Development Economics Fall 2002

Answers to Problem Set on Voluntary Contributions and Incentives

[1] [A] Consider the maximization problem:

$$\max \sum_{i=1}^{n} [u(c_i) - v(r_i)]$$

subject to

$$\sum_{i=1}^{n} c_i \le f(\sum_{i=1}^{n} r_i).$$

Of course you can use Lagrangeans to do this, but a simpler way is to first note that all c_i 's must be the same. For if not, transfer some from a larger c_i to a smaller c_j : by the strict concavity of uthe maximand must go up. The argument that all the r_i 's must be the same is just the same: again, proceed by contradiction and transfer some from larger r_i to smaller r_j . By the strict concavity of -v the maximand goes up. Note in both cases that the constraint is unaffected.

So we have the problem:

$$\max u\left(\frac{f(nr)}{n}\right) - v(r)$$

which (for an interior solution) leads to the necessary and sufficient first-order condition

$$u'(c^*)f'(nr^*) = v'(r^*).$$

[B] The (symmetric) equilibrium values \hat{c} and \hat{r} will satisfy the FOC

$$(1/n)u'(\hat{c})f'(n\hat{r}) = v'(\hat{r}),$$

[We showed in class that there are no asymmetric equilibria.] It is easy to see that this leads to underproduction (and underconsumption) relative to the first best. For if (on the contrary) $n\hat{r} \ge nr^*$, then $\hat{c} \ge c^*$ also. But then by the curvature of the relevant functions, both sets of FOCs cannot simultaneously hold.

[C] First think it through intuitively. As n is reduced there should be a direct accounting effect: total effort should come down simply because there are less people. But then there is the incentive effect: each person puts in more effort because they will have to share the output with a smaller number of people. Now let's see this a bit more formally. Let \hat{R} denote total equilibrium effort, and rewrite the FOC as

$$(1/n)u'(f(\hat{R})/n)f'(\hat{R}) - v'(\hat{R}/n) = 0.$$

Now we take derivatives. For ease in writing, we will write u', f'', etc., with the understanding that all these are evaluated at the appropriate equilibrium values. Doing this, we have

$$-\frac{1}{n^2}u'f' + \frac{1}{n}u''f'\left[-\frac{f}{n^2} + \frac{f'}{n}\frac{d\hat{R}}{dn}\right] + \frac{1}{n}u'f''\frac{d\hat{R}}{dn} - v''\left[\frac{1}{n}\frac{d\hat{R}}{dn} - \frac{\hat{R}}{n^2}\right] = 0,$$

and rearranging,

$$\frac{d\hat{R}}{dn} = \frac{\frac{1}{n^2}u'f' + \frac{1}{n^3}u''f'f - \frac{1}{n^2}v''\hat{R}}{\frac{1}{n^2}u''f'^2 + \frac{1}{n}u'f'' - \frac{1}{n}v''}.$$

The denominator is unambiguously negative. The numerator is ambiguous for the reasons discussed informally above.

[D] Each person chooses r to maximize

$$u\left(\left[\beta(1/n) + (1-\beta)\frac{r}{r+R^{-}}\right]f(r+R^{-})\right) - v(r)$$

where R^- denotes the sum of other efforts. Let (c, r) denote the best response. Write down the FOC (which are necessary and sufficient for a best response — why?):

$$u'(c)\left(\left[\beta(1/n) + (1-\beta)\frac{r}{r+R^{-}}\right]f'(r+R^{-}) + f(r+R^{-})\frac{(1-\beta)R^{-}}{(r+R^{-})^{2}}\right) = v'(r)$$

Now impose the symmetric equilibrium condition that $(c, r) = (\tilde{c}, tr)$ and $R^- = (n-1)\tilde{r}$. Using this in the FOC above, we get

$$u'(\tilde{c})\left[\frac{1}{n}f'(n\tilde{r}) + \frac{(1-\beta)(n-1)f(n\tilde{r})}{n^2\tilde{r}}\right] = v'(\tilde{r}).$$

Examine this for different values of β . In particular, at $\beta = 1$ we get the old equilibrium which is no surprise. The interesting case is when β is at zero (all output divided according to work points). Then you should be able to check that

$$u'(\tilde{c})f'(n\tilde{r}) < v'(\tilde{r})!$$

[Hint: To do this, use the strict concavity of f, in particular the inequality that f(x) > xf'(x) for all x > 0.]

But the above inequality means that you have *overproduction relative to the first best*. To prove this, simply run the underproduction proof in reverse and use the same sort of logic.

You should also be able to calculate the β that gives you exactly the first best solution. Notice that it depends only on the production function and not on the utility function.

[2] [A] Define a new function f by $f(s) \equiv F(s,s) = s^{\alpha}$. You can think of this as the "scale function" embodied in the Leontief function. Each individual effectively "has full access" to this function by his choice of effort, as long as his effort lies below that of the other agent.

Define s^* by

$$\frac{1}{2}\alpha s^{*\alpha - 1} = c'(s^*) = 1.$$

We claim that any symmetric $(r, r) \leq (s^*, s^*)$ is a Nash equilibrium of the game. For if one person chooses $r \leq s^*$, the other person — by the very construction of s^* — has an incentive to keep contributing all the way up to r, and no more.

[B] Obviously, the Nash equilibrium that is best for the agents, is given by (s^*, s^*) . In fact, we'll show something stronger: that it creates a higher sum of payoffs than any other Nash equilibrium from any other division of access shares. To prove this, first note that every Nash equilibrium (no matter what the shares are) must have equal provision of effort (r, r) (the higher effort guy would simply be wasting effort, a contradiction). Moreover, because social surplus is just f(s) - s which is strictly concave, (s^*, s^*) beats any (r, r) as long as $s^* > r$. So all we have to do is show that in any other Nash equilibrium, $s^* > r$.

This is easy. In any equilibrium, both FOC must satisfy:

$$\lambda_i f'(r) \ge 1.$$

If $r > s^*$, then we must conclude — remembering that one of the λ_i 's is less than 1/2 — that

$$\frac{1}{2}f'(s^*) > 1,$$

which is a contradiction.

[3] and [4] discussed in class. You should be able to work out the example with log utility on your own.

[5] Total payoff is given by

$$ka_i - c(r_i),$$

where $\sum_{i} a_{i} = f(R)$ and $R = \sum_{i} r_{i}$. Consider an expost situation in which (r_{1}, \ldots, r_{n}) are given. Let us maximize welfare

$$\sum_{i} w \left(k a_i - c(r_i) \right)$$

by choice of the allocation (a_1, \ldots, a_n) . If v_i denotes the payoff to agent i, then we get

$$w'(v_i)k = w'(v_j)k$$

for every i and j. This proves that ex-post utilities are equalized. Now the rest of the proof follows as in class.

[5] Suppose that the individual utility function in the Ray-Ueda model is given by

$$u(a_i) - c(r_i) = \ln a_i - r_i \equiv v_i,$$

and the social welfare function is given by

$$W = -\frac{1}{\alpha} \sum_{i} \{e^{-\alpha v_i} - 1\}.$$

[A] Standard; omitted.

[B] Now we work out the ex-post consumption allocations as a function of (r_1, \ldots, r_n) . That is, we maximize

$$-\frac{1}{\alpha}\sum_{i} \{e^{-\alpha[\ln a_i - r_i]} - 1\}$$

which is the same as minimizing

$$\sum_{i} a_i^{-\alpha} e^{\alpha r_i}.$$

From the FOC, we see that the solution involves

$$a_i/a_j = \frac{e^{\alpha r_i/(\alpha+1)}}{e^{\alpha r_j/(\alpha+1)}}$$

for all i and j, from which it is trivial to conclude that

$$a_i = F(\mathbf{r}) \frac{e^{\alpha r_i / (\alpha+1)}}{\sum_j e^{\alpha r_j / (\alpha+1)}}.$$

for all i.

[C] Now consider a symmetric Nash equilibrium of the effort game. Player i maximizes

$$\ln\left(F(\mathbf{r})\frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}\right) - r_i$$

or equivalently

$$\ln\left(F(\mathbf{r})\right) + \ln\left(\frac{e^{\alpha r_i/(\alpha+1)}}{\sum_j e^{\alpha r_j/(\alpha+1)}}\right) - r_i$$

by choosing r_i . Writing down the FOC, we have

$$\frac{\partial \ln F(\mathbf{r})}{\partial r_i} + \frac{\alpha}{\alpha+1} - \frac{\sum_j e^{\alpha r_j/(\alpha+1)}}{e^{\alpha r_i/(\alpha+1)}} \frac{e^{\alpha r_i/(\alpha+1)}}{\left(\sum_j e^{\alpha r_j/(\alpha+1)}\right)^2} \frac{\alpha}{\alpha+1} e^{\alpha r_i/(\alpha+1)} = 1,$$

and imposing symmetry, we may conclude that

$$\frac{\partial \ln F(\mathbf{r})}{\partial r_i} + \frac{\alpha}{\alpha+1} \left(1 - \frac{1}{n}\right) = 1.$$

Rearranging, we obtain the required result:

$$\frac{\partial \ln F}{\partial r_i} = \frac{1 + \alpha/n}{1 + \alpha} \tag{1}$$

for all i. Now, the FOC for the first-best is just the familiar condition

$$u'(c^*)F_i(\mathbf{r}^*) = v'(r^*),$$

which reduces in this special case to

$$\frac{1}{a^*}F_i(\mathbf{r}^*) = 1.$$

Using the fact that a^* is nothing but $F(\mathbf{r}^*)/n$, we see that

$$\frac{F_i(\mathbf{r}^*)}{F(\mathbf{r}^*)} = \frac{1}{n}.$$
(2)

Compare (1) and (2), noting that $\frac{F_i(\mathbf{r}^*)}{F(\mathbf{r}^*)}$ is nothing but $\frac{\partial \ln F(\mathbf{r}^*)}{\partial r_i}$. You will see that as α increases, the partial derivatives of F with respect to each input *decrease* to the first best level. It is a simple matter to conclude that output *increases* to the first-best level as α goes to infinity.

[D] Just computation, but do it just to make sure you are on top of the material.