Answers to Problem Set 2

[1] (a) Let $\delta(i)$ denote the discount factor of person i. Conditional on some trade occuring today, i can deviate. If he does so, his return is

$$D + \frac{\delta(i)}{1-\delta(i)}(1-s)M,$$

because he is in the market from tomorrow on where the expected return at each date is just $(1-s)M$.

If he does not do so, the return is

$$T + \frac{\delta(i)}{1-\delta(i)}sT,$$

where the first term is T and not sT because all this is *conditional* on a trade occuring today. The second expression must weakly exceed the first for there not to be a deviation; i.e.,

$$\frac{\delta(i)}{1-\delta(i)}[sT-(1-s)M] \geq D-T.$$

Now we show that if $j < i$, then the same inequality must hold as well. *Two* steps are involved. One is to note that $\frac{\delta}{1-\delta}$ goes up when δ goes up; the second is to note that $[sT-(1-s)M]$ is positive (otherwise the above inequality would not hold to start with). Combining, we see that the left-hand side goes up when δ goes up.

(b) The important part of this question is that I wanted you to recognize that there must be a flat part to the mapping before it starts to climb. In fact, if you did this carefully the next question would have dropped out as a bonus.

Let s be the expected number of people in the T-sector. We want to construct the map $f(s)$ which describes the measure of people who credibly want to be in that sector. Notice first that

$$f(s) = \text{ for all } s \text{ such that } sT \leq (1-s)M.$$

[This is the flat part.] for s beyond this threshold, $f(s)$ is given by the condition that

$$\frac{\delta(f(s))}{1-\delta(f(s))}[sT-(1-s)M] = D-T.$$

It is easy to see that this uniquely determines $f(s)$ for each s. To show that f is strictly increasing in this range, raise s. Then $[sT-(1-s)M]$ must rise. To maintain equality above, $\frac{\delta(f(s))}{1-\delta(f(s))}$ must fall. Because δ is decreasing in i, this means that $f(s)$ must rise.

Verbally, the fact that s increases has two effects: it raises the expected value of trades in the traditional sector and lowers it in the market sector. So now the traditional sector is more attractive. This by itself is *not* sufficient, Now one has to argue that because of this higher attraction, some slightly more impatient people can credibly stay in the traditional sector.

(c) If the market shuts down entirely, then even the most impatient people are in the traditional sector. Notice that the no-deviation constraint now reduces to

$$\frac{\delta}{1-\delta}T \geq D - T.$$

But now we have a contradiction, because for δ small enough this constraint *cannot*be satisfied. Therefore the market cannot shut down entirely. At the same time, the tradtional sector can shut down entirely. For then the expected return to being in the traditional sector is 0, while in the market it is $M > 0$. There is no paradox here because by assumption, contract-breaking is not possible in the market. *This* is the asymmetry which allows for one corner solution but not the other.

(d) For the traditional sector to be partially active we need the existence of some $s > 0$ such that

$$\frac{\delta(s)}{1-\delta(s)}[sT - (1-s)M] = D - T.$$

Rearranging, this is equivalent to the condition that

$$\delta(s) = \frac{D-T}{D-(1-s)(T+M)}.$$

One way to guarantee this is to have the discount factor going down very slowly as i goes up, with all the drop coming near the end. For then, while s is close to 1, the right-hand side is certainly less than one. But we can keep the left-hand side above 1 by having δ hover near one until the very end of the distribution. Verbally, this says that the condition for having a partially active traditional sector is implied by having lots of patient people and only a small fraction of impatient people.

[2] In both the games under consideration, let A stand for the generic strategy that involves play of L or U, and B for the generic strategy that involves play of R or D. In both cases note that playing B is likely to be "better" under low values of the signal, so that is how we will orient the calculations.

Suppose, then, that we imagine that a player will play B if the signal is some value X or less. Let us calculate the recursion value $\psi(X)$ such that *under this assumption*, someone will play B if his signal is $\psi(X)$ or less.

These examples have the same general structure. Suppose that the signal space is located on some interval $[\ell, h]$. For signals very close to ℓ playing B is dominant. For signals very close to h, playing A is dominant. So $\psi(\ell) > \ell$ and $\psi(h) < h$. Finally, we will show that ψ is nondecreasing but has a slope strictly less than one. This yields a unique intersection x^* (which depends on the extent of the noise ϵ). By *exactly* the same arguments as in Morris-Shin, there is a unique equilibrium of the imperfect observation game: play B iff the signal falls short of x^*. Finally, we describe x^* as $\epsilon \to 0$.

(a) In the first example, suppose that your opponent plays B if his signal is X or less. Suppose you see a signal x, and play B. if the true state is θ, the chance that your opponent plays B

is just the chance that your opponent's signal falls below the threshold X, given θ. This is given by the expression

$$\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\},$$

and so your expected payoff (now taking expectations over θ conditional on your signal) is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(b-\theta)\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}d\theta. \tag{1}$$

Likewise, if you play A, the chance that your opponent also plays A is

$$1-\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\},$$

and so your expected payoff conditional on x is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(a+\theta)\left[1-\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}\right]d\theta. \tag{2}$$

[Above, I am integrating from $x-\epsilon$ to $x+\epsilon$. I should be worrying about the lower and upper bounds on θ if I am too close to one edge of the signal space. But we can ignore this, because we know the behavior of Ψ at the edges of the signal space without having to write down the exact expressions.]

The equality of expressions (1) and (2) give you the threshold x for which you are indifferent between A and B, under the presumption that a signal below X results in a play of B for your opponent. In other words, $\psi(X)$ is the solution (in x) to the equation

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(b-\theta)\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}d\theta=\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(a+\theta)\left[1-\max\{\frac{X-(\theta-\epsilon)}{2\epsilon},0\}\right]d\theta. \tag{3}$$

By inspecting (3) it should be obvious that $\Psi(X)$ is nondecreasing in X. What is a little less obvious is the assertion that for all $X'>X$,

$$\psi(X')-\psi(X)<X'-X. \tag{4}$$

To prove (4), let X increase to $X+\Delta$. We want to show that the required solution to (3) in x increases by strictly less than Δ. Suppose this is false, then it must be that after raising X to $X+\Delta$, a rise from the previous solution x to $x+\Delta$ still does not (weakly) bring the LHS and RHS of (3) into new equality; i.e., we have

$$\frac{1}{2\epsilon}\int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon}(b-\theta)\max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon},0\}d\theta\geq\frac{1}{2\epsilon}\int_{x+\Delta-\epsilon}^{x+\Delta+\epsilon}(a+\theta)\left[1-\max\{\frac{X+\Delta-(\theta-\epsilon)}{2\epsilon},0\}\right]d\theta.$$

Now make the change of variables $\theta'\equiv\theta-\Delta$. Then, after all the substitutions, we may conclude that

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(b-\theta'-\Delta)\max\{\frac{X-(\theta'-\epsilon)}{2\epsilon},0\}d\theta'\geq\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(a+\theta'+\Delta)\left[1-\max\{\frac{X-(\theta'-\epsilon)}{2\epsilon},0\}\right]d\theta',$$

but this contradicts (3), the original relationship between X and x. So the claim in (4) is established. Now we have a unique equilibrium using exactly the same arguments as Morris and Shin.

Call this unique threshold x^*. Then, using this fixed point in (3) and noting that the "maxes" in that equation may now be dropped (why?), we have

$$\int_{x^*-\epsilon}^{x^*+\epsilon} \frac{(b-\theta)[x^* - (\theta-\epsilon)]}{2\epsilon} d\theta = \int_{x^*-\epsilon}^{x^*+\epsilon} (a+\theta)\left[1 - \frac{x^* - (\theta-\epsilon)}{2\epsilon}\right] d\theta.$$

Now pass to the limit as $\epsilon \to 0$ (use L'Hospital's Rule). It is easy to see that at the limit,

$$x^* = \theta^* = \frac{b-a}{2}.$$

[b] In the second example, make the same provisional assumption: your opponent plays B if his signal is X or less. Suppose you see a signal x, and play B. if the true state is θ, the chance that your opponent plays B is just the chance that your opponent's signal falls below the threshold X, given θ. This is given by the expression

$$\max\{\frac{X - (\theta-\epsilon)}{2\epsilon}, 0\},$$

just as in (a), and so your expected payoff (conditional on your signal) is

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} 4\max\{\frac{X - (\theta-\epsilon)}{2\epsilon}, 0\} d\theta.$$

[Again, I am integrating from $x-\epsilon$ to $x+\epsilon$ because we can neglect the edges of the state space (see discussion in part (a) above).]

On the other hand, if you play A, you're guaranteed θ (whatever it may turn out to be), so your expected payoff is just x, of course.

The equality of these two expressions give you the indifference threshold x. That is, $\psi(X)$ solves the equation (in x):

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} 4\max\{\frac{X - (\theta-\epsilon)}{2\epsilon}, 0\} d\theta = x. \tag{5}$$

Again, you can show that $\Psi(X)$ is nondecreasing in X and has slope less than one; i.e., that (4) holds for the ψ-function here as well. [Use the same sort of argument we did above; things here are even simpler.]

Call this unique threshold x^*. Then, using this fixed point in (5) and once again noting that the "maxes" may be dropped (why?), we have

$$\frac{1}{\epsilon}\int_{x^*-\epsilon}^{x^*+\epsilon} \frac{x^* - \theta + \epsilon}{\epsilon} d\theta = x^*.$$

Now pass to the limit as $\epsilon \to 0$. It is easy to see that

$$x^* = \theta^* = 2.$$

Observe the contrast between parts (a) and (b). In (a), equilibrium selection generally tracks the Pareto-dominant equilibrium. When $a = b$, the switch point is 0 (how *could* it be anything else, by symmetry and uniqueness?), and now if a and b depart from each other, the switch point moves in the "correct" direction. For example, when, if $b > a$, B will be played more often, because the switch point is now positive.

In part (b), the switch point is $\theta = 2$ (which is about its midpoint value, given the support of θ). At this point, $(4, 4)$ is still much better than $(\theta, \theta) = (2, 2)$. Why does $(4, 4)$ have so little attractive power? It is because the play of A has "insurance" properties: if your opponent does not play A, you still get something (in this example, you get full insurance in fact). But you get no insurance if you play B and your opponent does not. Thus the selection device not only looks at payoffs "at the equilibrium", it looks at payoffs "off the equilibrium" as well to make the selection.

[3] Obvious, if you have read my notes.

[4] (a) If $s > 1$, then being in sector B *guarantees* you a payoff of at least

$$\alpha s + (1 - \alpha)s = s > 1,$$

which is strictly higher than being in Sector A. So being in B is dominant. Similar trivial argument applies for the case in which $s < -1$.

(b) Suppose that an individual today believes that *tomorrow*, people will go to Sector B as long as $s > S$, where S is some threshold. Then *today*, an individual will choose sector B as long as

$$\alpha s + (1 - \alpha)[E(s'|s) + 2P(s' > S|s)] > 1.$$

To understand this, first note that we are only writing down a sufficient condition (not a necessary one). This is why I am pessimistic in the current period and assume that no one else is going to B. This explains the αs for the first period. In the second period, I certainly get the conditional expectation of s' given s, *plus* if the state exceeds S tomorrow, I will have a population of measure 2 coming in.

Now noting that $E(s'|s) = s$ (there is no drift), we may rewrite the above as

$$s + 2(1 - \alpha)P(s' > S|s) > 1.$$

This defines a mapping $g(S)$ from tomorrow's anticipated threshold to a threshold today, given by

$$g(S) + 2(1 - \alpha)P(s' > S|g(S)) = 1.$$

The only consistent expectation is one which replicates itself: that is, $g(S) = S$. Calling this s^*, we see that

$$s^* + 2(1 - \alpha)\frac{1}{2} = 1,$$

or that $s^* = \alpha$. The other case is proved similarly.

[5] This is one of those "exploratory" questions in which I would like to see you construct a model to the best of your ability. But the basic idea is very simple. Suppose that you are

thinking about supporting the potential equilibrium in which everybody is in sector A to start with, but everybody moves to sector B when they have the chance to do so, and stay there. This is an equilibrium which breaks free of history. Adserà-Ray provce that if there are lags in the return adjustment in sector B, no such equilibrium can come about provided that individuals can move whenever they like.

The idea is that people will postpone their movement as long as Sector B is not currently profitable. However, if individuals on average do not have that many chances to move, then they might move now in the fear that they will not get another chance to reap the benefits until far into the future. This suggests that the above equilibrium can be sustained if people have fewer chances to move.

However, here is an intriguing counterargument. If people have few chances to move, the rate of return in Sector B is going to climb very slowly anyway (because of the slowness of aggregate movement out of Sector A). So there are two forces at work: at the individual level, I may want to move because my next shot at moving is far away (in expected value), but at the same the aggregate slowness of movement means that Sector B's returns are climbing very slowly. The latter means I might as well wait. Is it possible to construct an example in which the second effect dominates the first.

Anybody interested in taking this further, see me. But remember you will have to study the relevant literature first, which includes Frankel-Pauzner.