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Answers to Problem Set 5

[1] and [2] Trivial as long as you've studied the basic concepts. For instance, in the very first question, the net return to the government is $0.5b because $20b has to be paid as debt service. This is a return of 5% which is below the threshold, so the investment will not be undertaken. This forms the basis for an argument to forgive some of the debt, which will result in a Pareto-improvement. Similar arguments apply to all the parts in these two questions.

[3] Suppose that p is strictly increasing and that there is a unique first-best choice of effort e^*. Then it must be the case that

$$p(e^*)Q - e^* > p(\hat{e})Q - \hat{e},$$

where $\hat{e}$ is the equilibrium solution. At the same time,

$$p(\hat{e})[Q - R] - [1 - p(\hat{e})]w - \hat{e} \geq p(e^*)[Q - R] - [1 - p(e^*)]w - e^*.$$

Adding these two inequalities and canceling all common terms, we see that

$$(R - w)[p(e^*) - p(\hat{e})] > 0.$$

Because $R > w$ and p is increasing, it follows that $e^* > \hat{e}$.

[4] If the borrower borrows B, needs to repay R ($< w_1$), and puts up an amount C as collateral, his net two-period utility is

$$W \equiv u(w_0 + B + A) + \delta[pu(w_1 - R + A) + (1 - p)u(A - C)],$$

while the lender's utility is
$$\Pi \equiv -B + \delta[pR + (1 - p)\beta C].$$

(a) The favorable effect of increasing C should be obvious: the lender is protected against a default to a greater extent. But there is a negative effect as well. Look at the state in which the borrower receives no income and consequently defaults. In that state a higher value of C will create even lower consumption ($A - C$), which leads ex ante to a greater variability of consumption. Since the borrower is risk-averse, this leads to a potential loss of social surplus because it is increasing the uninsurable risk in the system. [Well, not exactly uninsurable. We *could* have assumed that the lender allows the borrower some extra funding in this state. But this is mathematicaly identical in this model to a reduction of collateral.]

     If you think this second effect is weird, imagine taking a loan contract in which there is some chance that in some state you will lose *everything* ($A - C \simeq 0$). For you to participate willingly in such a contract you will have to be compensated for this risk in all the other states.

Indeed, the compensation may be so high that the lender may not be willing to lend at those terms. This effect is especially pronounced when $\beta < 1$. This diminishes the favorable effect of increased collateral while keeping the unfavorable effect as powerful as before.

To see this more formally, work out the "competitive solution" to the problem above: maximize W subject to $\Pi = 0$, for some *given* C. Set up the Lagrangean

$$\mathcal{L} \equiv u(w_0 + B + A) + \delta[pu(w_1 - R + A) + (1 - p)u(A - C)] + \lambda \left(\delta[pR + (1 - p)\beta C] - B \right),$$

and differentiate with respect to B to get

$$u'(w_0 + B + A) - \lambda = 0, \tag{1}$$

and then with respect to R to get

$$\delta p u'(w_1 - R + A) + \lambda \delta p = 0. \tag{2}$$

Now combine (1) and (2) to see that

$$w_0 + B + A = w_1 - R + A; \tag{3}$$

that is, we have complete consumption smoothing over date 0 and the "success state" in date 1, but the "failure" low-consumption state is delinked (basically by assumption, since we assume that the collateral C is a parameter which is unequivocally seized at this state).

The zero-profit condition tells us that $B = pR + (1 - p)\beta C$. Using this in (3), we can solve out for R as

$$R = \frac{w_1 - w_0}{1 + p} - \frac{1 - p}{1 + p}\beta C,$$

and therefore for the (common) consumption at date 0 and at the success phase — as

$$w_0 + B + A = w_1 - R + A = A + \frac{1}{1 + p}\left[pw_1 + w_0 + (1 - p)\beta C \right] \equiv \sigma.$$

This means that expected utility is given by

$$W \equiv (1 + \delta p)u \left(A + \frac{pw_1 + w_0 + (1 - p)\beta C}{1 + p} \right) + \delta p u(A - C).$$

Now we can take derivatives of borrower utility with respect to C. We see that

$$\frac{dW}{dC} = \frac{(1 + \delta p)(1 - p)\beta}{1 + p}u'(\sigma) - \delta p u'(A - C),$$

where σ is the common consumption calculated above. Notice that the smaller is the value of β or the ratio $u'(\sigma)/u'(A - C)$, the greater is the likelihood that the calculated derivative is negative, in line with our informal reasoning.

(b) Now introduce the debt overhang as we did in class. You should be able to do the exercise in a parallel way. The third effect is, of course, the moral hazard effect.

[5] Parts (a) and (b) are directly out of class and there is nothing to add. To do part (c) here is the basic idea which you can easily formalize. First, recall how we calculated a second-best package by fixing the lender's return at z and then calculating the maximum borrower's payoff. Here there were two possibilities: the loan is either first-best or incentive-constrained. Consider any z for which the latter situation applies. Then if we denote by $S(z)$ the total surplus generated at that z (the sum of the two discount-normalized payoffs), we know that

$S(z)$ is strictly decreasing in z.

[This should be apparent from class discussion, but if it isn't, make sure you understand it.]

Now we're going to show how to Pareto-improve this stationary package by using a nonstationary sequence while still maintaining all the enforcement constraints. Begin by writing down the enforcement constraint for any *sequence* of packages $\{L_t, R_t\}$:

$$(1-\delta)F(L_t) + \delta v \leq (1-\delta)\sum_{s=t}^{\infty} \delta^{s-t}[F(L_s) - R_s]$$

for all t, or equivalently,

$$(1-\delta)R_t + \delta v \leq (1-\delta)\sum_{s=t+1}^{\infty} \delta^{s-t}[F(L_s) - R_s] \tag{4}$$

for all t. Let's evaluate this constraint in a couple of different situations. First, study it for the second-best stationary package (L, R) that yields the lender z. Let's call the return to the borrower $B(z)$. [Notice that $S(z) = B(z) + z$.] Then (4) reduces to

$$(1-\delta)R + \delta v \leq \delta B(z). \tag{5}$$

Now consider the *nonstationary* sequence in which for some small $\epsilon > 0$, the borrower receives the package $(L, R + \epsilon)$ at date 0, and this is followed forever after by the stationary package that yields the lender $z' \equiv z - (1-\delta)\epsilon/\delta$. By construction, the lender is absolutely indifferent between the original stationary package and this new "two-pronged" substitute.

What about the borrower? Well, z is down to z' so the surplus $S(z') > S(z)$. Because $B(z) + z = S(z)$, this means that $B(z')$ is *strictly greater than* $(1-\delta)\epsilon/\delta$. It follows from (5) that

$$(1-\delta)(R + \epsilon) + \delta v \leq \delta B(z'),$$

so that this two-pronged sequence satisfies all the constraints. To complete the proof, notice that the borrower is strictly better off, because

$$\begin{aligned}(1-\delta)[F(L) - (R+\epsilon)] + \delta B(z') \quad &> \quad (1-\delta)[F(L) - R] + (1-\delta)\epsilon + \delta[B(z) + (1-\delta)\epsilon/\delta] \\ &= \quad (1-\delta)[F(L) - R] + \delta B(z) = B(z).\end{aligned}$$

[6] (a) Suppose that a borrower is revealed to be "normal", with a discount factor $\delta \in (0,1)$. Then we are back to the earlier model. With the lender having all the power, the optimal loan will solve

$$\delta F'(\hat{L}) = 1 + r,$$

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and repayment $\hat{R}$ will be chosen so that

$$\delta F(\hat{L}) - \hat{R} = \delta v.$$

Now look at the earlier stage where a borrower is only known to be normal with probability p. With probability $1 - p$ he has a discount factor of 0. So if a package (L, R) is offered, the enforcement constraint is simply

$$(1 - \delta)R + \delta v \leq \delta[F(\hat{L}) - \hat{R}], \tag{6}$$

(simply borrowed from (5) above), and we will also have to respect the participation constraint

$$(1 - \delta)[F(L) - R] + \delta[F(\hat{L}) - \hat{R}] \geq v. \tag{7}$$

The reason why the probability p does not enter above is that bad borrowers will default anyway, so we only have to respect the constraint for the normal borrowers.

Note that the lender's return (in the first phase) is given by

$$pR - (1 + r)L, \tag{8}$$

and because of his monopoly power, this is what he seeks to maximize, given the constraints (6) and (7).

If you draw the two constraints on a diagram, you will see that there are two possible solutions. If p is not too small, both (6) and (7) will hold with equality (or the latter will be slack but the loan will be zero). Because (6) also happens to be identical to the enforcement constraint in the full-information phase, this means that $R = \hat{R}$ in the testing phase as well. But the testing L must be lower, because either it is zero or (7) is met with equality and we know that in the stationary solution the participation constraint is always strictly slack.

If p is small enough, then (6) will become slack but some combination of R and L will be chosen so that (7) continues to hold with equality. [Intuitively, if the probability of repayment is very low, there is more to be gained from protecting the loan size than by asking for a lot of repayment. Now even R falls short of the full-information counterpart and of course the loan size continues to be smaller.

(b) If borrowers had not just two possible discount factors but a whole array of them, one would expect to see several testing phases, each with progressively increasing loan size. This turns ouyt to be a very hard problem to solve analytically by the way.

[7] (i) With a "large" number of people, total societal output is just p, and this should therefore by individual consumption as well, under the optimal scheme. So the optimal scheme involves a transfer $t = 1 - p$ when an output of 1 is produced. This means that the total transfer is $p(1 - p)$, which is divided among the $1 - p$ have-nots, giving everybody a consumption of precisely p. This is the (symmetric) optimum scheme.

(ii) In an infinitely repeated context with discount factor δ, the normalized payoff from participation is therefore just $u(p)$, the normalized payoff from perennial self-insurance is $pu(1) + (1 -$

$p)u(0)$, while the one-shot payoff from a deviation is $(1-\delta)u(1)$. So the enforcement constraint is

$$(1-\delta)u(1) + \delta[pu(1) + (1-p)u(0)] \leq u(p),$$

which is the same as

$$\delta \geq \frac{u(1) - u(p)}{u(1) - [pu(1) + (1-p)u(0)]}. \tag{9}$$

Note: for δ close enough to unity (9) is always satisfied.

(iii) Now suppose that (9) fails. We describe an approach to the optimal *stationary* second-best scheme. Let t be the common transfer made by all haves (not necessarily as large as in the optimal scheme). Then consumption when output is good is just $1-t$, and when output is bad it is $pt/(1-p)$. So the enforcement constraint now reads:

$$(1-\delta)u(1) + \delta[pu(1) + (1-p)u(0)] \leq (1-\delta)u(1-t) + \delta\{pu(1-t) + (1-p)u\left(\frac{pt}{1-p}\right)\}.$$

It is easy to check that the RHS of this expression is (a) strictly concave in t, and (b) coincides with the LHS when $t = 0$. Therefore the only way in which the RHS can exceed the PHS for some $t > 0$ is if (and only if) *the derivative of the RHS in t is strictly positive, evaluated at $t = 0$.* Writing out this condition yields the requirement that

$$-(1-\delta)u'(1) + \delta p[u'(0) - u'(1)] > 0,$$

or

$$\delta > \frac{u'(1)}{(1-p)u'(1) + pu'(0)} \tag{10}$$

You should be able to directly check that (10) is a strictly weaker condition than (**??**), as it should be.

[8] (i) as in 7(i).

(ii) Let t be a scheme as in question (7). Then expected utility is

$$pu(H - t) + (1-p)u\left(L + \frac{pt}{1-p}\right) - E$$

if effort is applied by *everybody*, and is simply

$$qu(H - t) + (1-q)u\left(L + \frac{pt}{1-p}\right)$$

if one player (of measure zero) deviates. This yields the incentive constraint

$$(p - q)\left[u(H - t) - u\left(L + \frac{pt}{1-p}\right)\right] \geq E$$

From this it is clear that perfect insurance is no longer incentive-compatible (the LHS of the above constraint would be zero).

[9] This question is *completely* parallel to the stationary credit market model with enforcement constraints studied in class. So I omit the answer but do work it out as it will give you separate insights into this sort of model. For a more general treatment of both models (and using nonstationary constracts), see Ray, *Econometrica* **70**, 547–582 (2002).

[10] (i) The laborer's lifetime utility — starting from a slack season — is

$$u(w_*) + \delta u(w^*) + \delta^2 u(w_*) + \delta^3 u(w^*) + \ldots = \frac{u(w_*)}{1 - \delta^2} + \delta \frac{u(w^*)}{1 - \delta^2}.$$

But of course, this evaulation is different if you begin from the peak season (this will be crucial in what follows):

$$u(w^*) + \delta u(w_*) + \delta^2 u(w^*) + \delta^3 u(w_*) + \ldots = \frac{u(w^*)}{1 - \delta^2} + \delta \frac{u(w_*)}{1 - \delta^2}.$$

(ii) Now suppose that a landlord-employer with a linear payoff function offers the laborer a contract (x_*, x^*), which is a vector of slack and peak payments. Presumably, the objective is to help the laborer smooth consumption (while still turning a profit for the landlord), so it makes sense to look at the case in which $x_* > w_*$ and $x^* < w^*$. Now, if the offer is made in the slack, there is a participation constraint to be met there, which is that

$$\frac{u(x_*)}{1 - \delta^2} + \delta \frac{u(x^*)}{1 - \delta^2} \geq \frac{u(w_*)}{1 - \delta^2} + \delta \frac{u(w^*)}{1 - \delta^2}. \tag{11}$$

But this is only one half of the story. In the peak season the laborer gets only x^* and therefore has an incentive (potentially) to break the contract, getting w^* on the spot market. By our assumptions, this breach will make him a spot laboreer ever thereafter. So his payoff contingent on breach is precisely his lifetime utility evaluated from the start of a peak season, so that the self-enforcement constraint simply boils down to

$$\frac{u(x^*)}{1 - \delta^2} + \delta \frac{u(x_*)}{1 - \delta^2} \geq \frac{u(w^*)}{1 - \delta^2} + \delta \frac{u(w_*)}{1 - \delta^2}. \tag{12}$$

These are the two constraints that have to be met. [Actually, one implies the other — see below.]

(iii) Using (11) and (12), we now show that a mutually profitable contract exists if and only if

$$\delta^2 u'(w_*) > u'(w^*). \tag{13}$$

First, remove the $(1 - \delta)^2$ terms in these constraints to obtain the inequalities

$$u(x_*) + \delta u(x^*) \geq u(w_*) + \delta u(w^*) \tag{14}$$

and

$$u(x^*) + \delta u(x_*) \geq u(w^*) + \delta u(w_*) \tag{15}$$

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respectively. Next, notice that (15) automatically implies (14) (this is just another instance of the enforcement constraint implying the participation constraint). This is because (15) is just equivalent to

$$\delta[u(x_*) - u(w_*)] \geq u(w^*) - u(x^*),$$

which *implies* that

$$u(x_*) - u(w_*) \geq \delta[u(w^*) - u(x^*)],$$

which in turn is equivalent to (14). So all we have to look for are conditions such that (15) alone is met for some $w_* \leq x_* \leq x^* \leq w^*$ *and* such that

$$x_* + \delta x^* > w_* + \delta w^*,$$

which is the profitability condition for the employer.

Equivalently, construct the zero-profit locus $x_* = w_* + \delta w^* - \delta x^*$ and plug this into (15) to ask if there is some $x^* < w^*$ such that

$$u(x^*) + \delta u (w_* + \delta w^* - \delta x^*) \geq u(w^*) + \delta u(w_*).$$

Notice that the LHS of this inequality is strictly concave in x^* and moreover at $x^* = w^*$ the LHS precisely equals the RHS. So the necessary and sufficient condition for the above inequality to hold at some x^* distinct from w^* is that the derivative of the LHS with respect to x^*, evaluated at $x^* = w^*$, be negative. Performing this calculation, we get the desired answer.