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Sketches of Answers to Problem Sets

The answers below are brief and try to give you the basic idea of how to approach these problems. You will gain a lot more from studying these answers if you spend some time independently trying to work on the problems.

Problem Set 1

(1) (a) Imagine that companies have some different costs of installing fax machines (perhaps because of different degrees of liquidity or access to credit) but that they all face a return from installing the machines that depends positively on how many *other* companies are installing. Then the graph that you draw will be upward-sloping. The more companies that are expected to install the machines, the more will actually do so. [Important: note that the intersections of this graph with the 45^0 line describes the equilibrium outcomes. Can you tell which intersections might be stable and which unstable, in the sense discussed in class?]

(b) The information given tells us that the number of companies y who actually install (as a function of the number of companies x who are expected to install) is given by the equation:

$$y = A + \frac{x^2}{1000},$$

provided that the upper bound of one million is not overstepped for either x or y (in this case, simply replace the corresponding values from the equation by 1 million).

Now let us calculate equilibria. First note that if x equals 1 million, then y as given by the equation will be way over 1 million, which simply means that *everybody* installing fax machines is always an equilibrium.

The remaining equilibria (if any) can be calculated by setting $y = x$, because this is where expected number and actual number coincide. Doing this, we get the equation

$$x = A + \frac{x^2}{1000},$$

(This is the same as looking at the intersection with the 45^0 line.)

The positive solutions are the other equilibria. Can you show that if A exceeds the value 250, there are *no* other equilibria? Use the graph, or your ability to solve quadratic equations.

(2) (a) If I am an evader, then I will be caught with probability $1/m$ where m is the total number of evaders. E.g., if $m = 3$, then there are three evaders and the chance of my getting caught is one out of three or 1/3. If I am not caught, then I pay nothing. But if I am caught, then I pay a fine of F. Thus my *expected* payout is $1/m$ times F, or simply F/m. As a potential evader, I will compare this loss with the sure payment of T (if I do not evade), and take the course of action that creates smaller losses.

(b) This situation is like a coordination game because if one person becomes an evader, she makes it *easier* for other people to evade. This is because the probability of getting caught comes down, so that the expected losses from evasion come down as well. In terms of part (a), m goes up if an additional evader enters the scene, so that F/m comes down. Thus an evader causes complementarities for other evaders.

(c) To see that "no evasion" is an equilibrium, suppose that nobody in the economy is evading. You are a potential evader. If you pay your taxes you will pay T. If you evade, then $m = 1$ (which is just another way of saying that you will be caught for sure), so that your expected loss is simply F. But $F > T$ by assumption. It follows that if nobody else is evading, you won't evade either. The same mental calculation holds for everybody, so that "no evasion" all around is an equilibrium.

What about everybody evading? Suppose that this is indeed happening, and you are considering evasion. If you do evade, then $m = N$, so that your expected losses are F/N. It follows that if $F/n < T$, you will jump on the bandwagon and evade as well. Thus "widespread evasion" is *also* an equilibrium provided that the consition $T > F/n$ holds.

(3) (a) Our formulation captures the following idea: a person's productivity is positively linked not only to his own skills, but also to that of his fellow workers. But more than that is true: note that $I_H - I_L = (1 + \theta)(H - L)$, which means that the *difference between the incomes from low and high skills* widens with more people acquiring high skills. It follows that whenever a person chooses to acquire skills, he increases the return to skill acquisition by everybody else. This is precisely the complementarity that underlies any coordination problem.

(b) Assume that $H - L < C < 2(H - L)$. First let us see if "no skill acquisition" can be an equilibrium. To this end, suppose that no one in society is acquiring skills: then $\theta = 0$. If you are thinking of becoming high-skilled, then the gain in your income is $I_H - I_L$, which is just $H - L$ (because $\theta \simeq 0$). If $H - L < C$ (which is assumed —see above), then it is not worthwhile for you to acquire skills. We have thus shown that if everybody believes that everybody else will not acquire skills, then no one will acquire skills. These beliefs thus form a self-fulfilling prophecy.

Now let us see if "universal skill acquisition" can be an equilibrium. Suppose that you believe that everybody else will acquire skills: then $\theta = 1$. Thus, if you are thinking of becoming high-skilled, then the gain in your income is $I_H - I_L$, which is $2(H - L)$ (because $\theta = 1$). If $2(H - L) > C$ (which is assumed —see above), then it is worthwhile for you to acquire skills. We have thus shown that if everybody believes that everybody else will acquire skills, then everyone will acquire skills. These beliefs also form a self-fulfilling prophecy.

Finally, there is a third equilibrium in which just the right amount of people invest in skill acquisition so that everybody is indifferent between acquiring or not acquiring skills. This is given by a fraction of skilled people θ^* such that $(1 + \theta^*)(H - L) = C$. This is an equilibrium because no one is doing anything suboptimal given his or her beliefs. But you can intuitively see why this equilibrium must be "unstable". If for some reason the fraction of skilled people exceeds θ^*, even by a tiny amount, then it becomes *strictly* preferable for everyone else to acquire skills, so that we rapidly move to the "universal skills" equilibrium.

If on the other hand, θ falls below θ^* (if only by a tiny amount), everyone will desist from acquiring skills, so that we move towards the "no skills" equilibrium.

(c) and (d) If the returns to low-skilled occupations is now given by $I_L = (1+\lambda\theta)L$, what this means is that we are changing the "sensitivity" of low-skill income to the fraction of high-skilled people. A higher λ means that low-skill income is more and more responsive to the fraction of high-skilled people. Note that the *difference* between high and low skill incomes thus becomes *less* responsive. To see this, observe that $I_H - I_L = (1+\theta)H - (1+\lambda\theta)L = [H - L] + \theta[H - \lambda L]$. Now see that if λ exceeds the value H/L, the difference between the two incomes will actually *fall* as θ goes up. In this case there cannot be any multiple equilibrium, for exactly the same reason as the traffic congestion example in the text cannot exhibit multiple equilibria.

(e) In this case, note that the cost of acquiring skills becomes infinitely high as θ becomes close to zero, while the cost declines to near zero as θ approaches one. Thus we see again that there are three equilibria. In the first, there is no skill acquisition because everyone, expecting that there is no skill acquisition, feels that the cost of acquiring high skills will be very high, and so desist from doing so. At the same time, the expectation that everyone acquires skills is also a self-fulfilling prophecy, because in this case the cost of education is very low. And there is a third equilibrium where people are indifferent between the two options. Just as in part (a), this equilibrium must be described by the condition that $I_H - I_L = \frac{1-\theta^*}{\theta^*}$ (why?).

(4) Other examples of coordination problems. To show that a situation gives rise to a coordination problem, what one needs to do is check if there are complementarities between the various agents concerned. In the first case (part (a)) the agents in question are the potential defaulting countries. The more defaulters there are, the harder it is to punish any one of them simply because it is harder for the creditor to give up trade with several countries. Thus each defaulter creates complementarities for other defaultors. [This may be one reason why we observed a sudden wave of defaults and renegotiations during the debt crisis of the 1980s, instead of sporadic isolated instances of default. See Chapter 17 of DE, Section 17.4.2 for more on the debt crisis.]

You should be able to do part (b) on your own. By this time you should also be thinking harder about the term "complementarities". Even though complementarities are sometimes associated with positive externalities for other people, this is not always the case, as part (b) shows. In its most abstract form, the term simply means: if one individual carries out an action, it tends to increase the propensity for others to carry out the same action. The action could mean buying a new computer, not paying taxes, defaulting on debt, or selling a stock in panic, as we have already learnt.

In part (c), think about what leads to a particular region turning into a full-fledged city. To some extent it is a question of location, but there are positive externalities at work here as well. If an area already has a conglomeration of businesses, it makes it easier for other businesses to set up there as well, because of access to a variety of infrastructural services. Likewise, individuals are more keen on moving to sucha place to work, because they know that the amenities of life are more likely to be available. Thus setting up life in a city creates positive externalities (up to a point at least: later there is pollution, congestion, and high cost-of-living to worry about), in the sense that it raises to return to others of setting-up in the city as well.

Thus think about concentrations of high tech companies in Slicon Valley or along Route 128 in the Boston area. It is easier for a new company to locate here because it will be easier to hire trained personnel, to have access to the latest in technological knowhow, to take advantage of the ancillary activities that have grown up around these firms. This is clearly a case of complementarities (and in this case the externalities are positive as well: they are also beneficial to society).

(5) The same idea as in the tax problem.

(6) (a) The gain from being your own self is S. If you are an L-type or an R-type, however, you will also feel a loss equal to $\frac{\alpha}{1-\alpha}$. Therefore the *net* gain from being your own type (L or R) is

$$S - \frac{\alpha}{1-\alpha}.$$

This is negative if $\alpha > \frac{S}{1+S}$. Above this threshold value, everybosy will say that they are type M.

(b) In this case, there are two possibilities. First, assume that $\alpha > \frac{S}{1+S}$, the threshold derived in part (a). Note that a fraction α (the true M-types) will always say that they are type M, because they have nothing to gain by stating any other position. But by part (a), the other types will hide their identity, which raises the value of β (the *announced* M-types) above the value of α. This process can only stop when everybody announces that they are type M.

On the other hand, if $\alpha < \frac{S}{1+S}$, there is an equilibrium in which everybody announces their true type, and so $\beta = \alpha$. You can check that nobody will want to deviate from their announcements. But at the same time, there is another "conformity" equilibrium in which everybody announces that they are type M (and in which β takes on the value of one).

(c) If there are potential conformist urges attached to each of the views L, M, and R (and not just M), then other equilibria appear. There may be conformist equilibria in which everybody announces L, or in which everybody announces R (try and provide a simple algebraic example of this).

(7) Discussed in class.

(8) Suppose that n people invest; then the return to investment is $R(n)$, an increasing function.

For no investment to be a unique equilibrium, it must be the case that if we look at the set of all potential investors, which is $(W-x)/W$, then R evaluated here must yield a value smaller than x:

$$R\left(\frac{W-x}{W}\right) < x.$$

Now suppose there is wealth equalization with average wealth $W/2$. the first condition for there to be any equilibrium with positive investment is $W/2 \geq x$.

We must also have $R(0) \leq x \leq R(1)$ for multiple equilibria. the second inequality will be strengthened further below.

Finally, we need the full-investment equilibrium here to Pareto-dominate the no-investment equilibrium *without redistribution*. This means that $R(1)-x-W/2 > W$, or $R(1)-x > W/2$.

Now satisfy yourself that these conditions are indeed potentially consistent with one another. That is, make sure you can write down at least one example in which all the conditions are simultaneously satisfied.

(9) (a) Define a correspondence by

$$\Gamma(b) \equiv \arg\max_a f(a, b).$$

[Note: this is generally a correspondence and not just a function.] To show that the game exhibits complementarities, we must show that if $b' > b$, and $a \in \Gamma(b)$ and $a' \in Gamma(b')$, then $a' \geq a$. Suppose this is false for some b and b' with $b' > b$, and for some $a \in \Gamma(b)$ and $a' \in Gamma(b')$. Then $a > a'$. Now observe that

$$f(a', b') \geq f(a, b')$$

while

$$f(a, b) \geq f(a', b).$$

Adding the two and transposing terms, we see that

$$f(a, b) - f(a', b) \geq f(a, b') - f(a', b')$$

In words, this means that f increases faster across a' to a when the second argument is bm, which is smaller than b'. This contradicts the assumption that $f_{ab} > 0$, which implies exactly the opposite.

(b) Standard.

(c) Let (a^*, b^*) be an interior Nash equilibrium. Then by the first-order condition for best responses, we know that

$$f_a(a^*, b^*) = 0 \text{ and } g_b(a^*, b^*) = 0.$$

Now suppose that we are given that f_b and g_a are nonzero, as we are. Think of a tiny change in a and b, mimc by da and db. Taking total derivatives around (a^*, b^*), we see that

$$df = f_b(a^*, b^*)db$$

where we've already used the fact that $f_a(a^*, b^*0 = 0$. Similarly,

$$dg = g_a(a^*, b^*)da$$

Now, depending on the signs of f_b and g_a, we may choose da and db to be positive or negative to assure ourselves that both payoffs go up.

(d)—(e). Easy. Try them yourself. Ask me if you cannot do them.

(10a) The solution depends on the assumption that the share of capital is common across the two countries (though it as well as relative TFPs may arbirtrarily vary). To see this, suppose that in country 1, the production function is given by

$$Y = AK^{\alpha}L^{1-\alpha},$$

while in country 2, it is given by

$$Y = BK^{\beta}L^{1-\beta}.$$

Then

$$r_1 = \alpha A k_1^{\alpha-1},$$

where k_1 is the capital per worker in country 1, and a similar formula holds for country 2, so that

$$\frac{r_1}{r_2} = \frac{\alpha}{\beta}\frac{A}{B}\frac{k_1^{\alpha-1}}{k_2^{\beta-1}}.$$

Meanwhile if per-wprker output is denoted by y_i, it is easy to see that

$$\frac{y_1}{y_2} = \frac{A}{B}\frac{k_1^{\alpha}}{k_2^{\beta}}.$$

Combining these two equations, we may conclude that

$$\frac{y_1}{y_2} = \frac{r_1}{r_2}\frac{\beta}{\alpha}\frac{k_1}{k_2}.$$

So knowing the ratios of the r's and the k's is sufficient to pin down the ratios of the y's provided that we assume that $\alpha = \beta$.

(b) Simple computation.

Problem Set 2

[1] (a) Let $\delta(i)$ denote the discount factor of person i. Conditional on some trade occuring today, i can deviate. If he does so, his return is

$$D + \frac{\delta(i)}{1-\delta(i)}(1-s)M,$$

because he is in the market from tomorrow on where the expected return at each date is just $(1-s)M$.

If he does not do so, the return is

$$T + \frac{\delta(i)}{1-\delta(i)}sT,$$

where the first term is T and not sT because all this is *conditional* on a trade occuring today. The second expression must weakly exceed the first for there not to be a deviation; i.e.,

$$\frac{\delta(i)}{1-\delta(i)}[sT-(1-s)M] \geq D-T.$$

Now we show that if $j < i$, then the same inequality must hold as well. *Two* steps are involved. One is to note that $\frac{\delta}{1-\delta}$ goes up when δ goes up; the second is to note that $[sT-(1-s)M]$ is positive (otherwise the above inequality would not hold to start with). Combining, we see that the left-hand side goes up when δ goes up.

(b) The important part of this question is that I wanted you to recognize that there must be a flat part to the mapping before it starts to climb. In fact, if you did this carefully the next question would have dropped out as a bonus.

Let s be the expected number of people in the T-sector. We want to construct the map $f(s)$ which describes the measure of people who credibly want to be in that sector. Notice first that

$$f(s) = \text{ for all } s \text{ such that } sT \leq (1-s)M.$$

[This is the flat part.] for s beyond this threshold, $f(s)$ is given by the condition that

$$\frac{\delta(f(s))}{1-\delta(f(s))}[sT-(1-s)M] = D-T.$$

It is easy to see that this uniquely determines $f(s)$ for each s. To show that f is strictly increasing in this range, raise s. Then $[sT-(1-s)M]$ must rise. To maintain equality above, $\frac{\delta(f(s))}{1-\delta(f(s))}$ must fall. Because δ is decreasing in i, this means that $f(s)$ must rise.

Verbally, the fact that s increases has two effects: it raises the expected value of trades in the traditional sector and lowers it in the market sector. So now the traditional sector is more attractive. This by itself is *not* sufficient, Now one has to argue that because of this higher attraction, some slightly more impatient people can credibly stay in the traditional sector.

(c) If the market shuts down entirely, then even the most impatient people are in the traditional sector. Notice that the no-deviation constraint now reduces to

$$\frac{\delta}{1-\delta}T \geq D-T.$$

But now we have a contradiction, because for δ small enough this constraint *cannot*be satisfied. Therefore the market cannot shut down entirely. At the same time, the tradtional sector can shut down entirely. For then the expected return to being in the traditional sector is 0, while in the market it is $M > 0$. There is no paradox here because by assumption, contract-breaking is not possible in the market. *This* is the asymmetry which allows for one corner solution but not the other.

(d) For the traditional sector to be partially active we need the existence of some $s > 0$ such that

$$\frac{\delta(s)}{1-\delta(s)}[sT-(1-s)M] = D-T.$$

Rearranging, this is equivalent to the condition that

$$\delta(s) = \frac{D - T}{D - (1-s)(T+M)}.$$

One way to guarantee this is to have the discount factor going down very slowly as i goes up, with all the drop coming near the end. For then, while s is close to 1, the right-hand side is certainly less than one. But we can keep the left-hand side above 1 by having δ hover near one until the very end of the distribution. Verbally, this says that the condition for having a partially active traditional sector is implied by having lots of patient people and only a small fraction of impatient people.

[2] In both the games under consideration, let A stand for the generic strategy that involves play of L or U, and B for the generic strategy that involves play of R or D. In both cases note that playing B is likely to be "better" under low values of the signal, so that is how we will orient the calculations.

Suppose, then, that we imagine that a player will play B if the signal is some value X or less. Let us calculate the recursion value $\psi(X)$ such that *under this assumption*, someone will play B if his signal is $\psi(X)$ or less.

These examples have the same general structure. Suppose that the signal space is located on some interval $[\ell, h]$. For signals very close to ℓ playing B is dominant. For signals very close to h, playing A is dominant. So $\psi(\ell) > \ell$ and $\psi(h) < h$. Finally, we will show that ψ is nondecreasing but has a slope strictly less than one. This yields a unique intersection x^* (which depends on the extent of the noise ϵ). By *exactly* the same arguments as in Morris-Shin, there is a unique equilibrium of the imperfect observation game: play B iff the signal falls short of x^*. Finally, we describe x^* as $\epsilon \to 0$.

(a) In the first example, suppose that your opponent plays B if his signal is X or less. Suppose you see a signal x, and play B. if the true state is θ, the chance that your opponent plays B is just the chance that your opponent's signal falls below the threshold X, given θ. This is given by the expression

$$\max\{\frac{X - (\theta - \epsilon)}{2\epsilon}, 0\},$$

and so your expected payoff (now taking expectations over θ conditional on your signal) is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} (b - \theta)\max\{\frac{X - (\theta - \epsilon)}{2\epsilon}, 0\} d\theta. \tag{1}$$

Likewise, if you play A, the chance that your opponent also plays A is

$$1 - \max\{\frac{X - (\theta - \epsilon)}{2\epsilon}, 0\},$$

and so your expected payoff conditional on x is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} (a + \theta)\left[1 - \max\{\frac{X - (\theta - \epsilon)}{2\epsilon}, 0\}\right] d\theta. \tag{2}$$

[Above, I am integrating from $x-\epsilon$ to $x+\epsilon$. I should be worrying about the lower and upper bounds on θ if I am too close to one edge of the signal space. But we can ignore this, because we know the behavior of Ψ at the edges of the signal space without having to write down the exact expressions.]

The equality of expressions (1) and (2) give you the threshold x for which you are indifferent between A and B, under the presumption that a signal below X results in a play of B for your opponent. In other words, $\psi(X)$ is the solution (in x) to the equation

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(b-\theta)\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}d\theta = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(a+\theta)\left[1-\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}\right]d\theta. \quad (3)$$

By inspecting (3) it should be obvious that $\Psi(X)$ is nondecreasing in X. What is a little less obvious is the assertion that for all $X' > X$,

$$\psi(X')-\psi(X) < X'-X. \quad (4)$$

To prove (4), let X increase to $X+\Delta$. We want to show that the required solution to (3) in x increases by strictly less than Δ. Suppose this is false, then it must be that after raising X to $X+\Delta$, a rise from the previous solution x to $x+\Delta$ still does not (weakly) bring the LHS and RHS of (3) into new equality; i.e., we have

$$\frac{1}{2\epsilon}\int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon}(b-\theta)\max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon},0\}d\theta \geq \frac{1}{2\epsilon}\int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon}(a+\theta)\left[1-\max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon},0\}\right]d\theta.$$

Now make the change of variables $\theta' \equiv \theta-\Delta$. Then, after all the substitutions, we may conclude that

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(b-\theta'-\Delta)\max\{\frac{X-(\theta'-\epsilon)}{2\epsilon},0\}d\theta' \geq \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(a+\theta'+\Delta)\left[1-\max\{\frac{X-(\theta'-\epsilon)}{2\epsilon},0\}\right]d\theta',$$

but this contradicts (3), the original relationship between X and x. So the claim in (4) is established. Now we have a unique equilibrium using exactly the same arguments as Morris and Shin.

Call this unique threshold x^*. Then, using this fixed point in (3) and noting that the "maxes" in that equation may now be dropped (why?), we have

$$\int_{x^*-\epsilon}^{x^*+\epsilon}\frac{(b-\theta)[x^*-(\theta-\epsilon)]}{2\epsilon}d\theta = \int_{x^*-\epsilon}^{x^*+\epsilon}(a+\theta)\left[1-\frac{x^*-(\theta-\epsilon)}{2\epsilon}\right]d\theta.$$

Now pass to the limit as $\epsilon \to 0$ (use L'Hospital's Rule). It is easy to see that at the limit,

$$x^* = \theta^* = \frac{b-a}{2}.$$

[b] In the second example, make the same provisional assumption: your opponent plays B if his signal is X or less. Suppose you see a signal x, and play B. if the true state is θ, the

chance that your opponent plays B is just the chance that your opponent's signal falls below the threshold X, given θ. This is given by the expression

$$\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\},$$

just as in (a), and so your expected payoff (conditional on your signal) is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}4\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}d\theta.$$

[Again, I am integrating from $x-\epsilon$ to $x+\epsilon$ because we can neglect the edges of the state space (see discussion in part (a) above).]

On the other hand, if you play A, you're guaranteed θ (whatever it may turn out to be), so your expected payoff is just x, of course.

The equality of these two expressions give you the indifference threshold x. That is, $\psi(X)$ solves the equation (in x):

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}4\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}d\theta=x. \tag{5}$$

Again, you can show that $\Psi(X)$ is nondecreasing in X and has slope less than one; i.e., that (4) holds for the ψ-function here as well. [Use the same sort of argument we did above; things here are even simpler.]

Call this unique threshold x^*. Then, using this fixed point in (5) and once again noting that the "maxes" may be dropped (why?), we have

$$\frac{1}{\epsilon}\int_{x^*-\epsilon}^{x^*+\epsilon}\frac{x^*-\theta+\epsilon}{\epsilon}d\theta=x^*.$$

Now pass to the limit as $\epsilon\to 0$. It is easy to see that

$$x^*=\theta^*=2.$$

Observe the contrast between parts (a) and (b). In (a), equilibrium selection generally tracks the Pareto-dominant equilibrium. When $a=b$, the switch point is 0 (how *could* it be anything else, by symmetry and uniqueness?), and now if a and b depart from each other, the switch point moves in the "correct" direction. For example, when, if $b>a$, B will be played more often, because the switch point is now positive.

In part (b), the switch point is $\theta=2$ (which is about its midpoint value, given the support of θ). At this point, $(4,4)$ is still much better than $(\theta,\theta)=(2,2)$. Why does $(4,4)$ have so little attractive power? It is because the play of A has "insurance" properties: if your oppoent does not play A, you still get something (in this example, you get full insurance in fact). But you get no insurance if you play B and your opponent does not. Thus the selection device not only looks at payoffs "at the equilibrium", it looks at payoffs "off the equilibrium" as well to make the selection.

[3] To describe this, calculate the threshold x^* as $\epsilon \to 0$. The easiest way to do this is the "sandwich" inequality:

$$[e - f(h(x^*, \epsilon))][1 - a(h(x^*, \epsilon)] \leq \frac{1}{2\epsilon} \int_{x^*-\epsilon}^{h(x^*,\epsilon)} [e - f(\theta)] d\theta \leq [e - f(h(x^*, \epsilon))][1 - a(h(x^*, \epsilon)],$$

which is obtained by noting that $f(x^* - \epsilon) \leq f(\theta) \leq f(h(x^*, \epsilon))$ for all $\theta \in [x^* - \epsilon, h(x^*, \epsilon)]$. Both sides of the sandwich go to the same limit, because x^* and $h(x^*, \epsilon)$ — as well as the realization of the state — all go to a common limit, call it θ^*. This limit solves the condition

$$[e - f(\theta^*)][1 - a(\theta^*)] = t.$$

[4] In sector A, the return is 0. In sector B, the return is $f(k, z)$, where k is aggregate capital and z is a state variable. Assume that f is continuous and increasing in both k and z, and that $f(1, 0) < 0$ while $f(0, 1) > 0$.

As in Morris-Shin, a noisy signal x of z is observed. This is distributed conditional on z uniformly on the interval $[z - \epsilon, z + \epsilon]$ with density $1/2\epsilon$. The realization of the signal is independent (conditional on z of course) across all agents. Now suppose that everyone uses the strategy: "put my one unit of capital into sector B if I get the signal $x \geq \bar{x}$, otherwise in sector A." Now we will calculate a "best reply" to this strategy.

Suppose I get a signal x. For each "truth" z consistent with that signal, how many people will move to B? Well, if $z - \epsilon \geq \bar{x}$, everyone will move. If $z - \epsilon \leq \bar{x}$, no one will. In the remaining (and central) case a measure of individuals given by

$$\frac{1}{2\epsilon}[z + \epsilon - \bar{x}] \tag{6}$$

will obtain a signal above $\bar{x}$, and so will move. Let $\psi(z, \bar{x})$ be a function that takes on these three values in each of these three cases. Then the value of f — conditional on the imagined truth z — will be

$$f(\psi(z, \bar{x}), z).$$

So my *expected* value of moving is

$$\int_{x-\epsilon}^{x+\epsilon} f(\psi(z, \bar{x}), z) \frac{1}{2\epsilon} dz.$$

So my "best-reply" threshold signal — call it $\phi(\bar{x})$ — is given by the solution to

$$\int_{\phi(\bar{x})-\epsilon}^{\phi(\bar{x})+\epsilon} f(\psi(z, \bar{x}), z) \frac{1}{2\epsilon} dz = 0. \tag{7}$$

When $\bar{x}$ is tiny — say $\bar{x} = 0$ — then everyone will always move, but nevertheless, because of the assumption that $f(1, 0) < 0$, I will still require a positive threshold to move. This means that $\phi(0) > 0$. By a similar argument, we may use the assumption that $f(0, 1) > 0$ to conclude that $\phi(1) < 1$. Finally, we must show that ϕ is nondecreasing but nevertheless has a slope less than one.

To see this, note that $\psi(z, \bar{x})$ is nonincreasing in $\bar{x}$ (indeed, strictly decreasing whenever $z - \epsilon < \bar{x} < z + \epsilon$ — inspect (6)). This means that the LHS of the equation (7) must decline in $\bar{x}$. To restore balance, $\phi(\bar{x})$ must rise. But it can't rise by the amount that $\bar{x}$ does (or by more than that), for if it did, it is easy to see that the LHS of (7) would rise as a consequence. [Note: this is precisely where the uniform noise assumption is being used.]

Therefore we have a unique fixed point x^* such that $\phi(x^*) = x^*$. This shows uniqueness in exactly the same way as we did in class.

[5] Omitted.

[6] (a) If $s > 1$, then being in sector B *guarantees* you a payoff of at least

$$\alpha s + (1 - \alpha)s = s > 1,$$

which is strictly higher than being in Sector A. So being in B is dominant. Similar trivial argument applies for the case in which $s < -1$.

(b) Suppose that an individual today believes that *tomorrow*, people will go to Sector B as long as $s > S$, where S is some threshold. Then *today*, an individual will choose sector B as long as

$$\alpha s + (1 - \alpha)[E(s'|s) + 2P(s' > S|s)] > 1.$$

To understand this, first note that we are only writing down a sufficient condition (not a necessary one). This is why I am pessimistic in the current period and assume that no one else is going to B. This explains the αs for the first period. In the second period, I certainly get the conditional expectation of s' given s, *plus* if the state exceeds S tomorrow, I will have a population of measure 2 coming in.

Now noting that $E(s'|s) = s$ (there is no drift), we may rewrite the above as

$$s + 2(1 - \alpha)P(s' > S|s) > 1.$$

This defines a mapping $g(S)$ from tomorrow's anticipated threshold to a threshold today, given by

$$g(S) + 2(1 - \alpha)P(s' > S|g(S)) = 1.$$

The only consistent expectation is one which replicates itself: that is, $g(S) = S$. Calling this s^*, we see that

$$s^* + 2(1 - \alpha)\frac{1}{2} = 1,$$

or that $s^* = \alpha$. The other case is proved similarly.

[7] This is one of those "exploratory" questions in which I would like to see you construct a model to the best of your ability. But the basic idea is very simple. Suppose that you are thinking about supporting the potential equilibrium in which everybody is in sector A to start with, but everybody moves to sector B when they have the chance to do so, and stay there. This is an equilibrium which breaks free of history. Adserà-Ray provce that if there are lags in the return adjustment in sector B, no such equilibrium can come about provided that individuals can move whenever they like.

The idea is that people will postpone their movement as long as Sector B is not currently profitable. However, if individuals on average do not have that many chances to move, then they might move now in the fear that they will not get another chance to reap the benefits until far into the future. This suggests that the above equilibrium can be sustained if people have fewer chances to move.

However, here is an intriguing counterargument. If people have few chances to move, the rate of return in Sector B is going to climb very slowly anyway (because of the slowness of aggregate movement out of Sector A). So there are two forces at work: at the individual level, I may want to move because my next shot at moving is far away (in expected value), but at the same the aggregate slowness of movement means that Sector B's returns are climbing very slowly. The latter means I might as well wait. Is it possible to construct an example in which the second effect dominates the first.

Problem Set 3

[1] Discussed in class.

[2] Suppose the required *minimum* collateral size is W_m. We are going to find a formula for W_m. If you repay the loan, you pay up 20,000 plus interest of 10%, which is a total of 22,000. If you do not repay, you lose the collateral (plus interest), you are fined 5000, and you lose 20% of business profits. If you put up W as collateral, the first of these is $W(1+r) = (1.1)W$. And business profits are $30,000 - 10w$, where w is the wage rate. So if you do not repay, your total losses are $(1.1)W + 5,000 + (0.2)(30,000 - w)$. You *will* repay, therefore, if this last expression is at least as great as 22,000 . This means that the *minimum* necessary collateral W_m is given by the formula

$$W_m = \frac{17,000 - (0.2)(30,000 - w)}{1.1}$$

Now you can calculate what happens to W_m for different values of the wage rate w. The minimum required collateral *increases* with the wage rate, because business profits are lower and therefore less valuable to the creditor in the event of nonrepayment. Thus a larger collateral is asked for to start with.

(ii) Just use the above formula to calculate W_m if the wage rate is at 500. Now let us go on to the second part. Let N be the total population, and let p be the fraction of people who cannot put up this kind of collateral. Then $p \times N$ people get into the labor market. The remainder become entrepreneurs: each of them demand 10 workers, so the total demand is $10 \times (1 - p) \times N$. If the total supply exceeds the total demand, then some people will be unable to get employment, whether as laborers or as entrepreneurs (in terms of our model in class, they would have to go into subsistence). The requirement for this is the condition

$$pN \geq 10(1 - p)N,$$

which means that the value of p should exceed 10/11. Translate this fraction into percentage terms to find the critical value x.

(iii) and (iv): Similar to what we do below in [3].

[3] (i) There is an initial distribution of wealth which we shall denote by $G(W)$. If for any person, $W \geq I$, then he can contemplate getting into business, in which case his labor demand is

$$L(w) \equiv \arg\max_L f(L) - wL,$$

while his profit is

$$\pi(w) \equiv \max_L \{f(L) - wL\} - I.$$

A person with sufficient upfront wealth will decide to become an entrepreneur if $\pi(w) > w$, a worker if $\pi(w) < w$, and will be indifferent if $\pi(w) = w$. Persons with insufficient upfront wealth must be workers.

Using this information we are ready to describe the market-clearing wage at any date (you can also use diagrams to supplement what follows). Define $\bar{w}$ by $\pi(\bar{w}) \equiv \bar{w}$. Clearly, the market-clearing wage can never strictly exceed $\bar{w}$, otherwise the demand for workers must drop to 0.

To describe the equilibrium, then, consider two scenarios:

Case 1. $w < \bar{w}$. In this case, the demand for workers is given by

$$[1 - G(I)]L(w),$$

while the supply is just $G(I)$, so the equilibrium wage rate is given by

$$[1 - G(I)]L(w) = G(I). \tag{8}$$

This construction is perfectly valid as long as the resulting wage rate stays short of $\bar{w}$. If (8) does not permit such a solution, then set $w = \bar{w}$ and move to Case 2.

Case 2. $w = \bar{w}$. Now potential entrepreneurs are indifferent between being workers and entrepreneurs, so we look for any fraction of entrepreneurs λ that satisfies the condition

$$\lambda \leq [1 - G(I)] \tag{9}$$

and solves

$$\lambda L(\bar{w}) = 1 - \lambda. \tag{10}$$

It is easy to see that *either* there is a solution to (8), *or* there is a solution to the twin conditions (9) and (10). In brief, there is a unique market clearing wage in $[0, \bar{w}]$ for every wealth distribution G.

Given the wealth distribution G_t at time t, we may then write $w(G_t)$ to be the market-clearing wage. Therefore the total resources to an entrepreneur with starting wealth W_t are

$$W_t + \pi(w(G_t)),$$

with the result that

$$W_{t+1} = s(1 + r)\{W_t + \pi(w(G_t))\}, \tag{11}$$

where s is the rate of savings and r is the rate of interest. Similarly, a worker at date t follows the difference equation

$$W_{t+1} = s(1+r)\{W_t + w(G_t)\}. \tag{12}$$

You can easily use these equations to generate multiple steady states, some with perfect equality and others without. If you also put in uncertainty into the wealth accumulation process you can easily get steady states such that the supports overlap. The reason there is no contradiction here (unlike in question [1]) is that the two steady states follow different laws of motion at the individual level, because the equilibrium wage rate is different across the steady states.

[4a] In a steady state, a standard single-crossing argument shows that no dynasty will switch skills, assuming that there are no financial bequests. (Why? Make sure you understand this.)

Therefore, if λ is a steady state, we may define the lifetime utility of a skilled and unskilled dynasty to be

$$\frac{u\left(w^s(\lambda) - x\right)}{1-\delta} \text{ and } \frac{u\left(w^u(\lambda)\right)}{1-\delta}$$

respectively. Because no skilled individual wishes to switch skills, it follows that

$$u\left(w^s(\lambda) - x\right) + \frac{\delta u\left(w^s(\lambda) - x\right)}{1-\delta} \geq u\left(w^s(\lambda)\right) + \frac{\delta u\left(w^u(\lambda)\right)}{1-\delta},$$

while

$$u\left(w^u(\lambda)\right) + \frac{\delta u\left(w^u(\lambda)\right)}{1-\delta} \geq u\left(w^u(\lambda) - x\right) + \frac{\delta u\left(w^s(\lambda) - x\right)}{1-\delta},$$

Combining these last two inequalities, we get the desired result. It is very easy to see that there is a continuum of steady states, following exactly the arguments used in class.

[b] To see that all steady states involve persistent inequality *in payoffs*, not just wages, simply note that the left hand term in the desired set of inequalities is strictly positive, and therefore so too must be the middle term.

[c] If a measure λ acquire skills, it is easy to see that net output (which is also per-capita net output) is given by

$$f(\lambda, 1-\lambda) - x\lambda.$$

Differentiate once with respect to λ to get the expression

$$f_1(\lambda, 1-\lambda) - f_2(\lambda, 1-\lambda) - x$$

and then twice to get the expression

$$f_{11} - f_{22} - 2f_{12}$$

which is negative by concavity of f. It follows that net output per capita is inverted-U shaped in λ, rising to a maximum at the value $\hat{\lambda}$ given by

$$f_1(\hat{\lambda}, 1-\hat{\lambda}) - f_2(\hat{\lambda}, 1-\hat{\lambda}) = w^s(\hat{\lambda}) - w^u(\hat{\lambda}) = x.$$

But we just proved in part (b) that in every steady state value of λ, $w^s(\lambda) - x > w^u(\lambda)$, so this means that every such value lies to the *left* of $\hat{\lambda}$. We have therefore proved that net output per-capita is rising over the set of steady states.

[5] (i) A steady state is Pareto-efficient if there is no way to have a feasible allocation *starting from the same initial allocation as the steady state*, which makes all generations just as better off as they were before, and some strictly better off.

The italicized phrase is important. It is true that net consumption is rising in λ over the set of steady states. That appears to suggest that each higher λ Pareto-dominates each lower λ. How do we square this with part (ii)? Answer: easy; appearances are deceptive. It is true that net consumption rises, but we are also changing initial conditions when making these comparisons! Pareto checks do not permit us to do this.

(ii) Suppose that a steady state has the property that $w(2) - w(1) > x/\delta$ (here 2 is obviously the skilled occupation).

We create a path that Pareto-dominates the steady state. Suppose that at date 0, a fraction λ is skilled (this is the fraction that will persist, of course, in steady state). What we do is create a path in which at date 1 a slightly larger measure, $\lambda + \epsilon$ is skilled. To create these extra skills we sacrifice some consumption at date 0 — $x\epsilon$ — which is divided equally among all households, so the consumption loss of each household is also $x\epsilon$. In period 2 we go back to the old proportion λ and stay there forever. This means that there is some extra consumption in period 1 (because there are more skilled people) — give this consumption equally to all. How much is the extra? It is

$$\begin{aligned} f(1-\lambda-\epsilon, \lambda+\epsilon) - f(1-\lambda, \lambda) &\simeq [f_2(1-\lambda, \lambda) - f_1(1-\lambda, \lambda)] \\ &= [w(2) - w(1)]\epsilon. \end{aligned}$$

[This is an approximation but can easily be made precise at the cost of obscuring the intuition, so we will keep it as it is.]

So to summarize: relative to the original steady state, this path displays a consumption shortfall of $x\epsilon$ in period 0, a consumption excess of (approximately) $[w(2) - w(1)]\epsilon$ in period 1, and no difference thereafter. Notice that agents after period 1 are unaffected, while all agents at period 1 are strictly better off. It therefore only remains to check agents at period 0. The utility loss for any such agent i at date 0 is

$$u(c(i)) - u(c(i) - x\epsilon) \simeq u'(c(i))x\epsilon,$$

while the utility gain is

$$\delta[u(c(i) + [w(2) - w(1)]\epsilon) - u(c(i))] \simeq \delta u'(c(i))[w(2) - w(1)]\epsilon.$$

Now use the condition $w(2) - w(1) > x/\delta$ to show that the gain outweighs the loss, and thereby complete the proof.

Proving that the opposite of the condition in (i) does imply Pareto-efficiency is not trivial (try it if you like). For a general treatment see Mookherjee and Ray (2002). But let us assume that this is the dividing line between Pareto-efficiency and inefficiency. The question asks us to show (under this assumption) that there is a continuum of both types of states in the two-occupation model.

Remembering that λ stands for the proportion of skilled labor, define λ^* by the condition $\bar{w}(\lambda) - \underline{w}(\lambda) = x/\delta$. By our discussion above, a steady state proportion λ is Pareto-efficient if and only if $\lambda \geq \lambda^*$. So it only remains to show that λ^* belongs to the interior of the set of steady states. This is done by verifying that the "double-inequality" condition is satisfied with *strict* inequality when $\lambda = \lambda^*$.

Here's the verification: exploit the strict concavity of u to see that $u\left(\bar{w}(\lambda^*)\right) - u\left(\bar{w}(\lambda^*) - x\right)$ $< u'\left(\bar{w}(\lambda^*) - x\right) x = u'\left(\bar{w}(\lambda^*) - x\right) \frac{\delta}{1-\delta}[\bar{w}(\lambda^*) - x - \underline{w}(\lambda^*)] < \frac{\delta}{1-\delta}[u\left(\bar{w}(\lambda^*) - x\right) - u\left(\underline{w}(\lambda^*)\right)]$ $< u'\left(\underline{w}(\lambda^*)\right) \frac{\delta}{1-\delta}[\bar{w}(\lambda^*) - x - \underline{w}(\lambda^*)] = u'\left(\underline{w}(\lambda^*)\right) x < u\left(\underline{w}(\lambda)\right) - u\left(\underline{w}(\lambda) - x\right)$.

[6] This is a different approach to the same thing we did in class. We make more assumptions here, such as the smoothness of the x and w functions, but the analysis is still of independent interest.

(i) At a steady state, an individual starting at h must choose h. By unimprovability, this means that the following expression has to be maximized:

$$u\left(w(h) - x(h')\right) + \frac{\delta}{1-\delta} u\left(w(h') - x(h')\right)$$

and the solution has to be $h' = h$.

(ii) For necessity, simply write down the first-order conditions. Sufficiency involves the following interesting theorem: if you have a differentiable real-valued function f defined on an interval of the real line with the property $f"(z) < 0$ whenever $f'(z) = 0$, then $f'(z) = 0$ is *sufficient* for checking a global maximum. Now go ahead and check that this condition precisely holds in the exercise above.

(iii): done in class. (iv): easy. (v)–(vii): see my notes.

[7] (i) Say the production function is $f(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. Read off wages as marginal products: define

$$w_i(\boldsymbol{\lambda}) \equiv f_i(\boldsymbol{\lambda})$$

for each i. Now if we define $x_1 = 0$, $x_2 = x$, and $x_3 = z$, we can write the steady state condition as follows: $\boldsymbol{\lambda}$ is a steady state if and only if for all occupations i and j,

$$u(w_i(bla) - x_j) - u(w_i(bla) - x_i) \leq \frac{\delta}{1-\delta}\left[u(w_j(bla) - x_j) - u(w_i(bla) - x_i)\right]. \tag{13}$$

(ii) Pick any w_1, and define $w_2 = w_1 + \frac{x}{\delta}$ and $w_3 = w_1 + \frac{z}{\delta}$. [The reason for doing this will become clear in a minute.] Now, we move w_1 up and down, notice that w_2 and w_3 move up and down as well. Observe that there is one and only one value such that the above relationships hold *and* the wage vector (w_1, w_2, w_3) supports profit-maximization for f at some $\boldsymbol{\lambda}$. For as $w_1 \to 0$, positive profits are certainly possible, and as $w_1 \to \infty$, so do the other wages so all positive production gives strictly negative profits. So there must be something in between — call it (w_1^*, w_2^*, w_3^*) — which gives exactly zero profits so that a profit-max is well-defined. Call the accompanying $\boldsymbol{\lambda}$ that goes with it, $\boldsymbol{\lambda}^*$.

Notice that this wage function has the property that the claimed inequality in this part actually *does not hold*. However, by exactly the same logic used in part (ii) of the previous question, this wage function and the accompanying $\boldsymbol{\lambda}^*$ constitute a *strict* steady state — all the inequalities in (13) hold strictly. This means that you can jiggle $\boldsymbol{\lambda}^*$ in any "direction" you want, move the accompanying wages as defined by $w_i(\boldsymbol{\lambda})$, and for small movements (13) must continue to hold. In particular, we can perturb $\boldsymbol{\lambda}^*$ to some $\boldsymbol{\lambda}$ so that it satisfies all the steady state conditions *and* the desired inequality

$$\frac{w_2 - w_1}{x} \neq \frac{w_3 - w_2}{z - x} \tag{14}$$

does hold.

(iii) We can do the same exercise as in the previous question, but this time it's easier. Take a steady state in which (14) does hold. Let us suppose, for concreteness, that

$$\frac{w_2 - w_1}{x} < \frac{w_3 - w_2}{z - x}. \tag{15}$$

Create a Pareto-improvement as follows. At date 0, withdraw a budget of ϵ from training type 2 workers and put it into the training of type 3 workers. There is no change in date 0 consumption. At date 1, divide any changed output equally. *Do not disturb the program from date 2 onwards.* To show this is an improvement, all we need to do is show that output in period 1 has increased. This is done using the same approximation techniques discussed earlier.

In date 1, increase λ_1 to $\lambda_1 + \epsilon$, and λ_3 to $\lambda_3 + \eta$. Reduce λ_2 to $\lambda_2 - \epsilon - \eta$. Do this as follows. the extra money released by the ϵ-transfer is ϵx; equalize this to the extra money used up by the η-transfer, which is $\eta(z - x)$. That is, make sure that

$$\epsilon x = \eta(z - x). \tag{16}$$

Now compute the output difference when ϵ and η are small. This is

$$\begin{aligned} f(\lambda_1 + \epsilon, \lambda_2 - \eta - \epsilon, \lambda_3 + \eta) - f(\lambda_1, \lambda_2, \lambda_3) &\simeq f_1(\boldsymbol{\lambda})\epsilon - f_2(\boldsymbol{\lambda})[\eta + \epsilon] + f_3(\boldsymbol{\lambda})\eta \\ &= w_1\epsilon - w_2[\eta + \epsilon] + w_3\eta \\ &= (w_3 - w_2)\eta - (w_2 - w_1)\epsilon. \end{aligned}$$

Using (15) and (16), it is easy to see that this expression is positive, and we are done.

[8] Here is a quick answer to this question. Suppose, for example, that we haven't checked whether an incumbent of occupation i wants to slide up, say, to occupation j, where j is more than one notch away. Say j is more expensive. Now look at a holder of occupation $j - 1$, which is also strictly more expensive than i. *We have checked the condition for this holder vis-a-vis* j, and he does not want to migrate to j by assumption. So for monotonicity not to be violated, $j - 1$ must want to migrate at least two notches further up. Now we are

back to the same scenario, but with $j-1>i$ as our starting point instead of i. This cannot go on forever, because there are only a finite number of occupations.

[9] (a) There is a typo in this part. I meant to say: show that you cannot have two distinct consumption levels c_i and c_j such that $\lambda(c_i)$ and $\lambda(c_j)$ are both strictly positive.

ANSWER. Suppose not, so that two different c_i and c_j are indeed assigned with $\lambda(c_i)$ and $\lambda(c_j)$ both positive. It must be that $\lambda(c_i)$ and $\lambda(c_j)$ are both strictly We have assumed that $\lambda(c)$ is locally strictly concave whenever $\lambda(c)>0$. Now we have a contradiction. Transfer from the smaller value of c to the bigger value. Output will go up and so will aggregate utility.

(b) So we can reformulate as follows. There will be m "productive" members are $n-m$ "unproductive" members, with consumptions a and b. The idea is that $\lambda(a)>0$ but $\lambda(b)=0$. [But the latter are still given something because they enter into social welfare.] So the idea is to choose (m,a,b) to maximize

$$mu(a)+(n-m)u(b)$$

subject to

$$f(m\lambda(a))\geq ma+(n-m)b.$$

Suppose the production function f is such that zero consumption is the *only* equal-division solution; that is, for all $c\geq n\hat{c}$,

$$f(n\lambda(c))<nc$$

Then there cannot be a positive-consumption equal-division solution. Yet, at the same time, it is possible that

$$f(\lambda(\hat{c}))>\hat{c}.$$

With large n and strictly concave f, this situation is easy to construct. Then an equal-division solution cannot be optimal. Moreover, this also answers part (c).

Problem Set 4

[1] and [2] Trivial as long as you've studied the basic concepts. For instance, in the very first question, the net return to the government is \$0.5b because \$20b has to be paid as debt service. This is a return of 5% which is below the threshold, so the investment will not be undertaken. This forms the basis for an argument to forgive some of the debt, which will result in a Pareto-improvement. Similar arguments apply to all the parts in these two questions.

[3] Suppose that p is strictly increasing and that there is a unique first-best choice of effort e^*. Then it must be the case that

$$p(e^*)Q-e^*>p(\hat{e})Q-\hat{e},$$

where $\hat{e}$ is the equilibrium solution. At the same time,

$$p(\hat{e})[Q-R]-[1-p(\hat{e})]w-\hat{e} \geq p(e^*)[Q-R]-[1-p(e^*)]w-e^*.$$

Adding these two inequalities and canceling all common terms, we see that

$$(R-w)[p(e^*)-p(\hat{e})] > 0.$$

Because $R > w$ and p is increasing, it follows that $e^* > \hat{e}$.

[4] (a) Obviously, $p(\theta)\theta$ is the expected return of the project θ.

(b) If a borrower puts down collateral $C < B$, and faces a rate of interest r, then his expected return from project θ is given by

$$p(\theta)[\theta - B(1+r)] - [1-p(\theta)]C = p(\theta)[\theta + \{C - B(1+r)\}] - C.$$

Therefore the borrower chooses θ to maximize

$$p(\theta)[\theta + K],$$

where $K \equiv C - B(1+r)$. Because $p(\theta)[\theta + K]$ is single-peaked by assumption, there is a well-defined unique solution to θ for every K. We show that θ is declining in K. This is our standard revealed preference argument — I leave it to you to fill in the details. Once we have this, unpack K to deduce that θ is strictly increasing in r and strictly decreasing in C.

(c) Notice that when $K = 0$ the choice of θ maximizes expected returns. It follows that for every $C < B$ the project choice is too risky relative to expected return maximization, and just as in the Stiglitz-Weiss model, the risk worsens and the expected payoff declines further as the rate of interest is increased.

Look at the optimal choice of an individual who has zero collateral, when he faces the interest rate $\bar{r}$; i.e., consider the value $\theta(0, \bar{r})$. If

$$p(\theta(0,\bar{r}))\theta(0,\bar{r}) < B(1+\bar{r}),$$

it follows that this person cannot be profitable to a bank for *any* rate of interest, because as we've just seen,

$$p(\theta)\theta < p(\theta(0,\bar{r}))\theta(0,\bar{r})$$

for values of r that exceed $\bar{r}$.

(d) Consider a borrower with collateral C, and suppose he is made a loan in equilibrium. Then we know that at the equilibrium rate of interest r,

$$p(\theta(C,r))\theta(C,r) = B(1+\bar{r}).$$

Now increase C. By part (b) and the first observations in part (c), the LHS of the above equation must go up. Because the loan market is competitive, the LHS must be made to decline to its former value by a change of r. Using part (b) again, this is only possible if r declines.

[5] If the borrower borrows B, needs to repay R ($< w_1$), and puts up an amount C as collateral, his net two-period utility is

$$W \equiv u(w_0 + B + A) + \delta[pu(w_1 - R + A) + (1 - p)u(A - C)],$$

while the lender's utility is

$$\Pi \equiv -B + \delta[pR + (1 - p)\beta C].$$

(a) The favorable effect of increasing C should be obvious: the lender is protected against a default to a greater extent. But there is a negative effect as well. Look at the state in which the borrower receives no income and consequently defaults. In that state a higher value of C will create even lower consumption ($A - C$), which leads ex ante to a greater variability of consumption. Since the borrower is risk-averse, this leads to a potential loss of social surplus because it is increasing the uninsurable risk in the system. [Well, not exactly uninsurable. We *could* have assumed that the lender allows the borrower some extra funding in this state. But this is mathematicaly identical in this model to a reduction of collateral.]

If you think this second effect is weird, imagine taking a loan contract in which there is some chance that in some state you will lose *everything* ($A - C \simeq 0$). For you to participate willingly in such a contract you will have to be compensated for this risk in all the other states. Indeed, the compensation may be so high that the lender may not be willing to lend at those terms. This effect is especially pronounced when $\beta < 1$. This diminishes the favorable effect of increased collateral while keeping the unfavorable effect as powerful as before.

To see this more formally, work out the "competitive solution" to the problem above: maximize W subject to $\Pi = 0$, for some *given* C. Set up the Lagrangean

$$\mathcal{L} \equiv u(w_0 + B + A) + \delta[pu(w_1 - R + A) + (1 - p)u(A - C)] + \lambda\left(\delta[pR + (1 - p)\beta C] - B\right),$$

and differentiate with respect to B to get

$$u'(w_0 + B + A) - \lambda = 0, \tag{17}$$

and then with respect to R to get

$$\delta pu'(w_1 - R + A) + \lambda\delta p = 0. \tag{18}$$

Now combine (17) and (18) to see that

$$w_0 + B + A = w_1 - R + A; \tag{19}$$

that is, we have complete consumption smoothing over date 0 and the "success state" in date 1, but the "failure" low-consumption state is delinked (basically by assumption, since we assume that the collateral C is a parameter which is unequivocally seized at this state).

The zero-profit condition tells us that $B = pR + (1 - p)\beta C$. Using this in (19), we can solve out for R as

$$R = \frac{w_1 - w_0}{1 + p} - \frac{1 - p}{1 + p}\beta C,$$

and therefore for the (common) consumption at date 0 and at the success phase — as

$$w_0 + B + A = w_1 - R + A = A + \frac{1}{1+p}\left[pw_1 + w_0 + (1-p)\beta C\right] \equiv \sigma.$$

This means that expected utility is given by

$$W \equiv (1+\delta p)u\left(A + \frac{pw_1 + w_0 + (1-p)\beta C}{1+p}\right) + \delta pu(A - C).$$

Now we can take derivatives of borrower utility with respect to C. We see that

$$\frac{dW}{dC} = \frac{(1+\delta p)(1-p)\beta}{1+p}u'(\sigma) - \delta pu'(A-C),$$

where σ is the common consumption calculated above. Notice that the smaller is the value of β or the ratio $u'(\sigma)/u'(A-C)$, the greater is the likelihood that the calculated derivative is negative, in line with our informal reasoning.

(b) Now introduce the debt overhang as we did in class. You should be able to do the exercise in a parallel way. The third effect is, of course, the moral hazard effect.

[6] Parts (a) and (b) are directly out of class and there is nothing to add. To do part (c) here is the basic idea which you can easily formalize. First, recall how we calculated a second-best package by fixing the lender's return at z and then calculating the maximum borrower's payoff. Here there were two possibilities: the loan is either first-best or incentive-constrained. Consider any z for which the latter situation applies. Then if we denote by $S(z)$ the total surplus generated at that z (the sum of the two discount-normalized payoffs), we know that

$S(z)$ is strictly decreasing in z.

[This should be apparent from class discussion, but if it isn't, make sure you understand it.]

Now we're going to show how to Pareto-improve this stationary package by using a nonstationary sequence while still maintaining all the enforcement constraints. Begin by writing down the enforcement constraint for any *sequence* of packages $\{L_t, R_t\}$:

$$(1-\delta)F(L_t) + \delta v \leq (1-\delta)\sum_{s=t}^{\infty}\delta^{s-t}[F(L_s) - R_s]$$

for all t, or equivalently,

$$(1-\delta)R_t + \delta v \leq (1-\delta)\sum_{s=t+1}^{\infty}\delta^{s-t}[F(L_s) - R_s] \tag{20}$$

for all t. Let's evaluate this constraint in a couple of different situations. First, study it for the second-best stationary package (L, R) that yields the lender z. Let's call the return to the borrower $B(z)$. [Notice that $S(z) = B(z) + z$.] Then (20) reduces to

$$(1-\delta)R + \delta v \leq \delta B(z). \tag{21}$$

Now consider the *nonstationary* sequence in which for some small $\epsilon > 0$, the borrower receives the package $(L, R + \epsilon)$ at date 0, and this is followed forever after by the stationary package that yields the lender $z' \equiv z - (1-\delta)\epsilon/\delta$. By construction, the lender is absolutely indifferent between the original stationary package and this new "two-pronged" substitute.

What about the borrower? Well, z is down to z' so the surplus $S(z') > S(z)$. Because $B(z) + z = S(z)$, this means that $B(z')$ *is strictly greater than* $(1 - \delta)\epsilon/\delta$. It follows from (21) that

$$(1 - \delta)(R + \epsilon) + \delta v \leq \delta B(z'),$$

so that this two-pronged sequence satisfies all the constraints. To complete the proof, notice that the borrower is strictly better off, because

$$\begin{aligned}(1 - \delta)[F(L) - (R + \epsilon)] + \delta B(z') &> (1 - \delta)[F(L) - R] + (1 - \delta)\epsilon + \delta[B(z) + (1 - \delta)\epsilon/\delta] \\ &= (1 - \delta)[F(L) - R] + \delta B(z) = B(z).\end{aligned}$$

[7] (a) Suppose that a borrower is revealed to be "normal", with a discount factor $\delta \in (0, 1)$. Then we are back to the earlier model. With the lender having all the power, the optimal loan will solve

$$\delta F'(\hat{L}) = 1 + r,$$

and repayment $\hat{R}$ will be chosen so that

$$\delta F(\hat{L}) - \hat{R} = \delta v.$$

Now look at the earlier stage where a borrower is only known to be normal with probability p. With probability $1 - p$ he has a discount factor of 0. So if a package (L, R) is offered, the enforcement constraint is simply

$$(1 - \delta)R + \delta v \leq \delta[F(\hat{L}) - \hat{R}], \tag{22}$$

(simply borrowed from (21) above), and we will also have to respect the participation constraint

$$(1 - \delta)[F(L) - R] + \delta[F(\hat{L}) - \hat{R}] \geq v. \tag{23}$$

The reason why the probability p does not enter above is that bad borrowers will default anyway, so we only have to respect the constraint for the normal borrowers.

Note that the lender's return (in the first phase) is given by

$$pR - (1 + r)L, \tag{24}$$

and because of his monopoly power, this is what he seeks to maximize, given the constraints (22) and (23).

If you draw the two constraints on a diagram, you will see that there are two possible solutions. If p is not too small, both (22) and (23) will hold with equality (or the latter will be slack but the loan will be zero). Because (22) also happens to be identical to the enforcement constraint in the full-information phase, this means that $R = \hat{R}$ in the testing phase as well. But the testing L must be lower, because either it is zero or (23) is met with equality and we know that in the stationary solution the participation constraint is always strictly slack.

If p is small enough, then (22) will become slack but some combination of R and L will be chosen so that (23) continues to hold with equality. [Intuitively, if the probability of repayment is very low, there is more to be gained from protecting the loan size than by asking for a lot of repayment. Now even R falls short of the full-information counterpart and of course the loan size continues to be smaller.

(b) If borrowers had not just two possible discount factors but a whole array of them, one would expect to see several testing phases, each with progressively increasing loan size. This turns ouyt to be a very hard problem to solve analytically by the way.

[8] (i) With a "large" number of people, total societal output is just p, and this should therefore by individual consumption as well, under the optimal scheme. So the optimal scheme involves a transfer $t = 1 - p$ when an output of 1 is produced. This means that the total transfer is $p(1-p)$, which is divided among the $1-p$ have-nots, giving everybody a consumption of precisely p. This is the (symmetric) optimum scheme.

(ii) In an infinitely repeated context with discount factor δ, the normalized payoff from participation is therefore just $u(p)$, the normalized payoff from perennial self-insurance is $pu(1) + (1-p)u(0)$, while the one-shot payoff from a deviation is $(1-\delta)u(1)$. So the enforcement constraint is

$$(1-\delta)u(1) + \delta[pu(1) + (1-p)u(0)] \leq u(p),$$

which is the same as

$$\delta \geq \frac{u(1) - u(p)}{u(1) - [pu(1) + (1-p)u(0)]}. \tag{25}$$

Note: for δ close enough to unity (25) is always satisfied.

(iii) Now suppose that (25) fails. We describe an approach to the optimal *stationary* second-best scheme. Let t be the common transfer made by all haves (not necessarily as large as in the optimal scheme). Then consumption when output is good is just $1-t$, and when output is bad it is $pt/(1-p)$. So the enforcement constraint now reads:

$$(1-\delta)u(1) + \delta[pu(1) + (1-p)u(0)] \leq (1-\delta)u(1-t) + \delta\{pu(1-t) + (1-p)u\left(\frac{pt}{1-p}\right)\}.$$

It is easy to check that the RHS of this expression is (a) strictly concave in t, and (b) coincides with the LHS when $t = 0$. Therefore the only way in which the RHS can exceed the PHS for some $t > 0$ is if (and only if) *the derivative of the RHS in t is strictly positive, evaluated at $t = 0$.* Writing out this condition yields the requirement that

$$-(1-\delta)u'(1) + \delta p[u'(0) - u'(1)] > 0,$$

or

$$\delta > \frac{u'(1)}{(1-p)u'(1) + pu'(0)} \tag{26}$$

You should be able to directly check that (26) is a strictly weaker condition than (25), as it should be.

[9] (i) as in 8(i).

(ii) Let t be a scheme as in question (7). Then expected utility is

$$pu(H-t)+(1-p)u\left(L+\frac{pt}{1-p}\right)-E$$

if effort is applied by *everybody*, and is simply

$$qu(H-t)+(1-q)u\left(L+\frac{pt}{1-p}\right)$$

if one player (of measure zero) deviates. This yields the incentive constraint

$$(p-q)\left[u(H-t)-u\left(L+\frac{pt}{1-p}\right)\right]\geq E$$

From this it is clear that perfect insurance is no longer incentive-compatible (the LHS of the above constraint would be zero).

[10] This question is *completely* parallel to the stationary credit market model with enforcement constraints studied in class. So I omit the answer but do work it out as it will give you separate insights into this sort of model. For a more general treatment of both models (and using nonstationary constracts), see Ray, *Econometrica* **70**, 547–582 (2002).

[11] (i) The laborer's lifetime utility — starting from a slack season — is

$$u(w_*)+\delta u(w^*)+\delta^2 u(w_*)+\delta^3 u(w^*)+\ldots=\frac{u(w_*)}{1-\delta^2}+\delta\frac{u(w^*)}{1-\delta^2}.$$

But of course, this evaluation is different if you begin from the peak season (this will be crucial in what follows):

$$u(w^*)+\delta u(w_*)+\delta^2 u(w^*)+\delta^3 u(w_*)+\ldots=\frac{u(w^*)}{1-\delta^2}+\delta\frac{u(w_*)}{1-\delta^2}.$$

(ii) Now suppose that a landlord-employer with a linear payoff function offers the laborer a contract (x_*, x^*), which is a vector of slack and peak payments. Presumably, the objective is to help the laborer smooth consumption (while still turning a profit for the landlord), so it makes sense to look at the case in which $x_* > w_*$ and $x^* < w^*$. Now, if the offer is made in the slack, there is a participation constraint to be met there, which is that

$$\frac{u(x_*)}{1-\delta^2}+\delta\frac{u(x^*)}{1-\delta^2}\geq\frac{u(w_*)}{1-\delta^2}+\delta\frac{u(w^*)}{1-\delta^2}. \tag{27}$$

But this is only one half of the story. In the peak season the laborer gets only x^* and therefore has an incentive (potentially) to break the contract, getting w^* on the spot market.

By our assumptions, this breach will make him a spot laboreer ever thereafter. So his payoff contingent on breach is precisely his lifetime utility evaluated from the start of a peak season, so that the self-enforcement constraint simply boils down to

$$\frac{u(x^*)}{1-\delta^2} + \delta\frac{u(x_*)}{1-\delta^2} \geq \frac{u(w^*)}{1-\delta^2} + \delta\frac{u(w_*)}{1-\delta^2}. \tag{28}$$

These are the two constraints that have to be met. [Actually, one implies the other — see below.]

(iii) Using (27) and (28), we now show that a mutually profitable contract exists if and only if

$$\delta^2 u'(w_*) > u'(w^*). \tag{29}$$

First, remove the $(1-\delta)^2$ terms in these constraints to obtain the inequalities

$$u(x_*) + \delta u(x^*) \geq u(w_*) + \delta u(w^*) \tag{30}$$

and

$$u(x^*) + \delta u(x_*) \geq u(w^*) + \delta u(w_*) \tag{31}$$

respectively. Next, notice that (31) automatically implies (30) (this is just another instance of the enforcement constraint implying the participation constraint). This is because (31) is just equivalent to

$$\delta[u(x_*) - u(w_*)] \geq u(w^*) - u(x^*),$$

which *implies* that

$$u(x_*) - u(w_*) \geq \delta[u(w^*) - u(x^*)],$$

which in turn is equivalent to (30). So all we have to look for are conditions such that (31) alone is met for some $w_* \leq x_* \leq x^* \leq w^*$ *and* such that

$$x_* + \delta x^* > w_* + \delta w^*,$$

which is the profitability condition for the employer.

Equivalently, construct the zero-profit locus $x_* = w_* + \delta w^* - \delta x^*$ and plug this into (31) to ask if there is some $x^* < w^*$ such that

$$u(x^*) + \delta u\left(w_* + \delta w^* - \delta x^*\right) \geq u(w^*) + \delta u(w_*).$$

Notice that the LHS of this inequality is strictly concave in x^* and moreover at $x^* = w^*$ the LHS precisely equals the RHS. So the necessary and sufficient condition for the above inequality to hold at some x^* distinct from w^* is that the derivative of the LHS with respect to x^*, evaluated at $x^* = w^*$, be negative. Performing this calculation, we get the desired answer.

Problem Set 5

[1] [A] Consider the maximization problem:

$$\max \sum_{i=1}^{n} [u(c_i) - v(r_i)]$$

subject to

$$\sum_{i=1}^{n} c_i \leq f(\sum_{i=1}^{n} r_i).$$

Of course you can use Lagrangeans to do this, but a simpler way is to first note that all c_i's must be the same. For if not, transfer some from a larger c_i to a smaller c_j: by the strict concavity of u the maximand must go up. The argument that all the r_i's must be the same is just the same: again, proceed by contradiction and transfer some from larger r_i to smaller r_j. By the strict concavity of $-v$ the maximand goes up. Note in both cases that the constraint is unaffected.

So we have the problem:

$$\max u\left(\frac{f(nr)}{n}\right) - v(r)$$

which (for an interior solution) leads to the necessary and sufficient first-order condition

$$u'(c^*)f'(nr^*) = v'(r^*).$$

[B] The (symmetric) equilibrium values $\hat{c}$ and $\hat{r}$ will satisfy the FOC

$$(1/n)u'(\hat{c})f'(n\hat{r}) = v'(\hat{r}),$$

[We showed in class that there are no asymmetric equilibria.] It is easy to see that this leads to underproduction (and underconsumption) relative to the first best. For if (on the contrary) $n\hat{r} \geq nr^*$, then $\hat{c} \geq c^*$ also. But then by the curvature of the relevant functions, both sets of FOCs cannot simultaneously hold.

[C] First think it through intuitively. As n is reduced there should be a direct accounting effect: total effort should come down simply because there are less people. But then there is the incentive effect: each person puts in more effort because they will have to share the output with a smaller number of people. Now let's see this a bit more formally. Let $\hat{R}$ denote total equilibrium effort, and rewrite the FOC as

$$(1/n)u'(f(\hat{R})/n)f'(\hat{R}) - v'(\hat{R}/n) = 0.$$

Now we take derivatives. For ease in writing, we will write u', f'', etc., with the understanding that all these are evaluated at the appropriate equilibrium values. Doing this, we have

$$-\frac{1}{n^2}u'f' + \frac{1}{n}u''f'\left[-\frac{f}{n^2} + \frac{f'}{n}\frac{d\hat{R}}{dn}\right] + \frac{1}{n}u'f''\frac{d\hat{R}}{dn} - v''\left[\frac{1}{n}\frac{d\hat{R}}{dn} - \frac{\hat{R}}{n^2}\right] = 0,$$

and rearranging,

$$\frac{d\hat{R}}{dn} = \frac{\frac{1}{n^2}u'f' + \frac{1}{n^3}u''f'f - \frac{1}{n^2}v''\hat{R}}{\frac{1}{n^2}u''f'^2 + \frac{1}{n}u'f'' - \frac{1}{n}v''}.$$

The denominator is unambiguously negative. The numerator is ambiguous for the reasons discusssed informally above.

[D] Each person chooses r to maximize

$$u\left(\left[\beta(1/n)+(1-\beta)\frac{r}{r+R^-}\right]f(r+R^-)\right)-v(r)$$

where R^- denotes the sum of other efforts. Let (c,r) denote the best response. Write down the FOC (which are necessary and sufficient for a best response — why?):

$$u'(c)\left(\left[\beta(1/n)+(1-\beta)\frac{r}{r+R^-}\right]f'(r+R^-)+f(r+R^-)\frac{(1-\beta)R^-}{(r+R^-)^2}\right)=v'(r)$$

Now impose the symmetric equilibrium condition that $(c,r)=(\tilde{c},tr)$ and $R^-=(n-1)\tilde{r}$. Using this in the FOC above, we get

$$u'(\tilde{c})\left[\frac{1}{n}f'(n\tilde{r})+\frac{(1-\beta)(n-1)f(n\tilde{r})}{n^2\tilde{r}}\right]=v'(\tilde{r}).$$

Examine this for different values of β. In particular, at $\beta=1$ we get the old equilibrium which is no surprise. The interesting case is when β is at zero (all output divided according to work points). Then you should be able to check that

$$u'(\tilde{c})f'(n\tilde{r})<v'(\tilde{r})!$$

[Hint: To do this, use the strict concavity of f, in particular the inequality that $f(x)>xf'(x)$ for all $x>0$.]

But the above inequality means that you have *overproduction relative to the first best.* To prove this, simply run the underproduction proof in reverse and use the same sort of logic.

You should also be able to calculate the β that gives you exactly the first best solution. Notice that it depends only on the production function and not on the utility function.

[2] [A] Define a new function f by $f(s)\equiv F(s,s)=s^\alpha$. You can think of this as the "scale function" embodied in the Leontief function. Each individual effectively "has full access" to this function by his choice of effort, as long as his effort lies below that of the other agent.

Define s^* by

$$\frac{1}{2}\alpha s^{*\alpha-1}=c'(s^*)=1.$$

We claim that any symmetric $(r,r)\leq(s^*,s^*)$ is a Nash equilibrium of the game. For if one person chooses $r\leq s^*$, the other person — by the very construction of s^* — has an incentive to keep contributing all the way up to r, and no more.

[B] Obviously, the Nash equilibrium that is best for the agents, is given by (s^*,s^*). In fact, we'll show something stronger: that it creates a higher sum of payoffs than any other Nash equilibrium *from any other division of access shares.* To prove this, first note that every Nash equilibrium (no matter what the shares are) must have equal provision of effort (r,r) (the higher effort guy would simply be wasting effort, a contradiction). Moreover, because social surplus is just $f(s) - s$ which is strictly concave, (s^*, s^*) beats any (r, r) as long as $s^* > r$. So all we have to do is show that in any other Nash equilibrium, $s^* > r$.

This is easy. In any equilibrium, both FOC must satisfy:

$$\lambda_i f'(r) \geq 1.$$

If $r > s^*$, then we must conclude — remembering that one of the λ_i's is less than $1/2$ — that

$$\frac{1}{2} f'(s^*) > 1,$$

which is a contradiction.

[3] and [4] discussed in class. You should be able to work out the example with log utility on your own.

[5] Total payoff is given by

$$ka_i - c(r_i),$$

where $\sum_i a_i = f(R)$ and $R = \sum_i r_i$.

Consider an ex post situation in which $(r_1, \ldots, r_n)$ are given. Let us maximize welfare

$$\sum_i w\left(ka_i - c(r_i)\right)$$

by choice of the allocation $(a_1, \ldots, a_n)$. If v_i denotes the payoff to agent i, then we get

$$w'(v_i)k = w'(v_j)k$$

for every i and j. This proves that ex-post utilities are equalized. Now the rest of the proof follows as in class.

[5] Suppose that the individual utility function in the Ray-Ueda model is given by

$$u(a_i) - c(r_i) = \ln a_i - r_i \equiv v_i,$$

and the social welfare function is given by

$$W = -\frac{1}{\alpha} \sum_i \{e^{-\alpha v_i} - 1\}.$$

[A] Standard; omitted.

[B] Now we work out the ex-post consumption allocations as a function of $(r_1, \ldots, r_n)$. That is, we maximize

$$-\frac{1}{\alpha} \sum_i \{e^{-\alpha[\ln a_i - r_i]} - 1\}$$

which is the same as minimizing

$$\sum_i a_i^{-\alpha} e^{\alpha r_i}.$$

From the FOC, we see that the solution involves

$$a_i/a_j = \frac{e^{\alpha r_i/(\alpha+1)}}{e^{\alpha r_j/(\alpha+1)}}$$

for all i and j, from which it is trivial to conclude that

$$a_i = F(\mathbf{r})\frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}.$$

for all i.

[C] Now consider a symmetric Nash equilibrium of the effort game. Player i maximizes

$$\ln\left(F(\mathbf{r})\frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}\right) - r_i$$

or equivalently

$$\ln\left(F(\mathbf{r})\right) + \ln\left(\frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}\right) - r_i$$

by choosing r_i. Writing down the FOC, we have

$$\frac{\partial \ln F(\mathbf{r})}{\partial r_i} + \frac{\alpha}{\alpha+1} - \frac{\sum_j e^{\alpha r_j/(\alpha+1)}}{e^{\alpha r_i/(\alpha+1)}}\frac{e^{\alpha r_i/(\alpha+1)}}{\left(\sum_j e^{\alpha r_j/(\alpha+1)}\right)^2}\frac{\alpha}{\alpha+1}e^{\alpha r_i/(\alpha+1)} = 1,$$

and imposing symmetry, we may conclude that

$$\frac{\partial \ln F(\mathbf{r})}{\partial r_i} + \frac{\alpha}{\alpha+1}\left(1-\frac{1}{n}\right) = 1.$$

Rearranging, we obtain the required result:

$$\frac{\partial \ln F}{\partial r_i} = \frac{1+\alpha/n}{1+\alpha} \tag{32}$$

for all i. Now, the FOC for the first-best is just the familiar condition

$$u'(c^*)F_i(\mathbf{r}^*) = v'(r^*),$$

which reduces in this special case to

$$\frac{1}{a^*}F_i(\mathbf{r}^*) = 1.$$

Using the fact that a^* is nothing but $F(\mathbf{r}^*)/n$, we see that

$$\frac{F_i(\mathbf{r}^*)}{F(\mathbf{r}^*)} = \frac{1}{n}. \tag{33}$$

Compare (32) and (33), noting that $\frac{F_i(\mathbf{r}^*)}{F(\mathbf{r}^*)}$ is nothing but $\frac{\partial \ln F(\mathbf{r}^*)}{\partial r_i}$. You will see that as α increases, the partial derivatives of F with respect to each input *decrease* to the first best level. It is a simple matter to conclude that output *increases* to the first-best level as α goes to infinity.

[D] Just computation, but do it just to make sure you are on top of the material.