Answers to Problem Set 3

[1] Discussed in class. For additional information, read the technical digression in my notes, or the analysis in Lucas-Stokey (p. 348–351), or the discussion by Banerjee-Newman (JPE 1993) in the context of inequality.

[2] Suppose the required *minimum* collateral size is W_m. We are going to find a formula for W_m. If you repay the loan, you pay up 20,000 plus interest of 10%, which is a total of 22,000. If you do not repay, you lose the collateral (plus interest), you are fined 5000, and you lose 20% of business profits. If you put up W as collateral, the first of these is $W(1+r) = (1.1)W$. And business profits are $30,000 - 10w$, where w is the wage rate. So if you do not repay, your total losses are $(1.1)W + 5,000 + (0.2)(30,000 - w)$. You *will* repay, therefore, if this last expression is at least as great as 22,000 . This means that the *minimum* necessary collateral W_m is given by the formula

$$W_m = \frac{17,000 - (0.2)(30,000 - w)}{1.1}$$

Now you can calculate what happens to W_m for different values of the wage rate w. The minimum required collateral *increases* with the wage rate, because business profits are lower and therefore less valuable to the creditor in the event of nonrepayment. Thus a larger collateral is asked for to start with.

(ii) Just use the above formula to calculate W_m if the wage rate is at 500. Now let us go on to the second part. Let N be the total population, and let p be the fraction of people who cannot put up this kind of collateral. Then $p \times N$ people get into the labor market. The remainder become entrepreneurs: each of them demand 10 workers, so the total demand is $10 \times (1 - p) \times N$. If the total supply exceeds the total demand, then some people will be unable to get employment, whether as laborers or as entrepreneurs (in terms of our model in class, they would have to go into subsistence). The requirement for this is the condition

$$pN \geq 10(1 - p)N,$$

which means that the value of p should exceed $10/11$. Translate this fraction into percentage terms to find the critical value x.

(iii) and (iv): Similar to what we do below in [3].

[3] (i) There is an initial distribution of wealth which we shall denote by $G(W)$. If for any person, $W \geq I$, then he can contemplate getting into business, in which case his labor demand is

$$L(w) \equiv \arg\max_L f(L) - wL,$$

while his profit is

$$\pi(w) \equiv \max_L \{f(L) - wL\} - I.$$

A person with sufficient upfront wealth will decide to become an entrepreneur if $\pi(w) > w$,a worker if $\pi(w) < w$, and will be indifferent if $\pi(w) = w$. Persons with insufficient upfront wealth must be workers.

Using this information we are ready to describe the market-clearing wage at any date (you can also use diagrams to supplement what follows). Define $\bar{w}$ by $\pi(\bar{w}) \equiv \bar{w}$. Clearly, the market-clearing wage can never strictly exceed $\bar{w}$, otherwise the demand for workers must drop to 0.

To describe the equilibrium, then, consider two scenarios:

Case 1. $w < \bar{w}$. In this case, the demand for workers is given by

$$[1 - G(I)]L(w),$$

while the supply is just $G(I)$, so the equilibrium wage rate is given by

$$[1 - G(I)]L(w) = G(I). \tag{1}$$

This construction is perfectly valid as long as the resulting wage rate stays short of $\bar{w}$. If (1) does not permit such a solution, then set $w = \bar{w}$ and move to Case 2.

Case 2. $w = \bar{w}$. Now potential entrepreneurs are indifferent between being workers and entrepreneurs, so we look for any fraction of entrepreneurs λ that satisfies the condition

$$\lambda \leq [1 - G(I)] \tag{2}$$

and solves

$$\lambda L(\bar{w}) = 1 - \lambda. \tag{3}$$

It is easy to see that *either* there is a solution to (1), *or* there is a solution to the twin conditions (2) and (3). In brief, there is a unique market clearing wage in $[0, \bar{w}]$ for every wealth distribution G.

Given the wealth distribution G_t at time t, we may then write $w(G_t)$ to be the market-clearing wage. Therefore the total resources to an entrepreneur with starting wealth W_t are

$$W_t + \pi(w(G_t)),$$

with the result that

$$W_{t+1} = s(1 + r)\{W_t + \pi(w(G_t))\}, \tag{4}$$

where s is the rate of savings and r is the rate of interest. Similarly, a worker at date t follows the difference equation

$$W_{t+1} = s(1 + r)\{W_t + w(G_t)\}. \tag{5}$$

You can easily use these equations to generate multiple steady states, some with perfect equality and others without. If you also put in uncertainty into the wealth accumulation process you can easily get steady states such that the supports overlap. The reason there is no contradiction here (unlike in question [1]) is that the two steady states follow different laws of motion at the individual level, because the equilibrium wage rate is different across the steady states.

[4] (also marked as [3] in the set). This is a different approach to the same thing we did in class. We make more assumptions here, such as the smoothness of the x and w functions, but the analysis is still of independent interest.

(i) At a steady state, an individual starting at h must choose h. By unimprovability, this means that the following expression has to be maximized:

$$u\left(w(h) - x(h')\right) + \frac{\delta}{1-\delta} u\left(w(h') - x(h')\right)$$

and the solution has to be $h' = h$.

(ii) For necessity, simply write down the first-order conditions. Sufficiency involves the following interesting theorem: if you have a differentiable real-valued function f defined on some subset of $I\!R$, with the property $f"(z) < 0$ whenever $f'(z) = 0$, then $f'(z) = 0$ is *sufficient* for checking a global maximum. Now go ahead and check that this condition precisely holds in the exercise above.

(iii): done in class. (iv): easy. (v)–(vii): see my notes.

[5] ([4] in the set). A steady state is Pareto-efficient if there is no way to have a feasible allocation starting from the same initial allocation as the steady state, which makes all generations just as better off as they were before, and some strictly better off.

(i) Suppose that a steady state has the property that $w(2) - w(1) > x/\delta$ (here 2 is obviously the skilled occupation).

We create a path that Pareto-dominates the steady state. Suppose that at date 0, a fraction λ is skilled (this is the fraction that will persist, of course, in steady state). What we do is create a path in which at date 1 a slightly larger measure, $\lambda + \epsilon$ is skilled. To create these extra skills we sacrifice some consumption at date 0 — $x\epsilon$ — which is divided equally among all households, so the consumption loss of each household is also $x\epsilon$. In period 2 we go back to the old proportion λ and stay there forever. This means that there is some extra consumption in period 1 (because there are more skilled people) — give this consumption equally to all. How much is the extra? It is

$$\begin{aligned} f(1-\lambda-\epsilon, \lambda+\epsilon) - f(1-\lambda, \lambda) &\simeq [f_2(1-\lambda, \lambda) - f_1(1-\lambda, \lambda)] \\ &= [w(2) - w(1)]\epsilon. \end{aligned}$$

[This is an approximation but can easily be made precise at the cost of obscuring the intuition, so we will keep it as it is.]

So to summarize: relative to the original steady state, this path displays a consumption shortfall of $x\epsilon$ in period 0, a consumption excess of (approximately) $[w(2) - w(1)]\epsilon$ in period 1, and no difference thereafter. Notice that agents after period 1 are unaffected, while all agents at period 1 are strictly better off. It therefore only remains to check agents at period 0. The utility loss for any such agent i at date 0 is

$$u(c(i)) - u(c(i) - x\epsilon) \simeq u'(c(i))x\epsilon,$$

while the utility gain is

$$\delta[u(c(i) + [w(2) - w(1)]\epsilon) - u(c(i))] \simeq \delta u'(c(i))[w(2) - w(1)]\epsilon.$$

Now use the condition $w(2) - w(1) > x/\delta$ to show that the gain outweighs the loss, and thereby complete the proof.

(ii) Proving that the opposite of the condition in (i) does imply Pareto-efficiency is not trivial (try it if you like). For a general treatment see Mookherjee and Ray (2002). But let us assume that this is the dividing line between Pareto-efficiency and inefficiency. The question asks us to show (under this assumption) that there is a continuum of both types of states in the two-occupation model.

Remembering that λ stands for the proportion of skilled labor, define λ^* by the condition $\bar{w}(\lambda) - \underline{w}(\lambda) = x/\delta$. By our discussion above, a steady state proportion λ is Pareto-efficient if and only if $\lambda \geq \lambda^*$. So it only remains to show that λ^* belongs to the interior of the set of steady states. This is done by verifying that the "double-inequality" condition is satisfied with *strict* inequality when $\lambda = \lambda^*$.

Here's the verification: exploit the strict concavity of u to see that $u\left(\bar{w}(\lambda^*)\right) - u\left(\bar{w}(\lambda^*) - x\right)$ $< u'\left(\bar{w}(\lambda^*) - x\right) x = u'\left(\bar{w}(\lambda^*) - x\right) \frac{\delta}{1-\delta}[\bar{w}(\lambda^*) - x - \underline{w}(\lambda^*)] < \frac{\delta}{1-\delta}[u\left(\bar{w}(\lambda^*) - x\right) - u\left(\underline{w}(\lambda^*)\right)]$ $< u'\left(\underline{w}(\lambda^*)\right) \frac{\delta}{1-\delta}[\bar{w}(\lambda^*) - x - \underline{w}(\lambda^*)] = u'\left(\underline{w}(\lambda^*)\right) x < u\left(\underline{w}(\lambda)\right) - u\left(\underline{w}(\lambda) - x\right)$.

(iii) First, verify that net consumption indeed always rises with λ over the set of steady states. For any proportion λ, look at

$$f(1-\lambda, \lambda) - x\lambda;$$

this is net consumption if the proportion is held fixed over time. It is easy to check that this expression, viewed as a function of λ, first rises and then falls, reaching its maximum when

$$f_2(1-\lambda, \lambda) - f_1(1-\lambda, \lambda) = x.$$

Now it is easy to see that any steady state of our model must have λ less than this (why?). So this proves that net consumption is rising in λ over the set of steady states.

That appears to suggest that each higher λ Pareto-dominates each lower λ. How do we square this with part (ii)? Answer: easy; appearances are deceptive. It is true that net consumption rises, but we are also changing initial conditions when making these comparisons! Pareto checks do not permit us to do this.

[6] ([5] in text). (i) Say the production function is $f(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. Read off wages as marginal products: define

$$w_i(\boldsymbol{\lambda}) \equiv f_i(\boldsymbol{\lambda})$$

for each i. Now if we define $x_1 = 0$, $x_2 = x$, and $x_3 = z$, we can write the steady state condition as follows: $\boldsymbol{\lambda}$ is a steady state if and only if for all occupations i and j,

$$u(w_i(bla) - x_j) - u(w_i(bla) - x_i) \leq \frac{\delta}{1-\delta}\left[u(w_j(bla) - x_j) - u(w_i(bla) - x_i)\right]. \qquad (6)$$

(ii) Pick any w_1, and define $w_2 = w_1 + \frac{x}{\delta}$ and $w_3 = w_1 + \frac{z}{\delta}$. [The reason for doing this will become clear in a minute.] Now, we move w_1 up and down, notice that w_2 and w_3

move up and down as well. Observe that there is one and only one value such that the above relationships hold *and* the wage vector (w_1, w_2, w_3) supports profit-maximization for f at some $\boldsymbol{\lambda}$. For as $w_1 \to 0$, positive profits are certainly possible, and as $w_1 \to \infty$, so do the other wages so all positive production gives strictly negative profits. So there must be something in between — call it (w_1^*, w_2^*, w_3^*) — which gives exactly zero profits so that a profit-max is well-defined. Call the accompanying $\boldsymbol{\lambda}$ that goes with it, $\boldsymbol{\lambda}^*$.

Notice that this wage function has the property that the claimed inequality in this part actually *does not hold*. However, by exactly the same logic used in part (ii) of the previous question, this wage function and the accompanying $\boldsymbol{\lambda}^*$ constitute a *strict* steady state — all the inequalities in (6) hold strictly. This means that you can jiggle $\boldsymbol{\lambda}^*$ in any "direction" you want, move the accompanying wages as defined by $w_i(\boldsymbol{\lambda})$, and for small movements (6) must continue to hold. In particular, we can perturb $\boldsymbol{\lambda}^*$ to some $\boldsymbol{\lambda}$ so that it satisfies all the steady state conditions *and* the desired inequality

$$\frac{w_2 - w_1}{x} \neq \frac{w_3 - w_2}{z - x} \tag{7}$$

does hold.

(iii) We can do the same exercise as in the previous question, but this time it's easier. Take a steady state in which (7) does hold. Let us suppose, for concreteness, that

$$\frac{w_2 - w_1}{x} < \frac{w_3 - w_2}{z - x}. \tag{8}$$

Create a Pareto-improvement as follows. At date 0, withdraw a budget of ϵ from training type 2 workers and put it into the training of type 3 workers. There is no change in date 0 consumption. At date 1, divide any changed output equally. *Do not disturb the program from date 2 onwards.* To show this is an improvement, all we need to do is show that output in period 1 has increased. This is done using the same approximation techniques discussed earlier.

In date 1, increase λ_1 to $\lambda_1 + \epsilon$, and λ_3 to $\lambda_3 + \eta$. Reduce λ_2 to $\lambda_2 - \epsilon - \eta$. Do this as follows. the extra money released by the ϵ-transfer is ϵx; equalize this to the extra money used up by the η-transfer, which is $\eta(z - x)$. That is, make sure that

$$\epsilon x = \eta(z - x). \tag{9}$$

Now compute the output difference when ϵ and η are small. This is

$$\begin{aligned} f(\lambda_1 + \epsilon, \lambda_2 - \eta - \epsilon, \lambda_3 + \eta) - f(\lambda_1, \lambda_2, \lambda_3) &\simeq f_1(\boldsymbol{\lambda})\epsilon - f_2(\boldsymbol{\lambda})[\eta + \epsilon] + f_3(\boldsymbol{\lambda})\eta \\ &= w_1\epsilon - w_2[\eta + \epsilon] + w_3\eta \\ &= (w_3 - w_2)\eta - (w_2 - w_1)\epsilon. \end{aligned}$$

Using (8) and (9), it is easy to see that this expression is positive, and we are done.

[7] ([6] in text) Here is a quick answer to this question. Suppose, for example, that we haven't checked whether an incumbent of occupation i wants to slide up, say, to occupation j, where

j is more than one notch away. Say j is more expensive. Now look at a holder of occupation $j-1$, which is also strictly more expensive than i. *We have checked the condition for this holder vis-a-vis j*, and he does not want to migrate to j by assumption. So for monotonicity not to be violated, $j-1$ must want to migrate at least two notches further up. Now we are back to the same scenario, but with $j-1 > i$ as our starting point instead of i. This cannot go on forever, because there are only a finite number of occupations.

[8] ([7] in text) (a) There is a typo in this part. I meant to say: show that you cannot have two distinct consumption levels c_i and c_j such that $\lambda(c_i)$ and $\lambda(c_j)$ are both strictly positive.

ANSWER. Suppose not, so that two different c_i and c_j are indeed assigned with $\lambda(c_i)$ and $\lambda(c_j)$ both positive. It must be that $\lambda(c_i)$ and $\lambda(c_j)$ are both strictly We have assumed that $\lambda(c)$ is locally strictly concave whenever $\lambda(c) > 0$. Now we have a contradiction. Transfer from the smaller value of c to the bigger value. Output will go up and so will aggregate utility.

(b) So we can reformulate as follows. There will be m "productive" members are $n-m$ "unproductive" members, with consumptions a and b. The idea is that $\lambda(a) > 0$ but $\lambda(b) = 0$. [But the latter are still given something because they enter into social welfare.] So the idea is to choose (m, a, b) to maximize

$$mu(a) + (n-m)u(b)$$

subject to

$$f(m\lambda(a)) \geq ma + (n-m)b.$$

Suppose the production function f is such that zero consumption is the *only* equal-division solution; that is, for all $c \geq n\hat{c}$,

$$f(n\lambda(c)) < nc$$

Then there cannot be a positive-consumption equal-division solution. Yet, at the same time, it is possible that

$$f(\lambda(\hat{c})) > \hat{c}.$$

With large n and strictly concave f, this situation is easy to construct. Then an equal-division solution cannot be optimal. Moreover, this also answers part (c).