Basic Game-Theoretic Concepts

Game in strategic form has following elements

Player set N

(Pure) strategy set for player i, S_i.

Payoff function f_i for player i

$f_i : S \to \mathbb{R}$, where S is product of S_i's.

Examples

various two-person games

	L	R
T	$2, 2$	$0, 3$
B	$3, 0$	$1, 1$

Prisoner's Dilemma

Examples

various two-person games

	L	R
T	$2,2$	$0,0$
B	$0,0$	$1,1$

Coordination Game

Examples

various two-person games

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Battle of the Sexes

Examples

various two-person games

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

Zero-Sum Game; matching pennies

economic games, such as Cournot oligopoly

n firms, so $N = \{1, \ldots, n\}$.

Homogeneous product x. Demand curve $P = P(x)$.

Output of firm i is s_i; $x = s_1 + \cdots + s_n$.

Payoff function for i is $f_i(\mathbf{s}) = P\left(\sum_j s_j\right) - C(s_i)$.

Be careful of strategies in sequential games . . .

Player 1 chooses from a set of actions A_1.

Player 2 observes this choice, then chooses from A_2.

What are S_1 and S_2?

Be careful of strategies in games with information resolution . . .

Player observes a signal from a set X, then chooses an action from a set A. What is her strategy set?

Mixed Strategies

Player i's *mixed strategy* is a probability distribution σ_i over S_i.

Space of i's mixed strategies is Σ_i.

Payoffs to i:

$$f_i(\sigma) \equiv \sum_{\mathbf{s}} f_i(\mathbf{s})\sigma_1(s_1) \cdot \ldots \cdot \sigma_n(s_n)$$

(use integrals if the strategy sets are not finite).

Be careful of mixed strategies; e.g., the sequential auditor game.

Best Responses

Fix strategy profile σ_{-i}

max i's payoff $f_i(s_i, \sigma_{-i})$.

Solution is a (pure) *best response*.

A mixed strategy can also be a best response: it must be a probability distribution over pure best responses.

Nash Equilibrium

σ^* is a *Nash equilibrium* if for every i, σ_i^* is a best response to σ_{-i}^*.

Interpreting Mixed Strategies:

as a deliberate choice

large populations

as beliefs

as pure strategies in an "extended game" (purification)

Look at the usual two-person games:

	L	R
T	$2, 2$	$0, 3$
B	$3, 0$	$1, 1$

Prisoner's Dilemma; unique equilibrium

Look at the usual two-person games:

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Coordination Game; three equilibria

Look at the usual two-person games:

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Battle of the Sexes; three equilibria

Look at the usual two-person games:

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, -1$

Matching pennies; no pure strategy equilibrium

Existence of Nash Equilibrium

Theorem. *Every game with finite strategy sets for each player has a Nash equilibrium, possibly in mixed strategies.*

Proof. Let Σ be product of all Σ_i's: set of all mixed strategy *profiles*.

For each $\sigma \in \Sigma$, each i, define

$$B_i(\sigma) = \{\sigma_i' \in \Sigma_i | \sigma_i' \text{ is a best response to } \sigma_{-i}\}$$

B_i nonempty and convex, and has closed graph.

Define $B : \Sigma \twoheadrightarrow \Sigma$ by $B(\sigma) = \bigtimes_{i \in N} B_i(\sigma)$.

Use Kakutani.

How General is That?

Infinite Strategy Spaces. If S_i is not finite but compact metric, then Nash equilibrium exists if each f_i is continuous (Glicksburg fixed point theorem).

Pure Strategy Existence. If S_i is compact and convex and f_i is continuous, and also quasiconcave in s_i, then a Nash equilibrium exists in pure strategies.

Discontinuous Payoffs. See Dasgupta-Maskin (1986) and Reny (1999).

Rationality, Knowledge and Equilibrium

Epistemic Analysis. What players know or believe about the game and about other players' knowledge or beliefs.

Observation. [Aumann-Brandenberger.] If each player

is rational

knows her own payoff, and

knows the strategies chosen by other players

Then the strategy profile chosen must be Nash.

Mutual knowledge of strategies is "enough".

Now recall notion of (mixed) strategies as beliefs.

Then mutual knowledge of those *beliefs* isn't enough.

Theorem. [Aumann-Brandenberger] *Assume two players. If the game, rationality and beliefs are mutual knowledge, then beliefs form a Nash equilibrium.*

(Need more, including common knowledge of beliefs, when there are more than two players.)

When Strategies are Not Mutually Known

Now need higher levels of knowledge about rationality and the game itself.

E.g., study the iteration leading to *rationalizability*.

Set $\Sigma_i^0 = \Sigma_i$ for all i. Recursion: given $\{\Sigma_j^k\}$, define

$$\Sigma_i^{k+1} = \{\sigma_i \in \Sigma_i^k | \sigma_i \text{ is a BR, within } \Sigma_i^k, \text{ to some } \sigma_{-i} \in \bigtimes_{j \neq i} \text{con}(\Sigma_j^k)\}.$$

Why convex hull

Independent conjectures

Define *rationalizable strategies*:

$$R_i = \cap_{k=0}^{\infty} \Sigma_i^k.$$

The rationalizable *pure* strategies are $P_i = \cup\{\text{supp } \sigma_i | \sigma_i \in R_i\}$.

Can be connected to a direct definition that looks a lot like Nash equilibrium:

A collection $(S_1^*, \ldots, S_n^*)$ of pure strategy subsets forms a *rationalizable family* if for every i

$$S_i^* \subseteq \{s_i \in S_i | s_i \text{ is a BR to some } \sigma_{-i} \text{ with support in } \bigtimes_{j \neq i} S_j^*\}.$$

Note: pure strategy NE forms a rationalizable family.

Theorem. *A pure strategy is rationalizable if and only if it belongs to a rationalizable family.*

Rationalizability doesn't imply Nash equilibrium . . .

. . . *even if the Nash equilibrium is unique.*

	L	M	R
T	$0, 7$	$2, 5$	$7, 0$
C	$5, 2$	$\mathbf{3}, \mathbf{3}^*$	$5, 2$
B	$7, 0$	$2, 5$	$0, 7$

Unique Nash equilibrium in pure *and* mixed strategies.

But $(\{L, R\}, \{T, B\})$ forms a rationalizable family, so each of these four strategies is rationalizable.

Sometimes rationalizability pins down the solution well.

Cournot example. $f_i(\mathbf{s}) = P(s_i)x - c(s_i)$, where $x = \sum_j s_j$.

Make all the assumptions to get "nice" reaction functions.

Do the iteration with pure strategies (mixing makes no difference).

Converges to Nash.

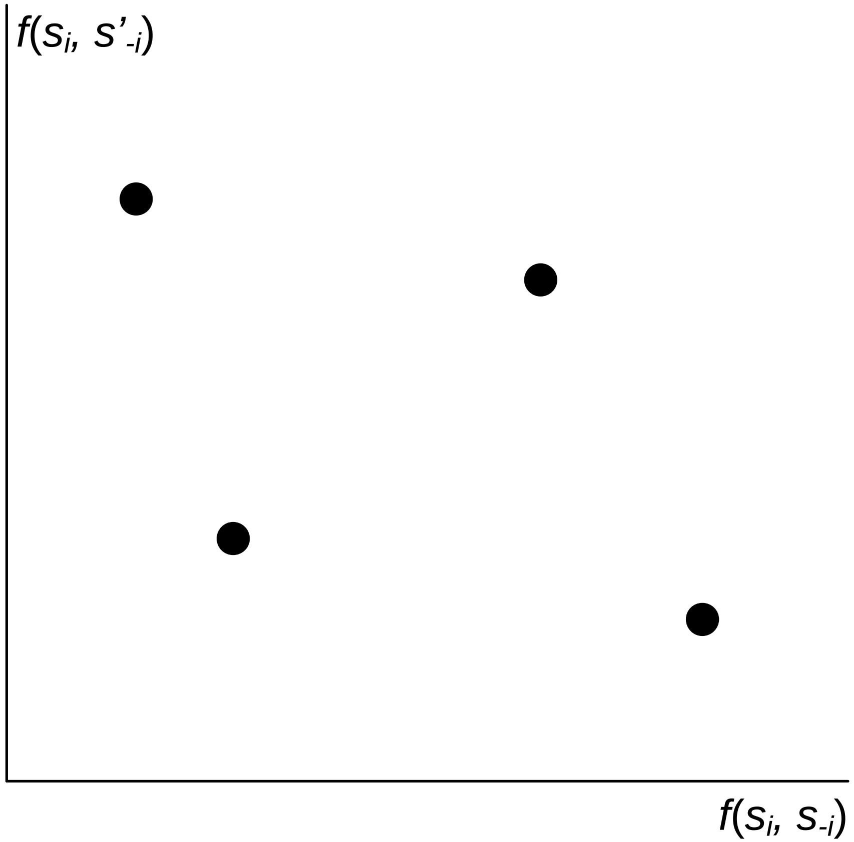
Things are different with three or more firms.

Related Notions

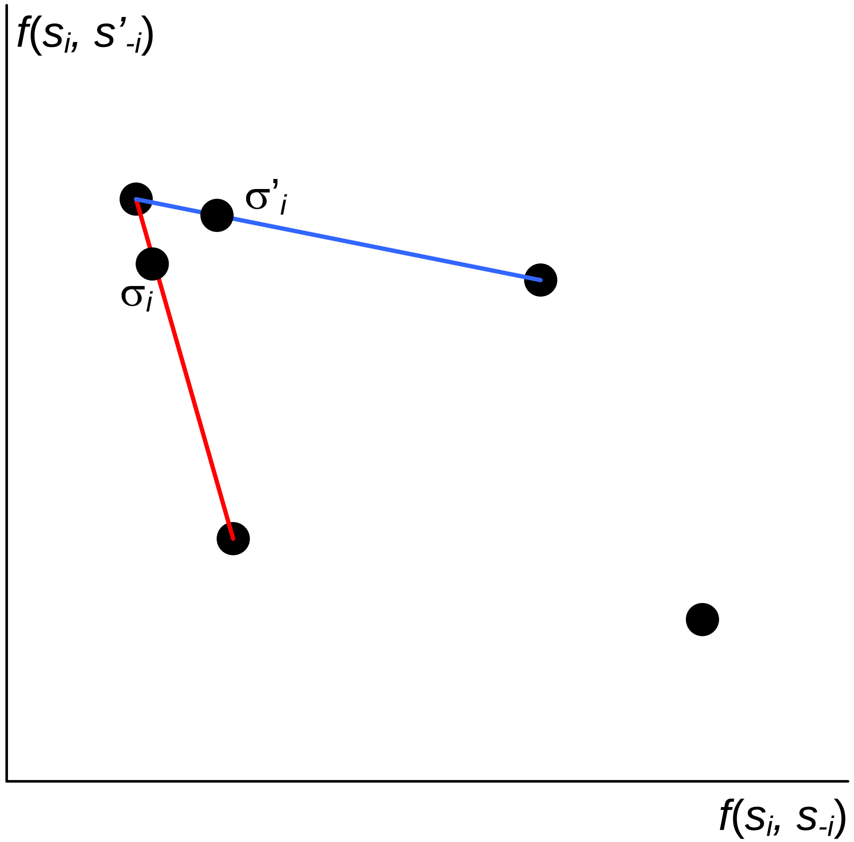
Strictly Dominated Strategies and Iterated Strict Dominance.

A strategy $\sigma_i \in \Sigma_i$ is *strictly dominated* if there exists $\sigma'_i \in \Sigma_i$ such that $f_i(\sigma'_i, s_{-i}) > f_i(\sigma_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

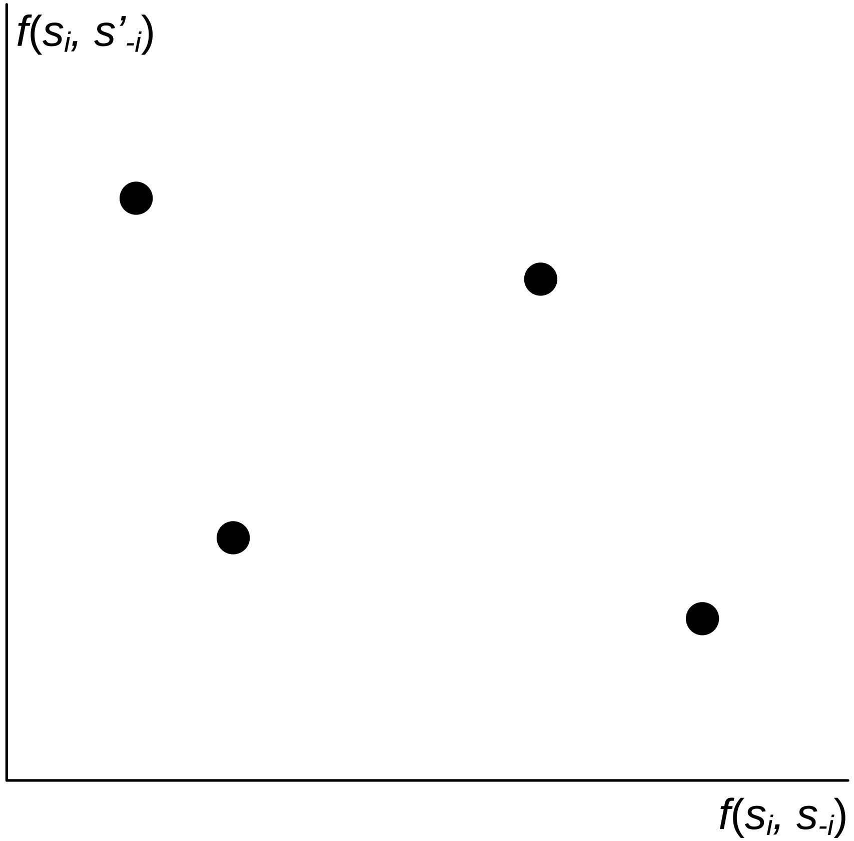
[Doesn't matter whether we use s_{-i} or σ_{-i} in the definition.]



If σ_i attaches positive probability to dominated s_i, it is also dominated.



But even otherwise, σ_i could be strictly dominated . . .



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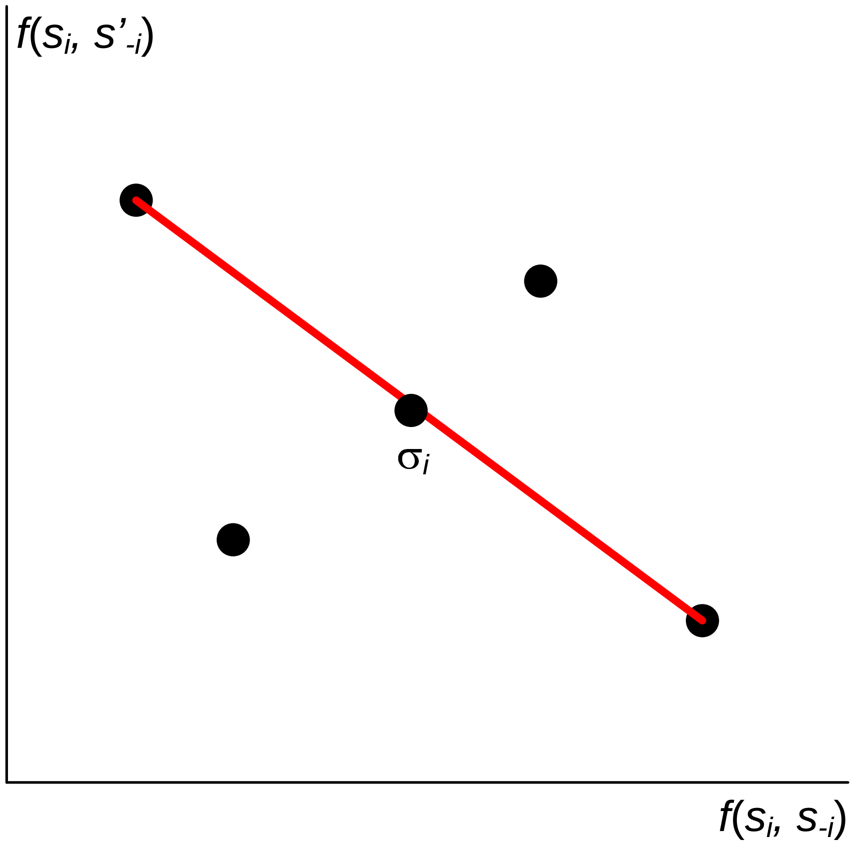
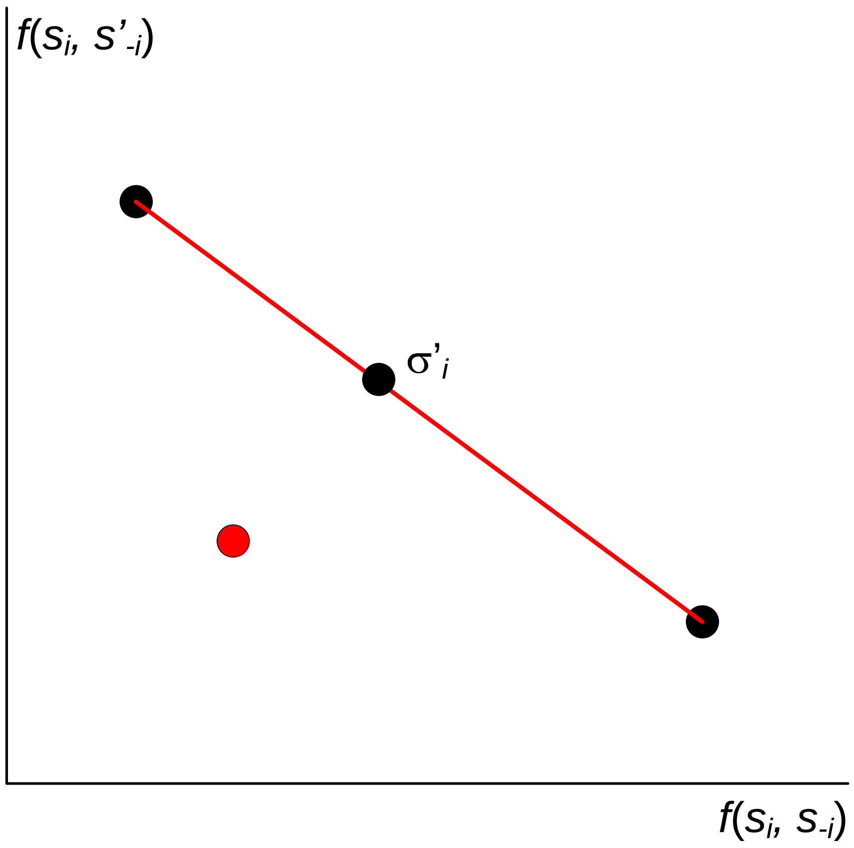


On the other hand, mixed strategies play a role in *dominating* other strategies:



Can use this definition to iteratively eliminate strictly dominated strategies, just as in rationalizability.

Why are the two concepts different then?

A best response to some belief is always an undominated strategy.

An undominated strategy always a best response to some *correlated* belief (separating hyperplane theorem).

With $n = 2$, coincides with rationalizability, otherwise weaker.

Weakly Dominated Strategies and Iterated Weak Dominance.

A strategy $\sigma_i \in \Sigma_i$ is *weakly dominated* if there exists $\sigma'_i \in \Sigma_i$ such that $f_i(\sigma'_i, s_{-i}) \geq f_i(\sigma_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, with strict inequality somewhere.

More problematic. Order of iterated elimination matters.

Efficiency

Fundamental fact. Nash equilibria in one-shot games "typically" inefficient.

Calculus the best way to see this.

$\frac{\partial f_i}{\partial s_i}(\mathbf{s}) = 0$ in NE, but

FOC for efficiency is

$$\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial s_i}(\mathbf{s}) = 0,$$

where the lambdas are weights (or Lagrangean multipliers).

Allows you to guess at the "direction" of inefficiency.

Cournot Example Again

n firms, constant marginal cost $c \geq 0$. Market price $P(x)$.

Joint monopoly output — m — the best outcome for the firms.

$$\max[P(x) - c]x.$$

$$\text{[FOC]} \qquad P(m) + mP'(m) - c = 0.$$

To check BR at m look at individual derivative evaluated at m:

$$P(m) + \frac{m}{n}P'(m) - c > 0$$

Understand where the externality lies.