Development Economics Fall 2005

Problem Set 5

[1] A family farm with n members produces a joint output using an increasing, smooth, strictly concave production function $f(R)$, where R denotes the sum of individual efforts r_i. Each individual has a utility function $u(c) - v(r)$, where c is his consumption and r is his effort. Assume that u and v are increasing and smooth, and that u is strictly concave while v is strictly convex.

The purpose of this question is to investigate two different reward systems: work points and equal sharing, and how they perform relative to the first best.

[A] Describe precisely the solution to the social planner's problem (in which all utilities are added up). Prove that it must involve equal effort and equal sharing of the output.

[B] Now suppose that r_i is chosen independently by agent i, under the assumption that the total output will be equally divided. write down the conditions characterizing a symmetric (interior) equilibrium. Compare these values with the first-best, and explain why they are different (and in which direction).

[C] Find out what happens to total contributions in this model when n is reduced (go ahead, take derivatives even though n is an integer). Under what conditions might it go *up*? Provide a verbal explanation.

Imagine, now, that effort decisions are taken selfishly. The following incentive scheme is in place. A fraction β of the output will be divided equally, and the remainder $1-\beta$ allocated according to "work points". That is, individual i gets a share $\frac{e_i}{E}$. *In what follows, continue to look only at symmetric Nash equilibria, where everybody puts in the same effort.*

[D] Under this sharing rule, describe what happens (relative to the first best) as β varies from 0 to 1, and provide intuition. Find a value of β such that the Nash equilibrium of the game coincides with the first best.

[E] Note that *in equilibrium*, all players share the output equally anyway regardless of the value of β. Explain intuitively why it is that we get different results for different values of β, despite this fact.

[2] Suppose that in a voluntary contributions game with two agents, output is given by the function

$$F(r_1, r_2) = [\min\{r_1, r_2\}]^\alpha$$

for some $0 < \alpha < 1$. [This is symmetric Leontief.] Suppose that the cost of supplying resources is linear: $c(r) = r$.

[A] With equal access shares $\lambda_1 = \lambda_2 = 1/2$, prove that there is a continuum of Nash equilibria.

[B] In part [A], pick the Nash equilibrium that is best for the agents. Show that it creates higher social surplus (sum of payoffs) than any other Nash equilibrium from any other division of access shares.

[3] A community of n individuals produces two goods. Individual i has resources w_i, which must be divided between a private good c_i and contributions to a public good r_i. The contributions together produce a pure public good g, according to the production function $g(r)$, where r is the *sum* of all the individual contributions r_i.

Each person has an identical utility function $u(c) + v(g)$, where these functions have all the usual properties (smoothness, strict concavity, unbounded steepness at zero).

Contributions are made selfishly: each individual takes as given the sum of all other contributions and maximizes with respect to his own.

[A] Prove that every individual who makes a positive contribution must get *the same* utility in equilibrium, regardless of wealth. Explain this result verbally.

[B] Now suppose that there are only two individuals. Take both u and v to be logarithmic, and $g(r) = \sqrt{r}$. Find a critical ratio of wealth levels such that if individual wealths are less dispersed than this ratio, then the conditions of [A] hold.

[C] Mancur Olson has argued that higher inequality may sometimes be better for the provision of public goods, because a greater portion of the marginal gains from the public good is internalized by the rich. Assuming two individuals and using parts [A] and [B], evaluate the Olson argument.

[D] Discuss whether the results in part [C] are robust with respect to dropping the *additive* specification of inputs in the production function for g.

[4] In the symmetric version of the Ray-Ueda model studied in class, prove that the first-best is fully insensitive to the form of the social welfare function as long as individual utility functions are strictly concave and the welfare function is symmetric and concave in individual payoffs.

[5] In the Ray-Ueda model, let us explore the possibility of dropping the strict concavity of the utility function in consumption. That is, suppose that $u(a_i) = ka_i$, while everything else is the same as before. Take the Bergson-Samuelson social welfare function to be any welfare function of the form $W = \sum_i w(v_i)$, where v_i is individual payoff and w is some smooth strictly concave Inada indicator. Now show that in this model, there is generally no underproduction in equilibrium.

Pay particular attention to the intuition behind this result: it is that all ex-post utilities move in the same direction (why?).

[5] Suppose that the individual utility function in the Ray-Ueda model is given by

$$u(a_i) - c(r_i) = \ln a_i - r_i \equiv v_i,$$

and the social welfare function is given by

$$W = -\frac{1}{\alpha} \sum_i \{e^{-\alpha v_i} - 1\}.$$

[A] Show that this welfare function varies from utilitarian to Rawlsian as α varies from 0 to ∞. [This is different from the CES parameterization; for a discussion, see Ray-Ueda, pp. 334–336.]

[B] Keeping the production function in the general format $F(r_1, \ldots, r_n)$ for now, explicitly work out the ex-post consumption allocations as a function of the vector $(r_1, \ldots, r_n)$, and show that

$$a_i = F(\mathbf{r}) \frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}.$$

for all i.

[C] Now work out the FOC for a symmetric Nash equilibrium of the effort game, and show that at that Nash equilibrium,

$$\frac{\partial \ln F}{\partial r_i} = \frac{1 + \alpha/n}{1 + \alpha}$$

for all i. Also record the FOC for the first-best. Compare the equilibrium to the first-best as $\alpha \to \infty$.

[D] Finally, for further tractability, take F to be the nth root of $(r_1, \ldots r_n)$, and compute equilibrium effort and first-best explicitly.