Games in Extensive Form

Extensive game described by following properties:

1. The set of players N. Nature sometimes a player.

2. The order of moves (a tree).

Three types of nodes: initial, (noninitial) decision (x), terminal (z)

Each node uniquely assigned to a player

Edges are the actions: $A(x)$ set of actions at nonterminal node x.

Formally, a *tree* is a set of nodes X endowed with a *precedence relationship* $\succ$.

$\succ$ is a partial order: transitive and asymmetric.

Initial node x_0: there is no $x \in X$ such that $x \succ x_0$.

Terminal node z: there is no $x \in X$ such that $z \succ x$.

Additional assumption: *exactly* one immediate predecessor for every noninitial node (*arborescence*).

Node Ownership

Each noninitial node assigned to one player.

$\iota : X \setminus Z \mapsto N \cup \{ \text{ Nature } \}$

Information Sets

$h \subseteq X$. H is collection of all h's.

Restriction: If x and x' are in same h, then $\iota(x) = \iota(x')$ and $A(x) = A(x')$.

So can write things like $\iota(h)$ and $A(h)$.

Interpreting Information Sets

Lack of information about something that has already happened

simultaneity

Flipping the future and past: moving Nature's moves up.

Perfect Recall

[1] x and x' in same h implies x cannot precede x'.

This isn't enough. Example.

[2] $x, x' \in h$, $p \succ x$ and $\iota(p) = \iota(h) \implies \exists p'$ (with $p = p'$ possibly) s.t.

p and p' are in the same information set, while $p' \succ x'$

Action chosen along p to x equals the action taken along p' to x'.

Strategies

Define $H_i \equiv \{h \in H | \iota(h) = i\}$ and $A_i \equiv \cup_{h \in H_i} A(h)$.

Pure strategy. Mapping $s_i : H_i \mapsto A_i$ such that $s_i(h) \in A(h)$ for all $h \in H_i$.

S_i set of pure strategies for i.

Can think of it as a mapping or collection of "giant vectors":

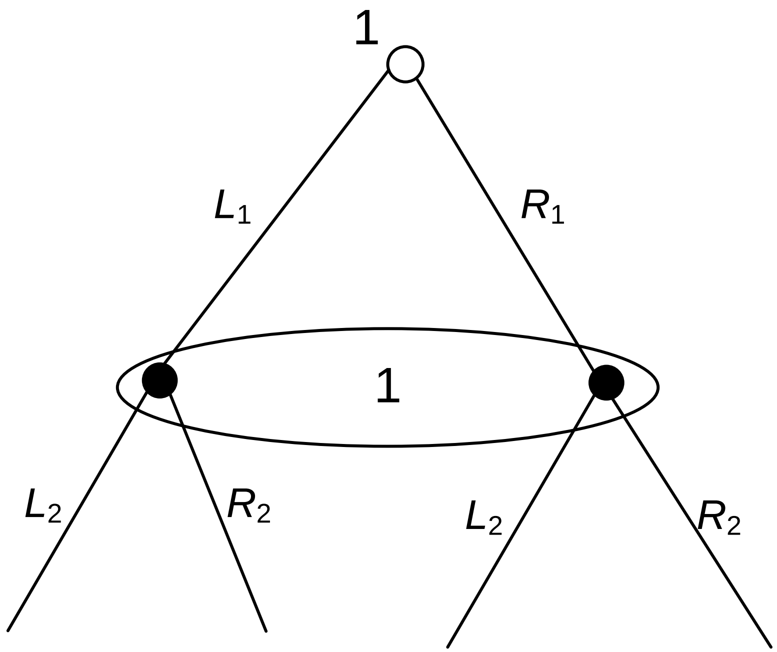
$$S_i = \bigtimes_{h \in H_i} A(h)$$

Behavior strategies. Mapping σ_i on H_i such that

$$\sigma_i(h) \in \mathcal{M}(A(h)) \text{ for all } i$$

Mixed strategies. Probability distribution m_i over all pure strategies.

Are mixed and pure strategies equivalent? Not without perfect recall!



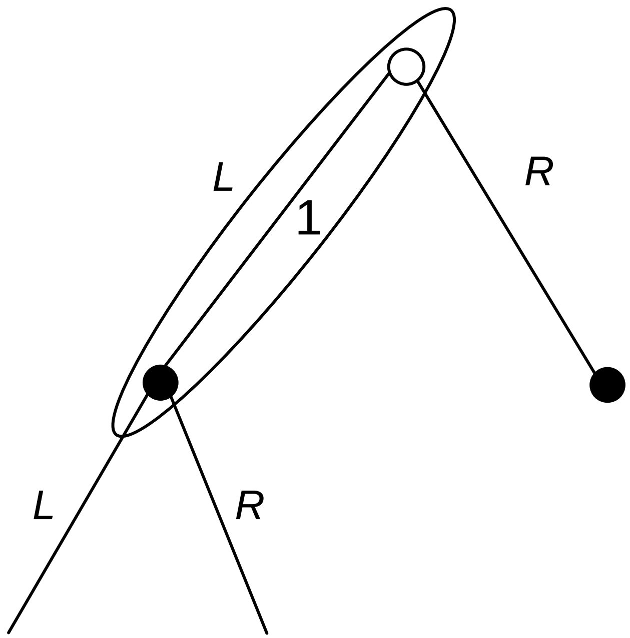
Pure strategy I: play (L_1, L_2).

Pure strategy II: play (R_1, R_2).

Mixed strategy plays these two with equal probability.

No behavior strategy mimics this.

On the other hand, behavior strategies may not be replicable by mixed strategies:



Behavior strategy: play L or R with equal probability.

But then, the path (L, R) is a possible outcome.

This is never possible with a mixed strategy, which can only yield paths (L, L) or (R, R).

This is ruled out as soon as you make [1] of the perfect recall assumption which is often built into the basic definition of a game.

Fix any behavioral strategy σ_i. For any pure strategy s_i simply define $m_i(s_i)$ by

$$m_i(s_i) = \prod_{h_i \in H_i} \sigma_i(s_i(h_i))$$

Kuhn's Theorem. *Assume perfect recall. Then for every mixed strategy there is an equivalent behavior strategy.*

"Proof."

Pick some mixed m_i and any h such that $\iota(h) = i$. Define:

$$P_i(h) = \{s_i \in S_i | \text{ under } s_i \text{ it is possible to reach } h\}$$

If $m_i(s) > 0$ for some $s \in P_i(h)$, define for every $a \in A(h)$,

$$\sigma_i(a) \equiv \frac{\sum_{s \in P(h), s(h)=a} m_i(s)}{\sum_{s \in P(h)} m_i(s)}$$

Otherwise, $m_i(s) = 0$ for all $s \in P_i(h)$. In that case, set

$$\sigma_i(a) \equiv \sum_{s \in S_i, s(h)=a} m_i(s)$$

Check this is ok with perfect recall.

Credibility and Subgame Perfection

Example. Chainstore paradox.

Subgame. Subtree of original tree with following properties:

initial node not a terminal node of original tree (nor its initial node if a *proper* subgame)

If x belongs to Subgame and $x \in h$, then y belongs to Subgame for all $y \in h$.

A (behavior) strategy profile is a *subgame perfect equilibrium* if it is Nash in every subgame.

Theorem. *In every finite extensive game with perfect recall, a subgame perfect equilibrium exists in behavior strategies.*

Proof. By induction on # of nonterminal nodes.

Assume existence in all trees with no more than k nonterminal nodes, for some $k \geq 1$.

For $k = 1$, assertion trivial to verify.

Take game with $k + 1$ nonterminal nodes.

If it has no proper subgames use standard theorem, then Kuhn's Theorem.

If it has, append subgame perfect payoffs to any nonterminal non-inital node and use induction. QED

Counterexample to existence (even of Nash equilibrium) in behavioral strategies when no perfect recall.

Backward Induction

Extensive game of perfect information. Every information set a singleton.

Subgame perfection reduces to *backward induction* for this case.

For any game G, define

$r(G) = \{x \in X | x$ is an immediate predecessor to terminal nodes alone$\}$

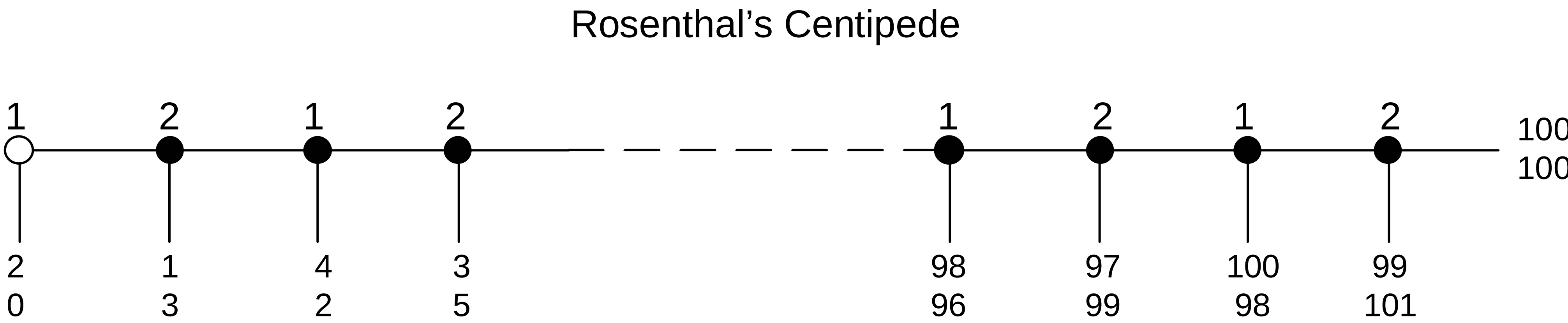
$$s(G) = \{z \in Z | \text{ z has no predecessor in } r(G)\}$$

Begin with $r(G)$. Optimize for i moving at some $x \in r(G)$.

Append resulting payoffs to $r(G)$; redefine as terminal nodes. Consider new game G' with terminal nodes $r(G) \cup s(G)$.

Repeat process until all nodes exhausted. QED

Backward Induction and a Rationality Paradox



Deviations as trembles

Irrational types