EC9AA Term 3: Lectures on Economic Inequality

Debraj Ray, University of Warwick, Summer 2022

- Slides 4: Measuring Upward Mobility

What the Term Might Mean

Mobility:

- ease of transition between various social categories:
- income, wealth, location, political persuasions …
- Centrally important in current debates
- The concept has several connotations, though.

What the Term Might Mean

Non-Directional:

- Mobility as pure movement:
  - Off-diagonal elements in a transition matrix defined on categories.

  Atkinson (1981), Bartholomew (1982), Chakravarty et al. (1985), Conlisk (1974), Dardanoni (1993), Hart (1976), Prais (1955), Shorrocks (1978a,b) …

Directional:

- Movement *up* $\succ$ movement *down*

  Chakravarty et al. (1985), Bénabou and Ok (2001), Chetty, Hendern, Kline, Saez (2014), Fields (2007), Bhattacharya (2011), Fields and Ok (1996, 1999), Mitra and Ok (1998) …

What the Term Might Mean

Relative:

Chakravarty et al. (1985), Bénabou and Ok (2001), Chetty, Hendern, Kline, Saez (2014), Fields (2007), Bhattacharya (2011)

Absolute:

Fields and Ok (1996, 1999), Mitra and Ok (1998), Chetty et al (2017)

- And of course, all combinations of
  - Directional/Non-directional
  - Absolute/Relative

An Incomplete List

Name	Measure	Directional	Non-directional	Absolute	Relative
King (1983)	$M_\kappa = 1 - \exp\left[-\frac{\gamma}{n}\sum \frac{\lvert z_i - y_i \rvert}{\mu_y}\right]$		✓		✓
Shorrocks index (1978)	$M_S = \frac{n - \mathsf{Tr}(P)}{n-1}$		✓		✓
Variability of the eigenvalues	$\sigma(\gamma_i)$		✓		✓
Bartholomew (1982)	$M_B = \frac{1}{n-1}\sum_i \sum_j \pi_i p_{ij} \mid i - j \mid$		✓		✓
IG Income Elasticity (IGE)	$\beta = \frac{\mathsf{Cov}(S_{it}, S_{it-1})}{\mathsf{Var}(S_{it-1})}$		✓	✓	
Correlation coefficient (CE)	$\rho_S = \frac{\mathsf{Cov}(S_{it}, S_{it-1})}{\sqrt{\mathsf{Var}(S_{it})}\sqrt{\mathsf{Var}(S_{it-1})}}$		✓	✓	
Slope rank-rank	$\rho_{PR} = \mathsf{Corr}(P_i, R_i)$		✓		✓
IG rank association (IRA)	$\beta = \frac{\mathsf{Cov}(p^y_{it}, p^X_{it})}{\mathsf{Var}(p^X_{it})}$		✓		✓
Mitra & Ok (1998)	$\mathsf{MO}_\alpha(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\gamma\left(\sum_i \lvert y_i - x_i \rvert^\alpha\right)^{1/\alpha}$		✓	✓	
Gini symmetric index of mobility	$GS = \frac{\sum_i (y_i - x_i)(F_{xi} - F_{yi})}{\sum_i (y_i - 1)F_{yi} + \sum_i (x_i - 1)F_{xi}}$		✓	✓	
Great Gatsby curve	Corr(Gini, IGE)		✓	✓	
Bhattacharya (2011)	$\nu = Pr(F_1(Y_1) - F_0(Y_0) > \tau \mid s_1 \leq F_0(Y_0) \leq s_2, X = x)$	✓			✓
Absolute upward mobility (1)	$p_{25} = \mathbb{E}(Y \mid X \leq 25)$	✓			✓
Absolute upward mobility (2)	$A = \Phi\left(\frac{\mu_c - \mu_p}{\sqrt{\sigma_p^2 + \sigma_c^2 + 2\rho\sigma_p\sigma_c}}\right)$	✓			✓
Chetty et al (2017)	$\mathsf{AM}(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\sum_i (1_{y_i] \geq x_i})$	✓		✓	
Rising up-up	$P_{20to100} = \mathbb{E}[Y = 100 \mid X = 20]$	✓			✓
Bottom half mobility	$\mu_0^{50} = \mathbb{E}(y \mid x \in [0, 50])$	✓			✓
Fields & Ok (1999)	$\mathsf{FO}(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\sum_i (\ln(y_i) - \ln(x_i))$	✓		✓	
Card (2018)	$\mathbb{E}(y > 50 \mid x \in [45, 70])$	✓		✓	
Pro-poor growth	$G = \sum_{k=1}^{5} w_k g_k$	✓		✓	

Axiomatic Approach

- Faced with this huge variety of measures, we proceed axiomatically.
  - We first develop a core measure of "instantaneous" mobility that is:

1. **Directional**: it rewards growth, and punishes decay;
2. **Progressive**: it rewards "growth transfers" from higher to lower incomes.

- Then a discrete-time extension that is also

3. **Panel-independent**: can be deployed on repeated cross-sections.

Instantaneous Upward Mobility

- **All finite populations**:
  - $\mathbf{z} = \{z_i\}$, where $z_i = (y_i, g_i)$, and:
  - $y_i > 0$ baseline income, g_i instantaneous growth rate of that income.
  - Can merge populations; $\mathbf{z} \oplus \mathbf{z}'$.
- **Instantaneous upward mobility index**:
  - $M(\mathbf{z})$
  - continuous, invariant to permutations of indices within $\mathbf{z}$.

Background Axioms

Zero Growth Anchoring.

- If under $\mathbf{z}$, $g_i = 0$ all i, then $M(\mathbf{z}) = 0$.

Local Merge.

- If $\mathbf{z}$, $\mathbf{z}'$ and $\mathbf{z}''$ identical *except* $g_k' = g_k - \epsilon$ and $g_k'' = g_k + \epsilon$ just one k;
- And if $[M(\mathbf{z}') + M(\mathbf{z}'')]/2 \neq M(\mathbf{z})$:

$\mapsto$ Then $M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z})$.

Core Axiom

Growth Progressivity.

- For any $\mathbf{z}$, i and j with $y_i < y_j$, and $\epsilon > 0$, send g_i to $g_i + \epsilon$ and g_j to $g_j - \epsilon$.
- Then $M(\mathbf{z}') > M(\mathbf{z})$.

Instantaneous Upward Mobillity

Theorem 1

Growth Progressivity, Zero Growth Anchoring and Local Merge hold if and only M can be written as

$$M(\mathbf{z}) = \sum_{i=1}^{n} \phi_i(\mathbf{y}) g_i,$$

for some continuous collection $\{\phi_i\}$, with $\phi_i(\mathbf{y}) = \phi_j(\tilde{\mathbf{y}})$ whenever i and j are permuted in $\mathbf{y}$ to get $\tilde{\mathbf{y}}$, and $\phi_i(\mathbf{y}) > \phi_j(\mathbf{y})$ when $y_i < y_j$. Proof Outline

Sharpening the Measure

Income Neutrality.

- $M(\mathbf{y}, \mathbf{g}) = M(\lambda \mathbf{y}, \mathbf{g})$ for all $\lambda > 0$.

Growth Alignment.

- If $\mathbf{g} > \mathbf{g}'$, then $M(\mathbf{y}, \mathbf{g}) > M(\mathbf{y}, \mathbf{g}')$ for all $\mathbf{y}$.
- If $\mathbf{g} = (g, g, \ldots, g)$, then $M(\mathbf{y}, \mathbf{g}) = M(\mathbf{y}', \mathbf{g})$ for all $\mathbf{y}$ and $\mathbf{y}'$,.

Binary Growth Tradeoffs.

- Is $M((y_i, g_i), (y_j, g_j), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij})) \geq M((y_i, g_i'), (y_j, g_j'), (\mathbf{y}_{-ij}, \mathbf{g}_{-ij}))$?
- Answer insensitive to $(\mathbf{y}_{-ij}, \mathbf{g}_{-ij})$.

Sharpening the Measure

Theorem 2

Above six axioms hold iff for every population $n \geq 3$, M can be written as:

$$M_\alpha(\mathbf{z}) = \frac{\sum_{i=1}^{n} y_i^{-\alpha} g_i}{\sum_{i=1}^{n} y_i^{-\alpha}}, \text{ for some } \alpha > 0.$$

Remarks:

- Actually, of the six, Zero Growth Anchoring and Local Merge are now automatically implied.
- Proof employs a substantial extension of Gorman's separability theorem; see Chatterjee ⓡ Ray ⓡ Sen (2021).

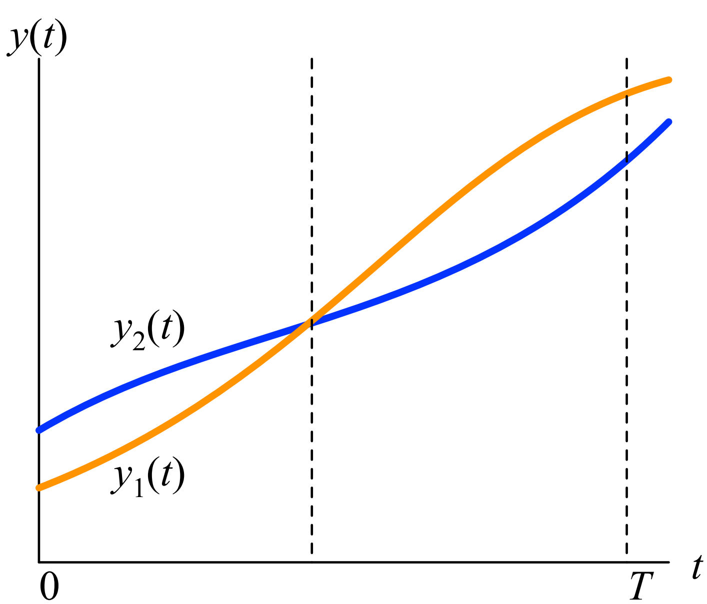
Upward Mobility in Discrete Time

- The "instantaneous measures" cannot be taken to the data as is.

Three considerations:

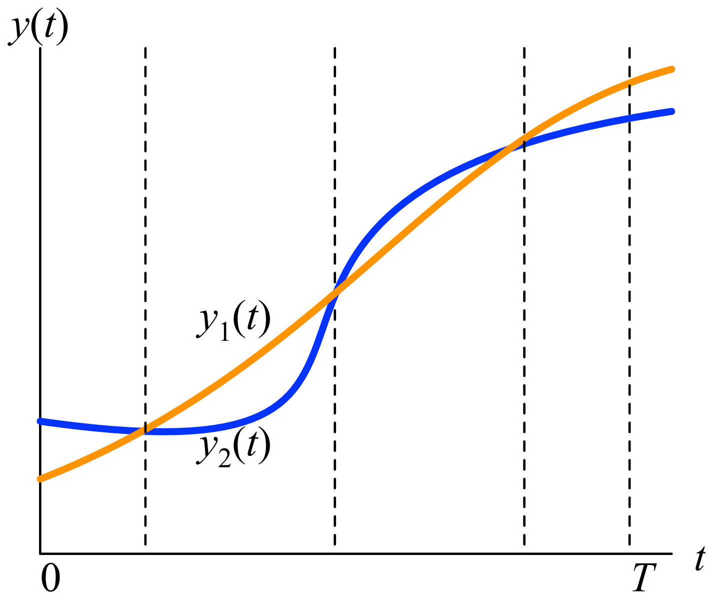
- The data come in discrete form: observations separated in time.
- Income trajectories can and do cross.
- We may or may not observe those crossings: often, no panel data.

Crossings



- Growth Progressivity does not directly apply to discrete observations

Crossings



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A Base for Discrete Mobility

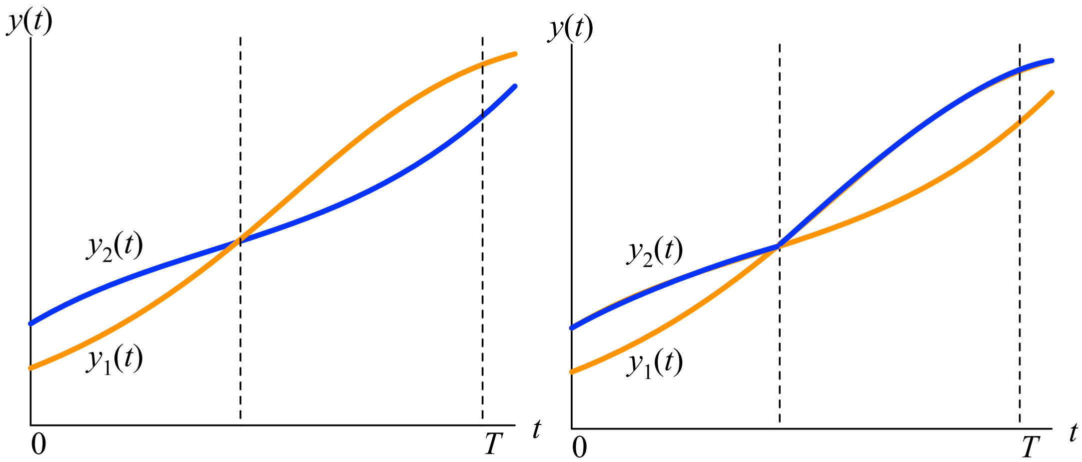
- Collection of income trajectories $\mathbf{y}(0, T)$.
  - Assume continuous and right-differentiable everywhere (will drop).
  - Induces $\mathbf{z}(\tau) = (\mathbf{y}(\tau), \mathbf{g}(\tau))$ for every $\tau \in [0, T)$.
- **Base condition**: for every $0 \leq s < t < T$,

$$\mu(\mathbf{y}(s,t)) = \Psi(\{M(\mathbf{z}(\tau))\}_s^t)$$

  for some Ψ, where $M =$ instantaneous measure.
  - Think Bergson-Samuelson.

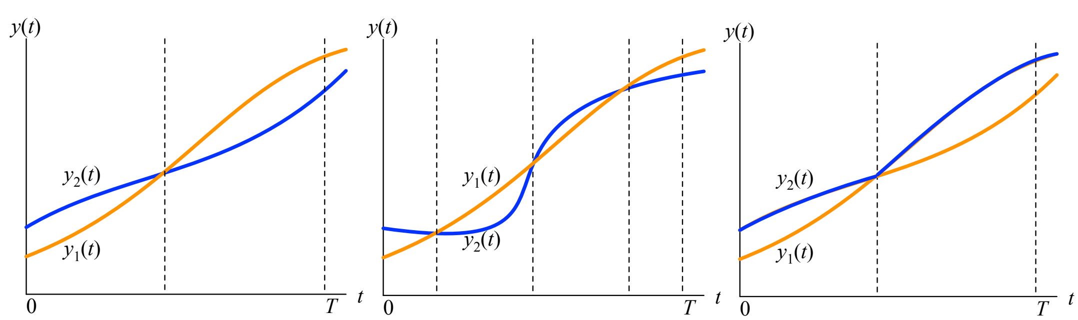
The Base Condition and Panel Independence

- **Panel independence**: spot the difference . . .



- **Cupola for transitions** typically constructed with much difficulty:
  - Data limited, often proprietary; Chetty et al (2017), Acciari et al (2021)

Path Independence



Path Independence: $\mu(\mathbf{y}(s,t)) = M^{\triangle}(\mathbf{y}(s), \mathbf{y}(t))$

for some $M^{\triangle}$ that depends only on $\mathbf{y}(s)$ and $\mathbf{y}(t)$.

- **Pragmatic**, of course, but also
- **Conceptual**: *upward* mobility is a directed object:

  With mobility as pure movement, path independence generally fails.

Discrete Upward Mobility

Theorem 3

Axioms 1–6, the base condition, and path independence hold iff over any collection of continuous right-differentiable trajectories $\mathbf{y}(s,t)$*,*

$$M_\alpha^\Delta(\mathbf{y}(s),\mathbf{y}(t)) = \frac{1}{t-s}\ln\left[\frac{\sum_{i=1}^n y_i^{-\alpha}(t)}{\sum_{i=1}^n y_i^{-\alpha}(s)}\right]^{-\frac{1}{\alpha}} \quad \textit{for some } \alpha > 0.$$

- Works on repeated cross-sections.

Some Remarks on Discrete Upward Mobility

I. Idea behind Theorem 3:

- By Theorem 1, instantaneous upward mobility $M(\mathbf{z}) = \sum_{i=1}^n \phi_i(\mathbf{y})g_i$.
  - So by the base condition:

$$\mu(\mathbf{y}(s,t)) = \Psi\left(\left\{\sum_{i=1}^n \phi_i(\mathbf{y}(\tau))g_i(\tau)\right\}_s^t\right) = \Psi\left(\left\{\sum_{i=1}^n \frac{\phi_i(\mathbf{y}(\tau))}{y_i(\tau)}\dot{y}_i(\tau)\right\}_s^t\right),$$

- Path independence $\Rightarrow \Psi$ has linear representation in $\{\dot{y}_i(\tau)\}_s^t$.
  - And so $\phi_i(\mathbf{y})/y_i$ becomes a vector field. Associated potential function?
- Same method can be extended to trajectories with simple jumps, using approximation by smooth functions.

Some Remarks on Discrete Upward Mobility

II. Upward Mobility as Change in Welfare:

- For arbitrary initial condition $\mathbf{x}(0)$, define $W(\mathbf{y})$ by

$$W(\mathbf{y}) \equiv \exp\left[M^{\Delta}(\mathbf{x}(0), \mathbf{y}(1))\right]$$

  - Then instantaneous mobility is the time derivative of this function.

- Solve to get: $W_{\alpha}(\mathbf{y}) = \left(\frac{1}{n}\sum_i y_i^{-\alpha}\right)^{-\frac{1}{\alpha}}$, for $\alpha > 0$.

  - $M_{\alpha}^{\Delta}(\mathbf{y}(s), \mathbf{y}(t)) =$ average growth of Atkinson equivalent income on $[s, t]$.

Some Remarks on Discrete Upward Mobility

III. Upward Mobility and Inequality

- Upward mobility rewards greater equalization of “terminal” incomes
  - But only any equalization implicit in the change of incomes.
  - Therefore it is **not a measure of equality**.
- For instance:
  - If all incomes grow equally, M_{α}^{Δ} insensitive to income distribution.
  - M_{α}^{Δ} values growth relative to no growth, no matter how disequalizing.

Some Remarks on Discrete Upward Mobility

IV. Upward Mobility as Pro-Poor Growth Chenery et al (1974)

- Discrete upward mobility measure can be written as:

$$M_\alpha^\Delta(\mathbf{y}(s), \mathbf{y}(t)) = \frac{1}{t-s} \ln \left[\frac{\sum_{j=1}^m n_j(t) y_j^{-\alpha}(t)}{\sum_{j=1}^m n_j(s) y_j^{-\alpha}(s)} \right]^{-\frac{1}{\alpha}}.$$

bunching repetitions into n_j. In contrast, growth given by:

$$\text{Log Growth} = \frac{1}{t-s} \ln \left[\frac{\sum_{j=1}^m n_j(t) y_j(t)}{\sum_{j=1}^m n_j(s) y_j} \right]$$

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$$\text{Log Growth} = \frac{1}{t-s} \ln \left[\frac{\sum_{j=1}^m n_j(t) y_j(t)}{\sum_{j=1}^m n_j(s) y_j} \right] = M_{-1}^\Delta(\mathbf{y}(s), \mathbf{y}(t))$$

- Isn't even on our "boundary" as $\alpha \to 0$.
  - Nevertheless, when all growth rates are the same, $M_\alpha^\Delta =$ log growth.

Some Remarks on Discrete Upward Mobility

V. The Relative Upward Mobility Kernel

- Upward mobility, while not the same as growth, is sensitive to it.
  - **Relative upward mobility** nets out average growth.

$$K_\alpha^\Delta(\mathbf{y}(s),\mathbf{y}(t)) = M_\alpha^\Delta(\mathbf{y}(s),\mathbf{y}(t)) - \frac{1}{t-s}\left[\ln(\bar{y}(t)) - \ln(\bar{y}(s))\right]$$
$$= \frac{1}{t-s}\ln\left[\frac{\sum_{i=1}^n e_i^{-\alpha}(t)}{\sum_{i=1}^n e_i^{-\alpha}(s)}\right]^{-\frac{1}{\alpha}} \quad (1)$$

  - where $e_i = y_i/\bar{y}$ is **excess growth factor** relative to per-capita income $\bar{y}$.
- K_α^Δ is admissible under Theorem 1; can be further axiomatized.

Some Remarks on Discrete Upward Mobility

VI. Alternative Measures

Extended class for general function h: $M_\alpha^h(\mathbf{z}) = \dfrac{\sum_{i=1}^n y_i^{-\alpha} h(g_i)}{\sum_{i=1}^n y_i^{-\alpha}}$.

- Set $\alpha = 0$ and h equal to indicator $I(g) = 0$ for $g < 0$, $I(g) = 1$ for $g \geq 0$:

$M_0^I(\mathbf{z}) =$ Population share under $\mathbf{z}$ for whom future $\succ$ present.

  - Influential measure used in Chetty et al (2017) and Berman (2021).
- Set $\alpha = 0$ and h equal to $\ln(g)$:

$$M_0^{\ln}(\mathbf{z}) = \frac{1}{n}\sum_{i=1}^n \ln g_i,$$

  - a measure introduced by Fields and Ok (1999).

Some Remarks on Discrete Upward Mobility

VI. Alternative Measures

Both the above measures violate growth progressivity.

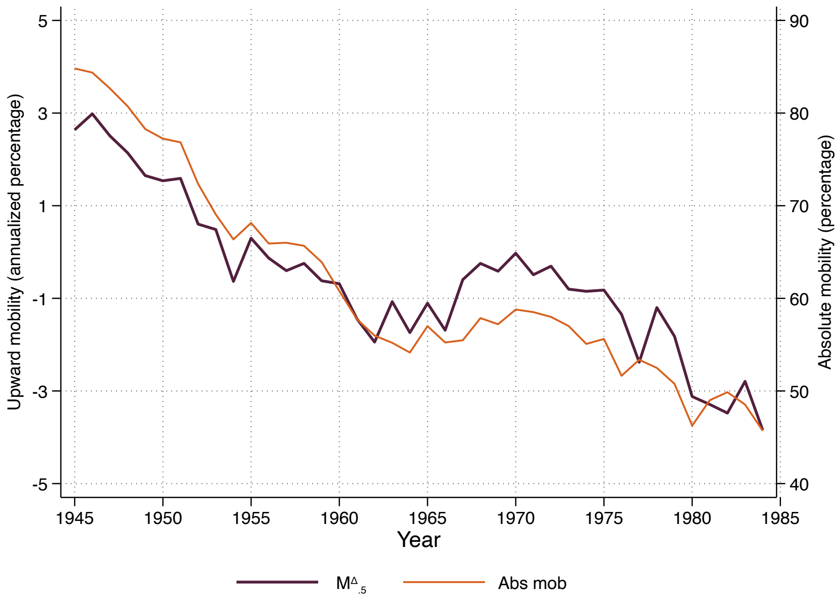
- Example:
  - Two equally-sized groups at incomes $10,000 and $20,000.
  - Growth rates 1% for each group.
  - Transfer 2 percentage points of growth from rich to poor.
  - Then both the previous measures go down.
- Indeed, growth progressivity is violated for any non-linear h function.

Upward Mobility in the Data

- **Chetty et al (2017) estimate $M_0^I(\mathbf{z})$** for US birth cohorts, 1940–84.
  - They estimate a copula from a unique panel of tax records.
  - For ordinary mortals, impossible to get hold of.
  - Combine with marginal income distributions from CPS and Census.
- ***In practice*, the dependence on exact copulas seems limited** Berman (2021)

"We use 28 different copulas measured for different cohorts, different countries …Estimating the absolute mobility in the United States with different copulas, some of which are very different from the one characterizing the United States, results in a similar evolution in time."

M_α^Δ Compared to Chetty et al (2017) for the United States

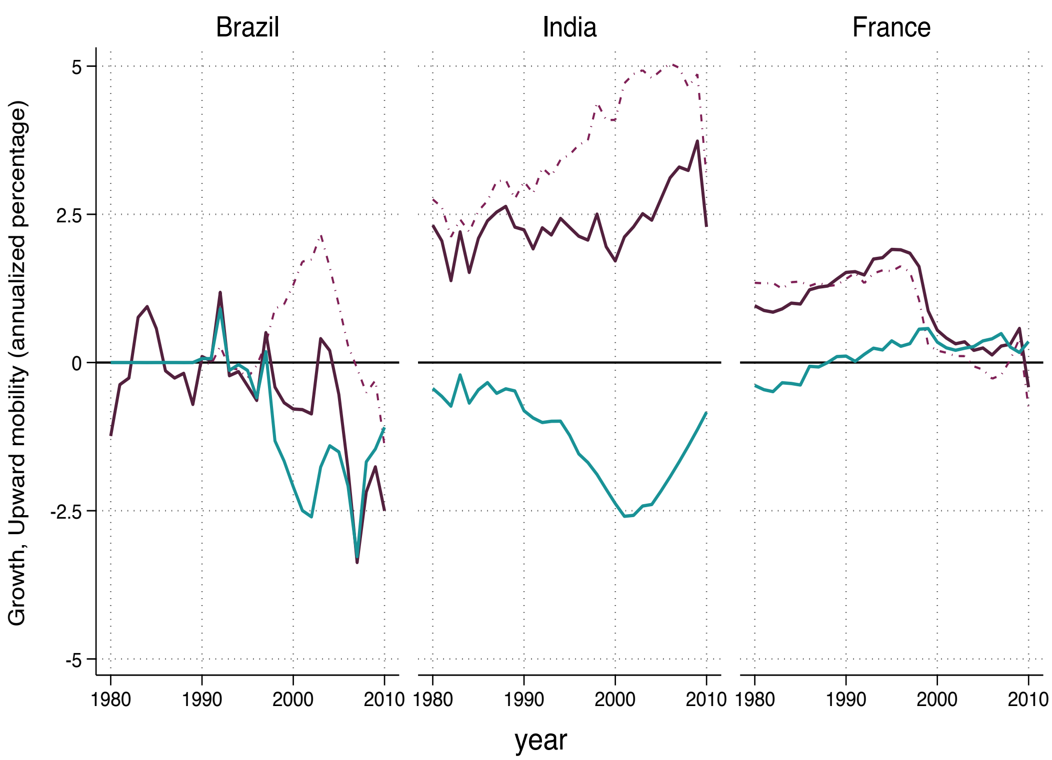


- Robust to different α.
- Robust to using other publicly available databases (e.g., WID).

Upward Mobility in Brazil, India and France

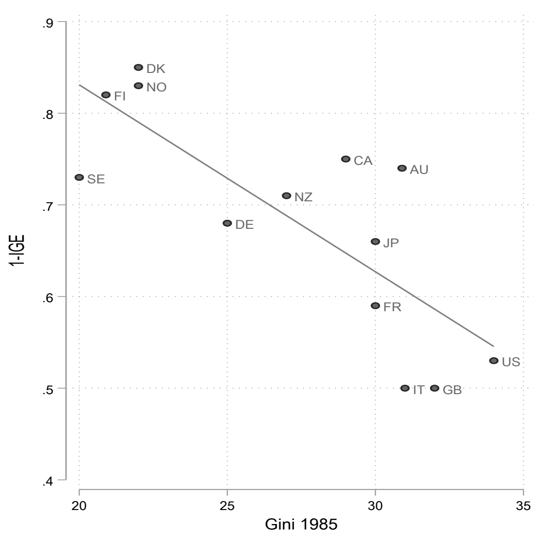
- Ten-year upward mobility in Brazil, India and France:
  - Data from the World Inequality Database (repeated cross-sections).
  - Measure $M_{0.5}^\Delta(\mathbf{y}(t), \mathbf{y}(t+10))$ and $K_{0.5}^\Delta(\mathbf{y}(t), \mathbf{y}(t+10))$.
  - Show robustness with respect to choice of α.

Upward Mobility in Brazil, India and France



The Great Gatsby Curve

- **High inequality is correlated with low mobility** Krueger (2012)

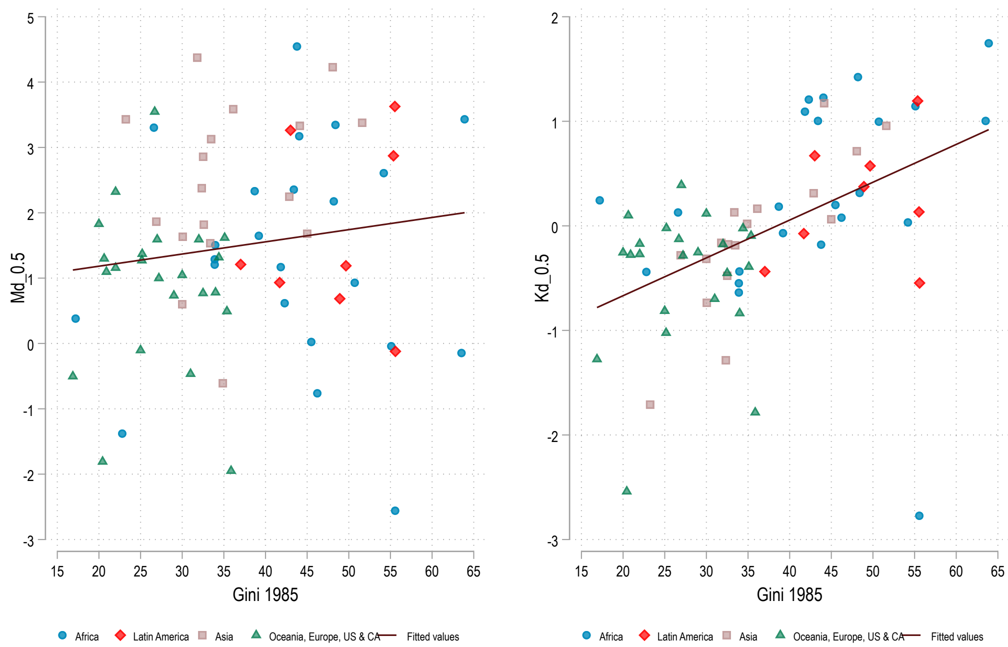


Krueger (2021) / Corak (2013)

Using $M^{\Delta}_{0.5}$

The Great Gatsby Curve

- But we can extend the analysis to 86 countries using WID data; 1985-2015:



Measuring Upward Mobility: A Summary

- A bewildering variety of mobility indices:
  - directional/non-directional; absolute/relative.
- We axiomatize a **class of directional indices**
  - At the core is the growth progressivity axiom.
  - Analogue of the Lorenz criterion for inequality measurement
- Our indices related to pro-poor growth measures
  - Satisfy a base condition, built from instantaneous mobility
  - **Panel-independent**
  - Path-independent
- If convincing, this **significantly expands the scope of empirical inquiry**

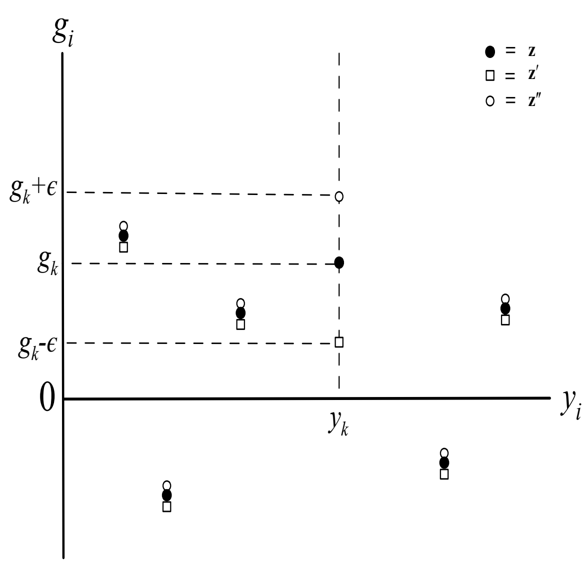
Appendix: Proof of Theorem 1

- **Step 1. For every k, $m(g_k) \equiv M(g_k|\mathbf{y}, \mathbf{g}_{-k})$ is affine in g_k, or equivalently:**

$$m(g_k) = \frac{1}{2}[m(g_k - \epsilon) + m(g_k + \epsilon)] \text{ for every } \epsilon > 0.$$

- Suppose false for some g_k and ϵ.
  - Define $\mathbf{z} = (\mathbf{y}, \mathbf{g}_{-k}, g_k)$, $\mathbf{z}' = (\mathbf{y}, \mathbf{g}_{-k}, g_k - \epsilon)$, and $\mathbf{z}'' = (\mathbf{y}, \mathbf{g}_{-k}, g_k + \epsilon)$.
  - Then $[M(\mathbf{z}') + M(\mathbf{z}'')]/2 \neq M(\mathbf{z})$.
  - By Local Merge, $M(\mathbf{z}' \oplus \mathbf{z}'') \neq M(\mathbf{z} \oplus \mathbf{z})$.

Appendix: Proof of Theorem 1



Appendix: Proof of Theorem 1

- **Step 2. (Gallier 1999) $M(z)$ multiaffine so can be written as**:

$$M(\mathbf{z}) = \sum_S \phi_S(\mathbf{y}) \left[\prod_{j \in S} g_j \right].$$

for a collection $\{\phi_S\}$ defined for every $\emptyset \neq S \subset \{1, \ldots, n\}$.

- **Step 3. All nontrivial product terms above *must have zero coefficients***.

Suppose $\{ij\} \subset S$ for some S with $\phi_S(\mathbf{y}) \neq 0$. We will only move g_i and g_j but with $g_i + g_j = G$, so hold all else fixed and write

$$M(\mathbf{y}, \mathbf{g}) = \alpha g_i (G - g_i) + \beta g_i + \gamma (G - g_i) + \delta.$$
$$\Rightarrow \frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_i} - \frac{\partial M(\mathbf{y}, \mathbf{g})}{\partial g_j} = \alpha G - 2\alpha g_i + \beta - \gamma.$$

Choose G and g_i to violate Growth Progressivity. back