiom

- **Examples:**

- $\mathbf{y} = (5000, 10000)$ + $\mathbf{g} = (8\%, 8\%) \succ \mathbf{y} = (5000, 10000)$ + $\mathbf{g} = (6\%, 10\%)$.
- $\mathbf{y} = (5000, 10000)$ + $\mathbf{g} = (2\%, -2\%) \succ \mathbf{y} = (5000, 10000)$ + $\mathbf{g} = (0\%, 0\%)$.
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  - No crossings in continuous time.
- **Growth Progressivity.**
  - For any $\mathbf{z}$, i and j with $y_i < y_j$, and $\epsilon > 0$, send g_i to $g_i + \epsilon$ and g_j to $g_j - \epsilon$.
  - Then $M(\mathbf{z}') > M(\mathbf{z})$.

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- **Notes:**
  - Measure tolerates lower growth if poor can grow faster.
  - Upward mobility $\neq$ overall welfare.

Upward Mobility Kernel

Theorem 1

An upward mobility kernel is growth progressive if and only if it can be written as

$$M(\mathbf{z}) = \sum_{i=1}^{n} \phi_i(\mathbf{y}) g_i$$

for continuous permutation-invariant $\{\phi_i\}$, with $\phi_i(\mathbf{y}) > \phi_j(\mathbf{y})$ when $y_i < y_j$.

Proof Outline

Sharpening the Kernel

- **Income Neutrality.** $M(\mathbf{y}, \mathbf{g}) = M(\lambda \mathbf{y}, \mathbf{g})$ for all $\lambda > 0$.

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- **Independent Pairwise Growth Tradeoffs:**

  Is $M((y_i, g_i), (y_j, g_j), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij})) \geq M((y_i, g_i'), (y_j, g_j'), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}))$?

  Answer insensitive to $(\mathbf{y}_{-ij}, \mathbf{g}_{-ij})$.

Sharpening the Kernel

Theorem 2

Under additional three axioms and $n \geq 3$, M can be written as:

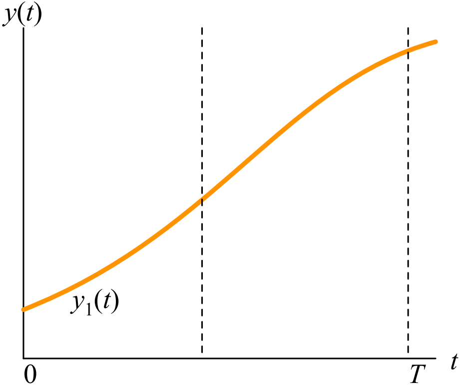
$$M_\alpha(\mathbf{z}) = \frac{\sum_{i=1}^n y_i^{-\alpha} g_i}{\sum_{i=1}^n y_i^{-\alpha}}, \text{ for some } \alpha > 0.$$

- Proof employs a substantial extension of Gorman's separability theorem;

  see Chatterjee ⓡ Ray ⓡ Sen (2021).

Income Trajectories

Towards a measure on trajectories:



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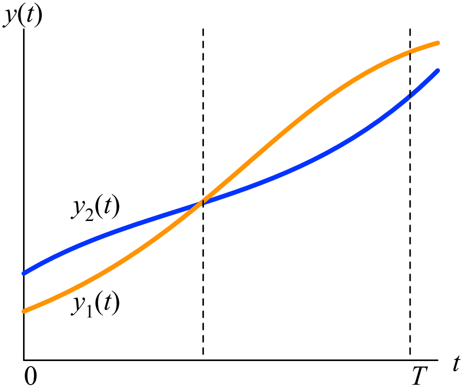

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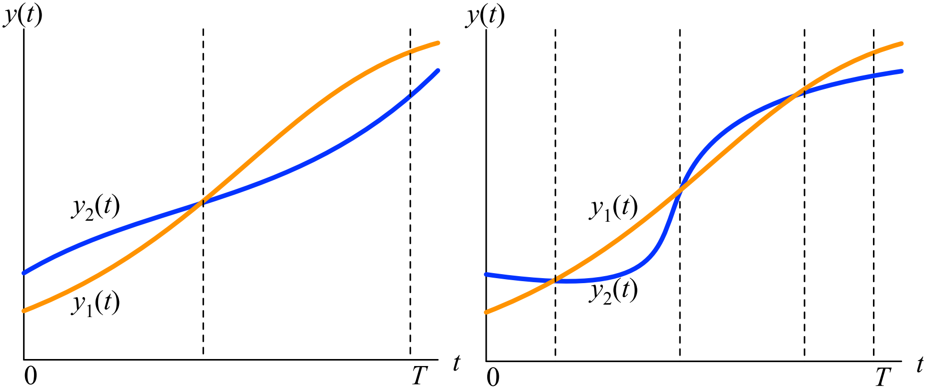

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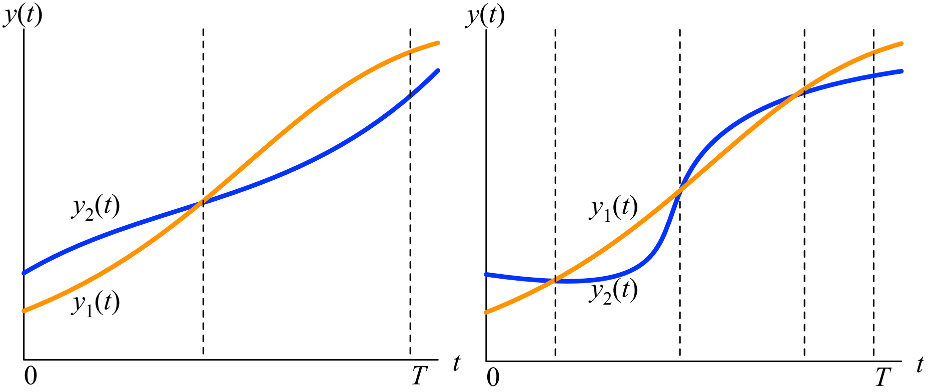

- $\mathbf{y}[s,t] = \{y_i(\tau)_s^t\}_{i=1}^n$
- **Upward mobility measure**: $\mu(\mathbf{y}[s,t])$.

Reducibility

- Assume $\mathbf{y}[s,t]$ continuously differentiable. Then:
  - Well-defined $\mathbf{z}(\tau) = (\mathbf{y}(\tau), \mathbf{g}(\tau))$ for each $\tau \in [s,t]$.
  - Well-defined $M(\mathbf{z}(\tau))$ for each $\tau \in [s,t]$.

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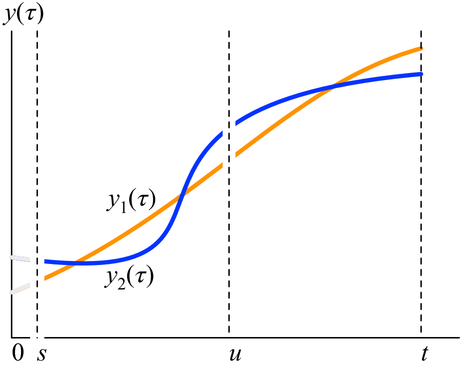
- μ is **reducible** if it's expressible as a function of all these M's:

$$\mu(\mathbf{y}[s,t]) = \Psi(\{M(\mathbf{z}(\tau))\}_s^t)$$

  - with $\mu(\mathbf{y}[s,t]) = m$ whenever $M(\mathbf{z}(\tau)) = m$ for all $\tau \in [s,t]$ (normalization)

Additivity



- μ is **additive** if for all $s < u < t$,
  - $(t-s)\mu(\mathbf{y}[s,t]) = (u-s)\mu(\mathbf{y}\,[s,u]) + (t-u)\mu(\mathbf{y}\,[u,t])$.

Upward Mobility

Theorem 3

Kernel axioms, reducibility, and additivity hold if and only if

$$\mu_\alpha(\mathbf{y}[s,t]) = \frac{1}{t-s} \ln \left[\frac{\sum_{i=1}^{n} y_i^{-\alpha}(t)}{\sum_{i=1}^{n} y_i^{-\alpha}(s)} \right]^{-\frac{1}{\alpha}} \text{ for some } \alpha > 0.$$

- **Remark:** Can also use income categories and population shares (see paper).

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- In what follows, we look at different aspects of this measure.

Upward Mobility as Change in Welfare

- **Mobility measure**:

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- Atkinson welfare function, or Atkinson equivalent income:

$$a_\alpha(\mathbf{y}) = \left(\frac{1}{n} \sum_{j=1}^{n} y_j^{-\alpha} \right)^{-\frac{1}{\alpha}},$$

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- $\mu_\alpha(\mathbf{y}[s,t]) =$ **average growth of Atkinson equiv income** on $[s,t]$.

- Not a measure of equality per se.

Upward Mobility as Pro-Poor Growth

- **Upward Mobility** $= \dfrac{1}{t-s} \ln \left[\dfrac{\sum_{j=1}^{n} y_j(t)^{-\alpha}}{\sum_{j=1}^{m} y_j(s)^{-\alpha}} \right]^{-\frac{1}{\alpha}}$
- **Growth** $= \dfrac{1}{t-s} \ln \left[\dfrac{\sum_{j=1}^{n} y_j(t)}{\sum_{j=1}^{m} y_j(s)} \right]$

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- Isn't even on our "boundary" as $\alpha \to 0$.

- Nevertheless, when all growth rates are the same, $\mu_\alpha =$ growth rate.

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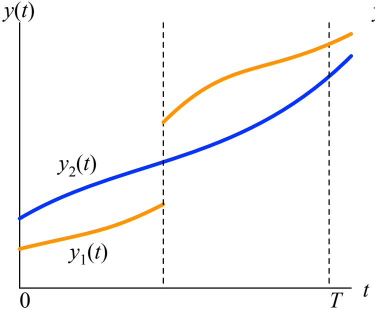

  - Approximate by smooth functions and use continuity: **same answer.**

Relative Upward Mobility

- **Relative upward mobility** nets out growth:

$$\rho_\alpha(\mathbf{y}[s,t]) = \mu_\alpha(\mathbf{y}[s,t]) - \frac{1}{t-s}\left[\ln(\bar{y}(t)) - \ln(\bar{y}(s))\right]$$
$$= \frac{1}{t-s}\ln\left[\frac{\sum_{i=1}^n e_i(t)^{-\alpha}}{\sum_{i=1}^n e_i(s)^{-\alpha}}\right]^{-\frac{1}{\alpha}} \tag{1}$$

  - where $e_i = y_i/\bar{y}$ is excess growth factor relative to per-capita income $\bar{y}$.

- ρ_α admissible under Theorem **??**; can be further axiomatized.

Upward Mobility and Panel Independence

- We now arrive at a central point of the paper:
- **Upward Mobility** $= \dfrac{1}{t-s} \ln \left[\dfrac{\sum_{j=1}^{n} y_j(t)^{-\alpha}}{\sum_{j=1}^{m} y_j(s)^{-\alpha}} \right]^{-\frac{1}{\alpha}}$ is **panel independent.**

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- **Answer:** To assess a family's changing fortunes, *that* family must be tracked.

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- **Answer:** To assess a family's changing fortunes, *that* family must be tracked.
  - But to assess upward mobility overall, it is *society* that must be tracked.
- A family receives time-varying weights depending on its relative location.
  - The impact on overall mobility feeds through the impact on mobility kernels.
  - Such nimble weight switches are central to our argument.

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- Specifically, study how the axioms work:
  - Growth Progressivity $\Rightarrow$ linearity of the kernel in growth rates.

Upward Mobility and Panel Independence

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- Specifically, study how the axioms work:
- Growth Progressivity $\Rightarrow$ linearity of the kernel in growth rates.
- Reducibility $\Rightarrow$

$$\mu(\mathbf{y}[s,t]) = \Psi\left(\left\{\sum_{i=1}^{n} \phi_i(\mathbf{y}(\tau)) g_i(\tau)\right\}_s^t\right) = \Psi\left(\left\{\sum_{i=1}^{n} \frac{\phi_i(\mathbf{y}(\tau))}{y_i(\tau)} \dot{y}_i(\tau)\right\}_s^t\right).$$

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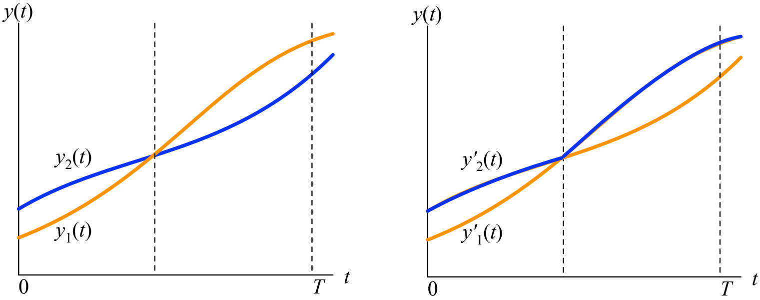
- $\dfrac{\phi_i(\mathbf{y})}{y_i} = \dfrac{y_i^{-\alpha-1}}{\sum_j y_j^{-\alpha}}$, which integrates out to Atkinson welfare. Jumps?

2. But what about mobility as pure movement "back and forth"?



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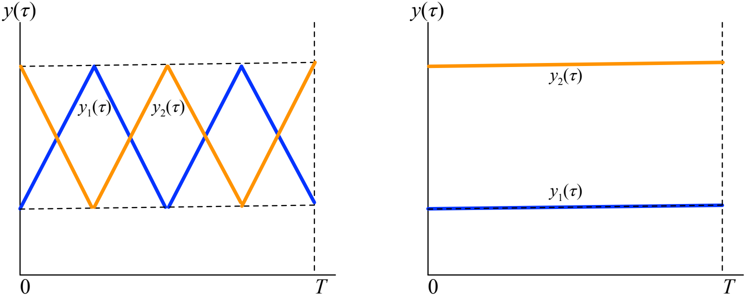

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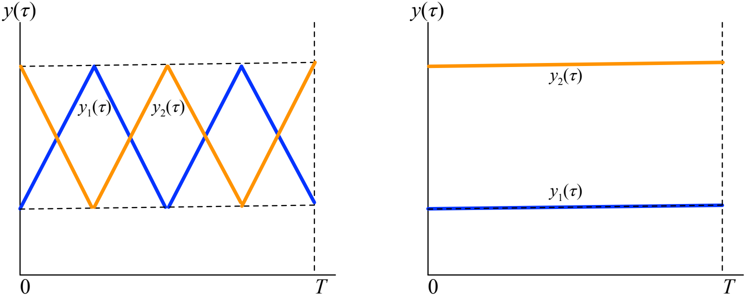

- Different exchange mobility or pure movement. ✓
- Different inequalities. ✓
- But **upward** mobility in both panels is zero.

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- where $M_\alpha^+(\mathbf{z}) = \sum_{i=1}^{n} \phi_i^+(\mathbf{y}) \max\{g_i, 0\}$ and $M_\alpha^-(\mathbf{z}) = \sum_{i=1}^{n} \phi_i^-(\mathbf{y}) \max\{-g_i, 0\}$.

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- $$E_\alpha(\mathbf{z}) = \sum_{i=1}^{n} \phi_i(\mathbf{y}) |g_i| = M_\alpha^+(\mathbf{z}) + M_\alpha^-(\mathbf{z})$$
- Our preferred approach to exchange mobility.
- Such a measure would not be panel-independent.

Upward Mobility and Panel Independence

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  - Or some other measure of permanent income (time-averaged?).
- Similar recommendations apply to poverty or inequality measurement.

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- **Answer:** K social groups. Each person i belongs to one $k(i) \in K$.
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Social Growth Progressivity. For any $\mathbf{z}$, i and j with $(y_i, w_{k(i)}) \leqslant (y_j, w_{k(j)})$, form $\mathbf{z}'$ by altering g_i to $g_i + \epsilon$ and g_j to $g_j - \epsilon$. Then $M(\mathbf{z}') > M(\mathbf{z})$.

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Social Income Neutrality. $M(\lambda\mathbf{y}, \mathbf{g}, \mathbf{w}) = M(\mathbf{y}, \mathbf{g}, \mathbf{w})$ & $M(\mathbf{y}, \mathbf{g}, \lambda\mathbf{w}) = M(\mathbf{y}, \mathbf{g}, \mathbf{w})$.

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Social Binary Growth Tradeoffs. For any i, j, any $(y_i, y_j, w_{k(i)}, w_{k(j)})$, comparing $((y_i, w_{k(i)}, g_i), (y_j, w_{k(j)}, g_j), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}, \mathbf{w}_{-k(i),k(j)}))$ and $((y_i, w_{k(i)}, g'_i), (y_j, w_{k(j)}, g'_j), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}, \mathbf{w}_{-k(i),k(j)})))$ is insensitive to $(\mathbf{y}_{-ij}, \mathbf{g}_{-ij}, \mathbf{w}_{-k(i),k(j)}))$.

Upward Mobility and Panel Independence

4, contd.

Theorem 4

The above axioms hold if and only if for $n \geq 3$ and groupings K,

$$\mu_{\alpha,\beta}(\mathbf{y}[s,t],K) = \frac{1}{t-s}\left\{ \ln\left[\frac{\sum_{i=1}^{n} y_i(t)^{-\alpha} w_{k(i)}(t)^{-\beta}}{\sum_{i=1}^{n} y_i(s)^{-\alpha} w_{k(i)}(s)^{-\beta}}\right]^{-1/\alpha} - \frac{\beta}{\alpha}\int_s^t \frac{\sum_{k\in K} n_k w_k(\tau)^{-\beta} a_k(\tau)^{-\alpha} g_k(\tau)}{\sum_{k\in K} n_k w_k(\tau)^{-\beta} a_k(\tau)^{-\alpha}} d\tau \right\},$$

for some $(\alpha,\beta) \gg 0$, where $a_k(\tau)$ is Atkinson equivalent group income.

- First term on RHS is panel-independent.
- Second term depends on trajectories, but **only at the group level**.
- Can approximate group Atkinson by standard inequality measures (see paper).

Upward Mobility and Panel Independence

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- **Answer:** No.
- For the United States, Chetty et al (2017) estimate:
- % population share: children $\succ$ parents (US birth cohorts, 1940–84).
- Transitions estimated from a unique panel of tax records
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  - % population share: children $\succ$ parents (US birth cohorts, 1940–84).
  - Transitions estimated from a unique panel of tax records
  - $\oplus$ marginal income distributions from CPS and Census.
- Generally very hard to get hold of.
  - Though similar studies exist for other countries; e.g., Acciari et al (2021).

Upward Mobility: Other Measures

Skip?

- The **Chetty et al** (2017) measure (also Berman 2021, Acciari et al 2021):

$$\mu^{\mathsf{c}}(\mathbf{y}[0,1]) = \sum_{i=1}^{n} I(y_i(0), y_i(1)).$$

  - where $I(y_i(0), y_i(1))$ is indicator for $y_i(0) < y_i(1)$.
  - Population share for whom future $\succ$ present.

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- The **Fields-Ok** (1999) measure:

$$\mu^{\mathsf{FO}}(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^{n}\left[\ln(y_i(0)) - \ln(y_i(1))\right] = \frac{1}{n}\sum_{i=1}^{n}\left[\int_0^1 g_i(\tau)d\tau\right].$$

- Both must fail growth progressivity.

Upward Mobility: Other Measures

- **Example for μ^{c}:**
  - Two persons at incomes \$10,000 and \$20,000.
  - Growth rates 1% for both. Then $\mu^{\mathsf{c}} = 1$.
  - Transfer 2 points of growth from rich to poor. Then $\mu^{\mathsf{c}} = 1/2$.
  - But growth progressivity asks that mobility must rise.

Upward Mobility: Other Measures

- **Rank-weighted measures:**

  - Such measures fail our axioms in a seemingly technical way:

  - They are not continuous — and this isn't just a technicality.

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  - And worse: large changes in *relative* income could go unnoticed.

Upward Mobility: Other Measures

- **Rank-weighted measures:**
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  - They are not continuous — and this isn't just a technicality.
- **Tiny changes** in incomes can generate **discrete jumps** in mobility.
  - And worse: large changes in *relative* income could go unnoticed.
- **Our measure is indeed correlated with rank-based measures**.
  - But is sensitive throughout, without being unduly affected by a rank switch.

Upward Mobility in the Data

- **Chetty et al (2017) estimate** $M^I(\mathbf{z})$ for US birth cohorts, 1940–84.
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Upward Mobility in the Data

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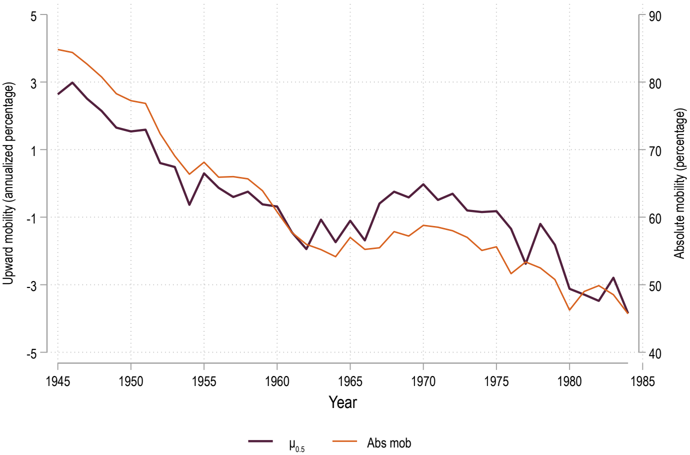
  - They estimate a copula from a unique panel of tax records.
- *In practice*, the dependence on exact copulas seems limited; Berman (2021)

"Estimating the absolute mobility in the United States with different copulas, some of which are very different from the one characterizing the United States, results in a similar evolution in time."

μ_α Compared to Chetty et al (2017) for the United States



- Robust to different α.
- Robust to using other publicly available databases (e.g., WID).

Upward Mobility in Brazil, India and France

- **Ten-year upward mobility in Brazil, India and France:**
  - Data from the World Inequality Database (repeated cross-sections).
  - Measure $\mu_{0.5}(\mathbf{y}[t, t+10])$ and $\rho_{0.5}(\mathbf{y}[t, t+10])$.
  - Robust with respect to choice of α (see paper).

Upward Mobility in Brazil, India and France

Ongoing Research: Distribution and Mobility

Esteban, Genicot, Mayoral, Ray (in preparation)

- **How does distribution affect subsequent mobility?**

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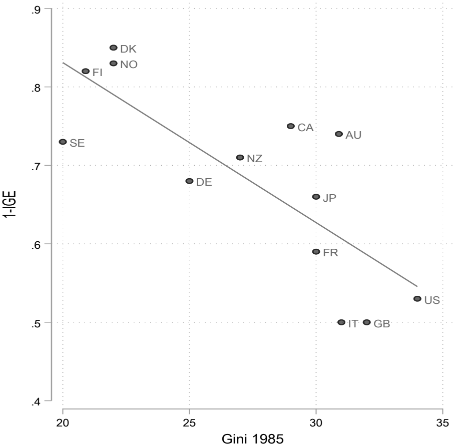
- **How does distribution affect subsequent mobility?**
- Distribution $\oplus$ future mobility?
  - Mechanical mean reversion
  - Classical convergence: convex technology
- Distribution $\ominus$ future mobility?
  - Classical poverty traps: missing credit markets, nonconvexities.
  - Psychological traps: β-δ, aspirations failure

The Great Gatsby Curve

- **High inequality is correlated with low mobility** Krueger (2012)



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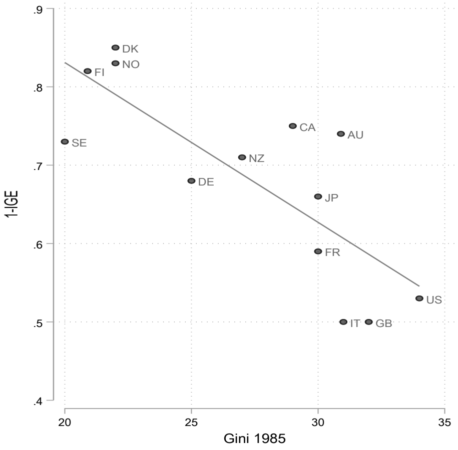
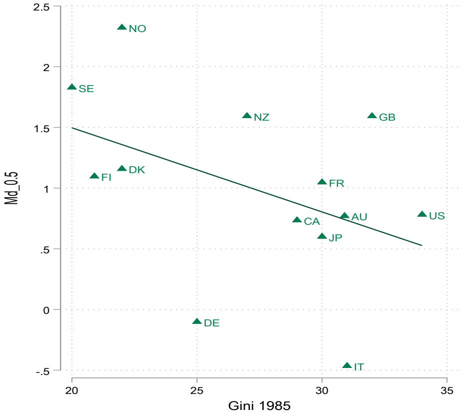




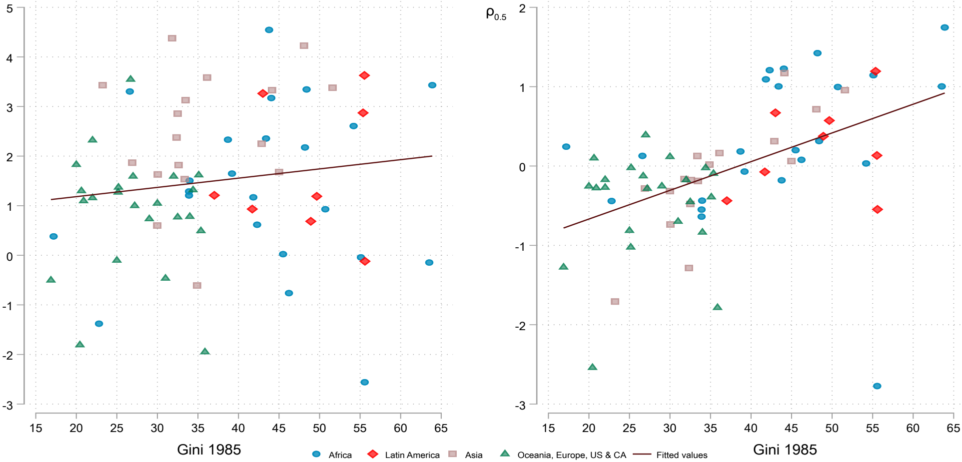
Krueger (2021) / Corak (2013)

Using $\mu_{0.5}$

The Great Gatsby Curve

- **Does the cross-section hold up? No.**

86 countries (WID); 1985-2015: Genicot ⓡ Ray ⓡ Concha-Arriagada



Absolute mobility$_{\alpha=0.5}$ Relative mobility$_{\alpha=0.5}$

The Great Gatsby Curve

- But the expansion of data allows us to **exploit panel structure**.
- Preliminary: 4-period panel (1980, 1990, 2000, 2010), 174 countries (WID)

	Absolute Upward Mobility, $\alpha = 0.5$ [t, t+10]			
	[1]	[2]	[3]	[4]
GINI	1.875	2.391		
	(0.000)	(0.000)		
ATKINSON			1.881	2.299
			(0.000)	(0.000)
LOG(INCOME)$_t$		-6.879		-6.873
		(0.000)		(0.000)
C	-10.096	5.795	-12.489	3.414
	(0.000)	(0.033)	(0.000)	(0.226)
R^2	0.096	0.404	0.104	0.411
Obs	696	696	696	696
Estimation	FE	FE	FE	FE

All regressions with year effects and country FE. Standard errors clustered at the country level. p-values in parentheses.

The Great Gatsby Curve

- But the expansion of data allows us to **exploit panel structure**.

- Preliminary: 4-period panel (1980, 1990, 2000, 2010), 174 countries (WID)

	Relative Upward Mobility, $\alpha = 0.5$ [t, t+10]			
	[1]	[2]	[3]	[4]
GINI$_t$	1.505	1.511		
	(0.000)	(0.000)		
ATKINSON$_t$			1.567	1.572
			(0.000)	(0.000)
LOG(INCOME)$_t$		-0.074		-0.081
		(0.532)		(0.523)
C	-8.324	-8.154	-10.640	-10.452
	(0.000)	(0.000)	(0.000)	(0.000)
R^2	0.164	0.164	0.213	0.213
Obs	696	696	696	696
Estimation	FE	FE	FE	FE

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Measuring Upward Mobility: A Summary

- A bewildering variety of mobility indices:
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- Our **trajectory-based measure** is pinned down by two conditions
  - reducibility and additivity.
  - It is panel-independent
- If convincing, this **significantly expands the scope of empirical inquiry**

Population Consistency

Given: $$\mathbf{z} = (y_1, g_1, \ldots, y_k, g_k, \ldots, y_n, g_n)$$

$$\mathbf{z}' = (y_1, g_1, \ldots, y_k, g_k - \epsilon, \ldots, y_n, g_n)$$

$$\mathbf{z}'' = (y_1, g_1, \ldots, y_k, g_k + \epsilon, \ldots, y_n, g_n)$$

and $\mathbf{z}'$ and $\mathbf{z}''$ have average mobility distinct from $\mathbf{z}$: $\frac{1}{2}[M(\mathbf{z}') + M(\mathbf{z}'')] \neq M(\mathbf{z})$,

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and $\mathbf{z}'$ and $\mathbf{z}''$ have average mobility distinct from $\mathbf{z}$: $\frac{1}{2}[M(\mathbf{z}') + M(\mathbf{z}'')] \neq M(\mathbf{z})$,

Then: $$M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z}).$$

back

Appendix: Proof of Theorem 1

- **Step 1. For every k, $m(g_k) \equiv M(g_k|\mathbf{y}, \mathbf{g}_{-k})$ is affine in g_k, or equivalently**:

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$$m(g_k) = \frac{1}{2}\Big[m(g_k - \epsilon) + m(g_k + \epsilon)\Big] \text{ for every } \epsilon > 0.$$
- Suppose false for some g_k and ϵ.
  - Define $\mathbf{z} = (\mathbf{y}, \mathbf{g}_{-k}, g_k)$, $\mathbf{z}' = (\mathbf{y}, \mathbf{g}_{-k}, g_k - \epsilon)$, and $\mathbf{z}'' = (\mathbf{y}, \mathbf{g}_{-k}, g_k + \epsilon)$.
  - Then $M(\mathbf{z}') + M(\mathbf{z}'') \neq M(\mathbf{z}) + M(\mathbf{z})$.
  - By Local Merge, $M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z})$.
  - Say $M(\mathbf{z}' \oplus \mathbf{z}'') > M(\mathbf{z} \oplus \mathbf{z})$.

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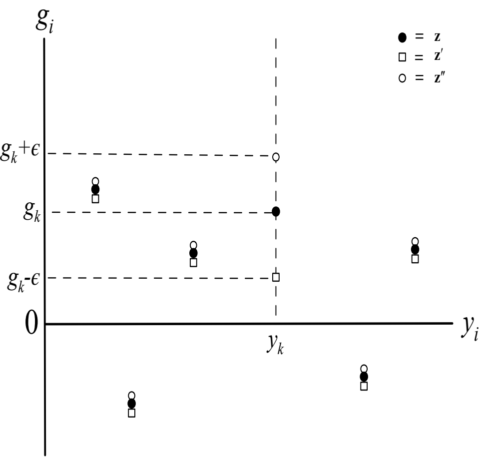

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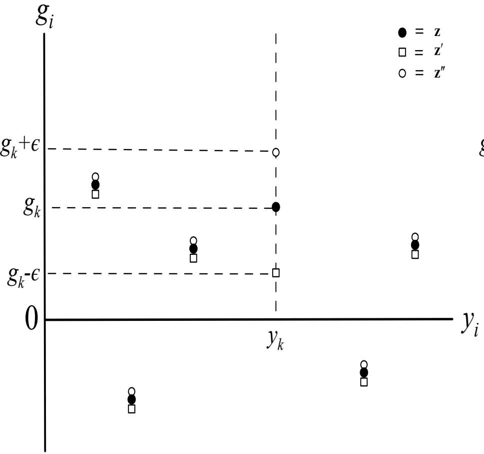

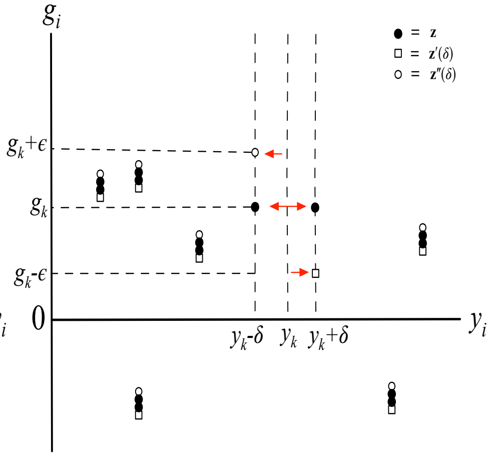

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$$M(\mathbf{z}) = \sum_{S} \phi_S(\mathbf{y}) \left[\prod_{j \in S} g_j \right].$$

for a collection $\{\phi_S\}$ defined for every $\emptyset \neq S \subset \{1, \ldots, n\}$.

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- **Step 3. All nontrivial product terms above *must have zero coefficients*.**

Suppose $\{ij\} \subset S$ for some S with $\phi_S(\mathbf{y}) \neq 0$. We will only move g_i and g_j but with $g_i + g_j = G$, so hold all else fixed and write

$$M(\mathbf{y}, \mathbf{g}) = \alpha g_i (G - g_i) + \beta g_i + \gamma (G - g_i) + \delta.$$

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$$M(\mathbf{z}) = \sum_S \phi_S(\mathbf{y}) \left[\prod_{j \in S} g_j \right].$$

for a collection $\{\phi_S\}$ defined for every $\emptyset \neq S \subset \{1, \ldots, n\}$.

- **Step 3. All nontrivial product terms above *must have zero coefficients*.**

Suppose $\{ij\} \subset S$ for some S with $\phi_S(\mathbf{y}) \neq 0$. We will only move g_i and g_j but with $g_i + g_j = G$, so hold all else fixed and write

$$M(\mathbf{y}, \mathbf{g}) = \alpha g_i(G - g_i) + \beta g_i + \gamma(G - g_i) + \delta.$$

$$\Rightarrow \frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_i} - \frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_j} = \alpha G - 2\alpha g_i + \beta - \gamma.$$

Choose G and g_i to violate Growth Progressivity. back