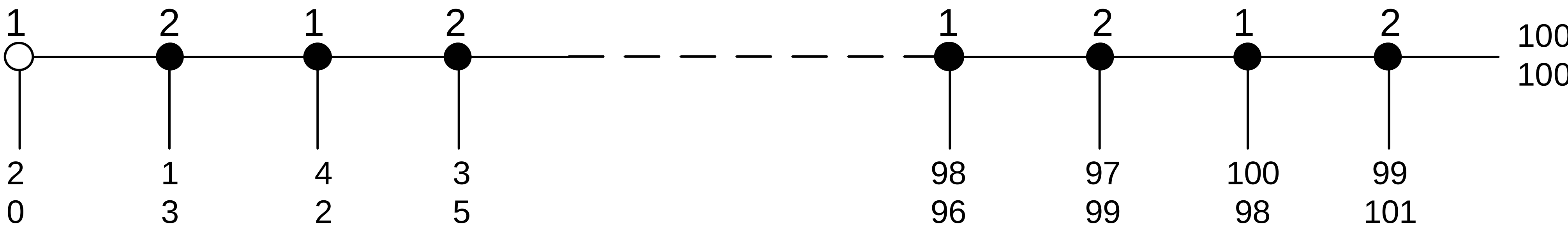
More On Extensive Form Games

Rosenthal's Centipede



Continue the study of backward induction in the centipede.

Deviations as "trembles". Then can maintain rationality assumption on deviation.

Irrational Types. Suppose with probability ϵ each player is a passer; otherwise rational.

Define a *stage* to be an epoch in which players 1 and 2 move once each.

A Trivial Observation. If $T > (1/\epsilon)$, rational player 1 will pass in the first stage.

A More Subtle Observation. As long as $T \geq 2$, rational player 1 never exits for sure at stage 1.

Full solution of equilibria uses indifference condition for taking/passing and Bayes' Rule.

Strategies. Passing type passes for sure. Rational player 1 passes w.p. a_t at stage t and rational player 2 passes w.p. b_t at stage t.

Beliefs. At stage t, 2 believes that 1 is a passer w.p. α_t and 1 believes that 2 is a passer w.p. β_t.

$\beta_1 = \epsilon$, and for all $t \geq 1$

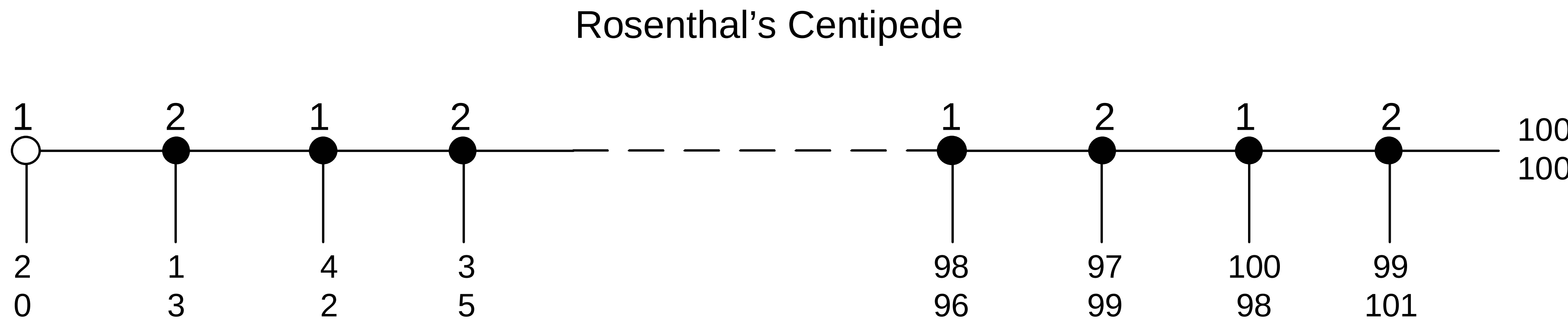
$$\beta_{t+1} = \frac{\beta_t}{\beta_t + (1 - \beta_t) b_t} \geq \beta_t.$$

$\alpha_0 = \epsilon$, and for all $t \geq 0$

$$\alpha_{t+1} = \frac{\alpha_t}{\alpha_t + (1 - \alpha_t) a_{t+1}} \geq \alpha_t.$$

[Bayes' Rule]

Payoffs.



At stage t, if 1 takes: 1 gets $2t$ and 2 gets $2t-2$.

If 2 takes: 1 gets $2t-1$ and 2 gets $2t+1$.

Preliminary Steps.

If $b_t = 1$, then $a_t = 1$. (Obvious.)

If $a_{t+1} = 1$, then $b_t = 1$. (Obvious again.)

Combine. Once a_t falls below 1, all subsequent a_s's and b_s's fall below 1.

In this *interior phase*, taking is always optimal.

But for all $t \leq T - 1$, $a_t > 0$ and $b_t > 0$. So in interior phase, indifference condition holds.

Review: If 1 takes: 1 gets $2t$ and 2 gets $2t-2$.

If 2 takes: 1 gets $2t-1$ and 2 gets $2t+1$.

So in the interior phase, for player 1:

$$2t = (2t-1)(1-\beta_t)(1-b_t) + [1-(1-\beta_t)(1-b_t)](2t+2),$$

$$\boxed{(1-\beta_t)(1-b_t) = 2/3}$$

and for player 2:

$$2t+1 = 2t(1-\alpha_t)(1-a_t) + [1-(1-\alpha_t)(1-a_t)](2t+3),$$

$$\boxed{(1-\alpha_t)(1-a_t) = 2/3}$$

The passing probabilities have to steadily fall in the interior phase.

Recap the four equations:

$\beta_1 = \epsilon$, and for all $t \geq 1$

$$\beta_{t+1} = \frac{\beta_t}{\beta_t + (1 - \beta_t) b_t} \geq \beta_t.$$

$\alpha_0 = \epsilon$, and for all $t \geq 0$

$$\alpha_{t+1} = \frac{\alpha_t}{\alpha_t + (1 - \alpha_t) a_{t+1}} \geq \alpha_t.$$

$$(1 - \beta_t)(1 - b_t) = 2/3$$

$$(1 - \alpha_t)(1 - a_t) = 2/3$$

Theorem. If T is large enough, the interior phase *must* be preceded by one *in which both players pass for sure*.

Summary

Backward induction (and so subgame perfection) is a problematic argument.

But many situations where it makes sense: e.g., assessing credibility.

Move on to other aspects of subgame perfection, and then applications.

An Infinite Extensive Form

Not fully general, but captures a lot.

Ambient action sets for individual i: a sequence $A_i(t)$, each compact metric.

Action $a_i(t)$. Action *profile* $\mathbf{a}(t) \in A(t) \equiv \bigtimes_i A_i(t)$.

A *path* is a sequence of action profiles $\mathbf{a} = \{\mathbf{a}(t)\}$.

Assume all $a(t)$ commonly observed and remembered.

So for $t \geq 1$, t-history may be identified with a finite sequence of action profiles up to $t - 1$.

Arbitrary singleton $h(0)$ for 0-history.

$H(t)$ set of all (feasible) t-histories, will be defined recursively.

For all i and $t \geq 0$, a *feasibility correspondence* $C_i(t, .)$ maps elements in $H(t)$ to subsets of $A_i(t)$. [$C_i(0, h(0)) = A_i(0)\ \forall\ i$.]

[C.1] For each i and date t, $C_i(t, .)$ is a nonempty-valued, continuous correspondence.

$H(0)$ just a singleton set. Recursively

$$H(t+1) = \{(h(t), a(t)) | h(t) \in H(t) \text{ and } a_i(t) \in C_i(t, h(t)) \text{ for all } i\}.$$

Feasible path has truncations $(h(0), \mathbf{a}(0), \ldots \mathbf{a}(t-1))) \in H(t)$ for all $t \geq 1$.

Let H be the set of all feasible paths.

Can be viewed as subset of $\bigtimes_t A(t)$.

Observation. *Under* [C.1], H *is compact in the product topology*.

Proof.

$\bigtimes_t A(t)$ compact by Tychonoff. So prove H is closed.

Payoff functions $F_i : H \mapsto \mathbb{R}$.

[F.1] For each i, F_i is a continuous function on H.

All-important discounting assumption. Illustrate with cake-eating.

Strategy $\sigma_i = \{\sigma_i(t,.)\}$: for all $t \geq 0$ and $h(t) \in H(t)$, $\sigma_i(t, h(t)) \in C_i(t, h(t))$.

Strategy profile is $\sigma = \{\sigma_i\}$.

Generates path $\mathbf{a}(\sigma)$, and more generally, $\mathbf{a}(\sigma, h(t))$ for each $h(t)$.

Equilibrium

σ is a *Nash equilibrium* if for every i and σ'_i,

$$F_i(\mathbf{a}(\sigma)) \geq F_i(\mathbf{a}(\sigma_{-i}, \sigma'_i)).$$

And σ is a *subgame perfect Nash equilibrium* if for i and $h(t)$ and σ'_i,

$$F_i(\mathbf{a}(\sigma, h(t))) \geq F_i(\mathbf{a}(\sigma_{-i}, \sigma'_i, h(t))).$$

Special Cases

Finite-Horizon Games. Horizon T.

Set $A_i(t)$ to a singleton for all i and all $t \geq T$.

Perfect Information.

At every t and $h(t)$, *at most one* set $C_i(t, h(t))$ is not a singleton.

Repeated Games With Discounting.

$A_i(t) = A_i$ for all t, and $C_i(t, h(t)) = A_i$ for all $h(t)$.

"One-period" utility $f_i : \bigtimes_j A_j \mapsto \mathbb{R}$ and discount factor $\delta_i \in (0, 1)$ s.t.

$$F_i(\mathbf{a}) = (1 - \delta_i) \sum_{t=0}^{\infty} \delta_i^t f_i(\mathbf{a}_t).$$

Games with Termination Threats.

E.g., A principal offers a contract at each date to an agent.

Agent can comply or deviate. If he complies, the relationship continues. If he deviates, the relationship is over.

Games Against Different Selves.

E.g., The Strotz model and its later descendants — such as hyperbolic discounting.

Games with State Variables.

E.g. Oil cartel.

State variable: going vector of oil stocks.

Actions: oil extraction, obviously affected by the stock.

The One-Shot Deviation Principle.

Originally formulated by Blackwell (1965) for dynamic programming.

Fix σ_{-i} for all agents other than i.

Induces a "one-player game".

σ_i is a *perfect best response* if for no σ_i' and $h(t)$ is

$$F(\mathbf{a}(\sigma', h(t))) > F(\mathbf{a}(\sigma, h(t))),$$

where $\sigma = (\sigma_i, \sigma_{-i})$ and $\sigma' = (\sigma_i', \sigma_{-i})$.

σ^a: strategy obtained by substituting the choice a at $h(t)$.

σ_i is *unimprovable* for i if for no $h(t)$ and corresponding σ_i^a is

$$F_i(\mathbf{a}(\sigma^a, h(t))) > F_i(\mathbf{a}(\sigma, h(t)))$$

where $\sigma^a = (\sigma_i^a, \sigma_{-i})$.

Theorem. *Under* [F.1], *an unimprovable strategy must be a best response*.

Why [F.1]?

1-player game. At each node can choose a or b.

Get 1 if choose a i.o., otherwise get 0.

Consider strategy: choose b at every node. Unimprovable but suboptimal.

Proof of One-Shot Deviation Principle.

Suppose that σ_i is not a best response.

For some σ_i' and $h(t)$: $F_i(\mathbf{a}(\sigma', h(t))) > F(\mathbf{a}(\sigma_i, h(t)))$.

So there is $\epsilon > 0$ and path $\mathbf{a}$ (coincides with $h(t)$ up to $t - 1$) s.t.

$$F_i(\mathbf{a}) \geq F_i(\mathbf{a}(\sigma_i, h(t))) + 2\epsilon$$

By [F.1], $\exists$ integer M s.t. if any feasible path $\mathbf{a}'$ shares the first $(M + t)$-histories as $\mathbf{a}$,

$$F_i(\mathbf{a}') \geq F_i(\mathbf{a}) - \epsilon.$$

Combine:

$$F_i(\mathbf{a}') \geq F(\sigma, h(t)) + \epsilon.$$

Now complete the proof.

Existence Questions in Infinite Games

Problem with the backward induction argument

Example [Hellwig, Leininger, Reny and Robson (1990)]: $n = 2$.

Player 1 chooses a_1 from $[-1, 1]$.

Then 2 chooses a_2 from $[-1, 2]$.

Player 1's payoff function is $a_1 - a_2$.

Player 2's payoff function is $a_1 a_2$.

Obvious that 2's strategy must be a selection from

$$\begin{aligned} C(a_1) &= \{-1\} \text{ if } a_1 < 0 \\ &= [-1, 2] \text{ if } a_1 = 0 \\ &= \{2\} \text{ if } a_1 > 0. \end{aligned}$$

So to guarantee existence, the selection must be chosen carefully.

But unclear how to make the selection (e.g. introduce dummy player between 1 and 2).

The "Markovian" nature of backward induction may fail.

Example from the Strotz model [Peleg-Yaari (1973)]:

Four-period cake-eating: $c(t)$, $t = 0, 1, 2, 3$.

Four persons (or personalities) in charge . . . Payoffs:

$$f_3(\mathbf{c}) = c_3$$

$$f_2(\mathbf{c}) = \min\{2c(2), \frac{c(2)+3}{2}\} + c(3)$$

$$f_1(\mathbf{c}) = \min\{2c(1), \frac{c(1)+3}{2}\} + c(3)$$

$$f_0(\mathbf{c}) = [c(0)c(1)c(2)]^{1/3} + c(1)$$

$y(t) =$ stock of uneaten cake, so $y(t+1) = y(t) - c(t)$.

$$\begin{aligned} f_3(\mathbf{c}) &= c_3 \\ f_2(\mathbf{c}) &= \min\{2c(2), \frac{c(2)+3}{2}\} + c(3) \end{aligned}$$

$$c(3) = y(3) \text{ for all } y(3).$$

$$\begin{aligned} c_2(y) &= y \text{ for } y \leq 1 \\ &= 1 \text{ for } y > 1. \end{aligned}$$

$$
\begin{aligned}
f_1(\mathbf{c}) &= \min\{2c(1), \frac{c(1)+3}{2}\} + c(3) \\
f_0(\mathbf{c}) &= [c(0)c(1)c(2)]^{1/3} + c(1)
\end{aligned}
$$

$$
\begin{aligned}
c_1(y) &= \{y\} \text{ for } y < 3 \\
&= \{1,3\} \text{ for } y = 3 \\
&= \{1\} \text{ for } y > 3.
\end{aligned}
$$

Correspondence. Let selection attach probability p to choice of 1 when $y(1) = 3$.

$$
\begin{aligned}
f_0 &= c(0)^{1/3} + 1 \text{ for } 0 \leq c(0) < y(0) - 3 \\
&= p[c(0)^{1/3} + 1] + 3(1-p) \text{ for } c(0) = y(0) - 3 \\
&= y(0) - c(0) \text{ for } y(0) - 3 < c(0) \leq y(0).
\end{aligned}
$$

If $p < 1$, no maximum whenever $y(0) > 11$.

But if $p > 0$ no maximum whenever $3 \leq y(0) < 11$!