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A Game-Theoretic Analysis of Bargaining with Reputations

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In this paper we introduce and discuss some models of bargaining. These have the form of repeated plays of a game among pairs of individuals, with the opponents in each particular game drawn randomly from a large population. The players' information about one another is limited to a single quantity, termed reputation, which summarizes the behavior of a player in previous trials of the game, and so changes endogenously. We distinguish some possible decision rules or "customs" which players might use to determine their moves in the game as a function of their own and their opponent's reputation, and investigate whether or not these actions lead to a suitably defined social equilibrium. We then compare the equilibrium customs from the point of view of the welfare of the population as a whole.

INTRODUCTION

For the sake of simplicity, game-theoretic models are often trimmed of psychological and sociological variables which might, in realistic circumstances, be expected to have significant effects on game outcomes. This starkness of models is especially noticeable in the game-theoretic literature on bargaining. The goal of our paper is to incorportate one such sociological variable, reputation, endogenously into a game-theoretic model of bargaining in such a way that it plays a major and, we hope, realistic role. To accomplish this, we shall explore a stochastic-process model of bargaining and, using recently-developed concepts, study its steady-state equilibria.

We consider a society consisting of a large number of individuals. Throughout their lifetimes on a regular basis, the individuals of this society find themselves in situations where they must bargain with each other. We view bargaining as the playing of a game whose form is that two players can have a fixed quantity of money provided they can agree on how to divide it; but otherwise they get nothing. To make this as simple as possible, we postulate that in a single trial of the game each player has only two options: Y (to yield), or $\bar{Y}$ (not to yield) and that the consequent payoffs to the two players are given below.

	Y	$\bar{Y}$
Y	2,2	1,3
$\bar{Y}$	3,1	0,0

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The numbers preceding the commas are the row player's payoffs while those following the commas are the column player's. We interpret the numbers as von-Neumann-Morgenstern utilities. Thus four units of utility are available to the pair if at least one player yields, but the opponent of a yielder is better off not yielding. Of course, each player makes his choice in ignorance of his opponent's choice.

Let us note that if the bargaining game were thought of as a one-shot noncooperative game between two individuals, with no ramifications for future or from past plays, the traditional noncooperative-game analysis would focus on the game's Nash equilibria—the outcomes from which neither player has an incentive to move unilaterally. This game has three Nash equilibria: the off-diagonal pure strategy combinations in which one player yields while the other does not, producing payoffs (3,1) and (1,3), respectively; and the mixed strategy combination at which both players choose Y or $\bar{Y}$ with probability 1/2 each, producing expected payoffs (3/2, 3/2). The symmetric equilibrium is inefficient in the sense that the sum of the payoffs is 3, not 4. The asymmetric equilibria, while efficient, seem to be playable only if there is some coordinating device available to the players.

The bargaining game is to be played between pairs of individuals who have had little or no prior contact. However, we assume that the players do have as information about one another a single quantity, termed reputation, which is meant to summarize the behavior of a player in previous trials of the game against other opponents (higher reputation signifying a tendency not to yield). Hence reputations change endogenously as time progresses. We assume there is a fixed number n of distinct reputations. The exact definition of reputation and the way in which it responds to players' actions will be specified later and will have an important role in our models.

At times $t = 1, 2, \ldots$ the individuals of the society are paired off randomly (independently of the past, each player having an equal chance of being matched with anyone else) to play the bargaining game. The players' choices of Y or $\bar{Y}$ determine their current payoffs and may also change their reputations for the next round of the game. We suppose that players evaluate their own streams of payoffs u_1, $u_2, \ldots$ stretching into the future by discounting with a factor σ, $0 \leq \sigma < 1$, the same for all the players; that is, the value of the stream is taken to be $\sum_{t=1}^{\infty} \sigma^{t-1} u_t$. The objective for each player is to maximize this sum. In order to do so, he generally needs to consider the influence of a move in a bargaining game not only on his current payoff but also, via its effect on his reputation, on subsequent payoffs.

We focus attention on possible decision rules which members of the society might unanimously employ to determine their moves in the games. Such unanimously applied rules are referred to as customs. We restrict ourselves to those customs which depend only on the participants' current reputations and which use these reputations as a device to coordinate the players' choices between following a pure or a mixed strategy. In particular we consider the following general customs:

C1: players yield to opponents of higher reputation, do not yield to opponents of lower reputation, and play some mixed strategy against opponents of the same reputation (the mixed strategy possibly depending on the reputation level);

C2: players yield to opponents of lower reputation, do not yield to opponents of higher reputation, and play some mixed strategy against opponents of equal reputation (the mixed strategy possibly depending on the reputation level).

In both C1 and C2 unequal reputations serve to coordinate strategy choices among the asymmetric equilibria of the one-shot game.

For the sake of a simple comparison, we also include a custom C3, in which players ignore reputation and play throughout only the same fixed mixed strategy.

Our first question regarding each custom is whether or not it generates a steady-state equilibrium, i.e. a steady-state condition in which each player is acting individually to maximize his own discounted sum of utilities. Here we require that when a player assumes all others to be following the custom in question and, on the strength of that assumption, derives particular conclusions about (steady-state) values of certain parameters of the society, then not only does his own interest lead him to abide by the same custom, but also the conclusions on which he acts are correct ones (or at least are approximately correct when the population of players is large). Thus we insist that the behavior of all players be based on fulfilled expectations about the custom under consideration. When a custom such as C1 or C2 is a steady-state equilibrium, the coordinating device absent in the one-shot game has, in a sense, arisen endogenously. The raw materials necessary for the coordinating device are the reputations.

Although individuals in the game contend with one another for their shares of the payoff, society as a whole receives 4 units from each game, except when both players choose not to yield. For any custom, we can ask for the steady-state probability of this last event, which quantity we term the social loss attributable to the custom. We will use this loss to rank different customs from the point of view of society as a whole.

Customs of the form C1 seem to us to be more prevalent in real-life bargaining situations than those of the form C2. We had initially wondered whether this phenomenon could be associated, along social-Darwinist lines, with a greater social loss for C2 customs. This has proved not to be the case. We have discovered instead that both a custom's being in equilibrium and the social ranking for the equilibrium customs are sensitive to the way reputation is defined in our models.

In particular, we envision two different definitions of reputation. One, termed relative, corresponds to a percentile ranking: it requires that the population be at all times divided equally among the n reputation groups. The other, termed absolute, allows reputation groups to vary in size. We show that with reputation defined relatively, both C1 and C2 generate equilibria for each value of the discount factor σ, provided only that the number n of distinct reputations is sufficiently large; and that C2 is always socially preferable to C1 (except at $\sigma = 0$ where they both generate the same social loss). With reputation defined absolutely, we show that equilibrium exists for σ sufficiently small, and that, when σ is close to zero, C1 is socially preferable to C2. Under both reputation regimes, C1 and C2 equilibria, when they exist, are each socially preferable to C3 equilibria, which always exis..

After describing in Section 2 the general form of the steady-state equilibrium conditions, we treat the two reputation regimes separately. In Section 3, we describe the relative reputation regime and state the results for it. These results are proved in Section 4. In Section 5, we describe the absolute reputation regime and state results obtained for it. These results are proved in Section 6. Section 7 is devoted to additional remarks and conclusions.

The basic structure of the models and the notion of equilibrium employed in this paper are closely related to those in Rosenthal (1979), but the discussion here will be almost entirely self-contained. Although the treatment is game-theoretic, it differs significantly from the extensive game-theoretic literature on bargaining. Most of that literature deals with one-shot bargaining games of the sort pioneered in Nash (1950) and Nash (1953). There have been dynamic models of bargaining (e.g. Harsanyi and Selten (1972)), but these typically deal with a time-dependent process involving the same players. Our model, by way of contrast, limits the information which a player possesses about the opponent he faces at any time. This reduces his strategic possibilities, since he has less available on which to condition his action, but emphasizes the roles of reputation and societal custom as coordinating devices.

2. THE GENERAL FORMULATION

Denote the set of reputations (or states) by $\{1,\dots, n\}$. Each player (the player set consists of a large even number K of individuals) is initially assigned a reputation. This reputation may change over time in a way which is understood by all players. At times 1, 2,... the players are matched pairwise so that each player is equally likely to be matched with any of the other players. Each matched pair of players must play the bargaining game. Each player is told the current reputation of his current opponent but nothing else about him. As a result of the play of each game, payoffs are distributed, reputations are changed according to the known rules, time advances, and the process is repeated.

The details of reputation formation are the source of the differences between variants of our model and will be spelled out in Sections 3 and 5, respectively. The general situation, however, is as follows. Each player's reputation at time t ($t > 1$) is a random variable with range $\{1,\dots, n\}$, whose conditional distribution (given all that has gone before) is a function of (at most) the player's own reputation and action at $(t - 1)$ and some aggregated measures of all other players' reputations and actions at $(t - 1)$. The same functions are assumed for all players and all times t, which is to say that reputation is symmetric across players, stationary, and Markovian. In both variants reputation changes slowly, i.e. at time t its value will differ by at most one step from its value at $(t - 1)$.

We now describe a simple approach to studying the motivations of individual players. Suppose that a player assumes that the reputation of the opponent he will draw at each future time t will be an independent selection from a probability distribution $\pi = (\pi_1,\dots, \pi_n)$ on reputation states. Suppose he further assumes that all his future opponents will always play against him according to a known decision rule (possibly admitting randomized choices), the same for all players, which depends only on their own and his own current reputations. That is, all his future opponents will use a stationary, Markovian decision rule. Then the player's problem is to maximize the discounted sum of his expected utilities starting at any time period. This is a Markovian Decision Problem (see Derman (1970), for example) (now with n^2 states), for which it is known that there exists a solution which is itself a stationary Markovian decision rule. Furthermore, the same rule is optimal independently of the players' initial reputation states. (Of course, the optimal rule is a function defined on reputation pairs.) In light of this and the symmetry of the model itself, it seems natural to look for a stationary Markovian decision rule with the property that when all players use it and begin with initial reputation states independently distributed according to π, then the rule solves each individual's own Markovian decision problem and generates π as the individual's invariant distribution over reputation states. Such a decision rule will be called a societal equilibrium.

When the population size K is large, if everyone plays according to some societal equilibrium rule the individual's assumption that his opponents are drawn independently from the distribution π will be approximately realized. (Here we mean that, at any fixed time, if K is sufficiently large there is a high probability that the individual's next opponent will have no connection with the individual's past actions. Thus any updating of π by a player to take account of observed history will not result in much of a change.) By an appropriate modification of a result in Rosenthal (1979) it can be shown that the loss in utility the individual suffers by solving the Markovian problem at time one instead of his actual non-Markovian problem, which depends on the population size, declines to zero as $K \to \infty$.

We shall model the situation from the viewpoint of a player who makes the simplifying Markovian assumptions. A fixed number n of reputation states, a specific custom, reputation regime, and vector π define an $n \times n$ transition matrix Q, whose ith row represents the probabilities of transition from the ith reputation state to each of the n possible reputation states. Since in our model these transitions generally depend on the reputation of an opponent drawn randomly from the population, the quantities π_i enter into Q, as do the numbers p_i, where p_i denotes (under C1 and C2) the probability of playing Y in the mixed strategy adopted by players who find both themselves and their opponents in state i. For the vector $\pi = (\pi_1, \dots, \pi_n)$ to be an invariant distribution for the Markov chain, we evidently require

$$\pi Q(\pi, p), \tag{1}$$

where $p = (p_1, \dots, p_n)$.

The custom, the probabilities $\{p_i\}$, and $\{\pi_i\}$, likewise define the expected return r_i from a single trial of the game for a player in state i, before he is informed of his current opponent's state. Thus,

$$r = r(\pi, p), \tag{2}$$

where $r = (r_1, \dots, r_n)^T$ (superscript T representing transpose). However, to compute the expected (discounted) value v_i of the entire payment stream for a player who begins the game in state i, we must take into account not only the immediate payoff r_i, but also the fact that reputations change as a result of play in accordance with the transition matrix Q. Thus $\nu = (\nu_1, \ldots, \nu_n)^T$ satisfies the equation

$$+ \sigma Q\nu. \tag{3}$$

This has the explicit solution

$$v = r + \sum_{j=1}^{\infty} \sigma^j Q^j r, \tag{4}$$

which converges since $\sigma < 1$ and Q is a stochastic matrix.

Finally, and crucially, we must verify that the custom under consideration is in equilibrium—that each player would in his own interest abide by it, given that the other players do. In accordance with Derman (1970), it is sufficient to check that a one-time-only deviation from the custom will not be profitable for any individual. This requirement leads to additional equations and inequalities. In particular, in order that a player choose a mixed strategy when facing an opponent of equal reputation, as our customs dictate, he must be indifferent between playing Y or $\bar{Y}$, given that the opponent plays Y with probability p_i. This fact generates equations of the form

$$p_i = p_i(\sigma, r, v), \tag{5}$$

which in turn will prove sufficient to guarantee that the inequalities alluded to are also satisfied. This will be clarified further in the next section.

If for a given custom, reputation regime, number n of states, and discount factor σ, the system of equations (1), (2), (3), (5) has a solution (π, p, v), we say that the custom in question is a societal equilibrium. To compare different equilibrium customs, we have selected the criterion of social loss, but it might seem equally natural to take $\pi \cdot v$, the expected value of the sequence of games when the initial reputation is drawn randomly from its steady-state distribution. It is easy to see that these are equivalent. For from (3),

$$\pi \cdot v = \pi \cdot r + \sigma\pi \cdot Qv,$$

so that by (1)

$$(1 - \sigma)\pi \cdot v = \pi \cdot r.$$

Now under the custom C1, (2) becomes

$$\begin{aligned} r_i &= 3 \sum_{j<i} \pi_j + \sum_{j>i} \pi_j + \pi_i[2p_i^2 + p_i(1 - p_i) + 3(1 - p_i)\, p_i] \\ &= 3 \sum_{j<i} \pi_j + \sum_{j>i} \pi_j + 2\pi_i[1 - (1 - p_i)^2]; \end{aligned} \tag{6}$$

while for C2 we find

$$r_i = \sum_{j<i} \pi_j + 3 \sum_{j>i} \pi_j + 2\pi_i[1 - (1 - p_i)^2]. \tag{7}$$

Thus with C1

$$\begin{aligned}(1 - \sigma)\pi \cdot v = \pi \cdot r &= \sum_{i=1}^{n} \pi_i \left[3 \sum_{j<i} \pi_j + \sum_{j>i} \pi_j\right] + 2 \sum_{i=1}^{n} \pi_i^2 - 2 \sum_{i=1}^{n} \pi_i^2(1 - p_i)^2 \\ &= 4 \sum_{i=1}^{n} \pi_i \sum_{j<i} \pi_j + 2 \sum_{i=1}^{n} \pi_i^2 - 2 \sum_{i=1}^{n} \pi_i^2(1 - p_i)^2 \\ &= 2 \left[\sum_{i=1}^{n} \pi_i\right]^2 - 2 \sum_{i=1}^{n} \pi_i^2(1 - p_i)^2 \\ &= 2 - 2 \sum_{i=1}^{n} \pi_i^2(1 - p_i)^2.\end{aligned}$$

The sum appearing on the right is precisely the social loss (as defined earlier) under the rules of C1. The identical calculation applies to C2. We will therefore refer to $\pi \cdot v$ as the social value of the custom.

As to C3, since it represents an equilibrium in a single trial of the game, with all $p_i = 1/2$ and since reputation has no role, it clearly is a steady-state equilibrium in every variant of the model. Here $r_i = 3/2$ for all i, $\pi \cdot v = 3/2(1 - \sigma)$, and the social loss is $1/4$.

3. First Variant of the Model

In the first variant of the model, reputations are thought of as being relative: we require that, with n distinct reputation states, n divides K and at each time exactly $1/n$ of the population is in each state. Now according to our conception of reputation, when an individual in state i plays Y his reputation should tend to decrease, while if he plays $\bar{Y}$ it should tend to increase. But it may not be possible to accommodate each individual with actual increases or decreases while maintaining the population equally distributed among states. Thus we arrange matters so that an individual in state i who plays Y moves to some intermediate, potentially lower, state $(i - 1/2)$, from which his actual reputation in the following time period is determined to be either $(i - 1)$ or i, the assignment made randomly but so as to maintain the desired distribution of individuals across states. Similarly, playing $\bar{Y}$ places the player in auxiliary state $(i + 1/2)$ from which he moves either to i or $(i + 1)$. Intuitively, all players from the auxiliary state $1/2$ are placed in state 1. Then enough players are added from auxiliary state $3/2$ to bring $1/n$ of the population into state 1, the remainder from state $3/2$ going to state 2, and so forth. Of course, dividing the players from an auxiliary state $(i - 1/2)$ into two groups is done randomly so that each player in it has the same probability, δ_i^t (at time t) of moving into state i. Figure 1 represents the situation schematically.

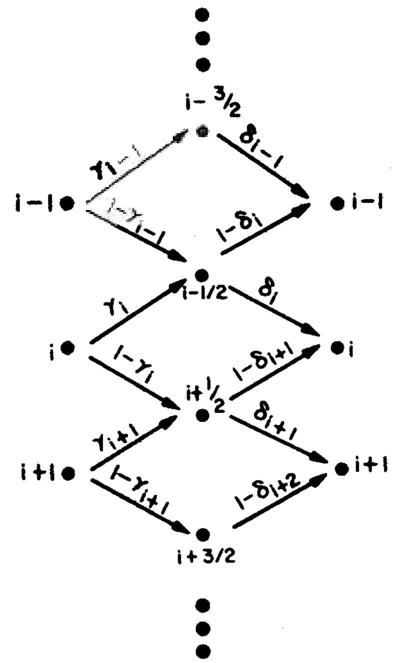

FIG. Relative reputation scheme.

More precisely, denoting by γ_i^t the proportion of the population of state i at time t which plays Y, we obtain

$$\begin{gathered}\gamma_1^t + (1 - \delta_2^t)(1 - \gamma_1^t + \gamma_2^t) = 1 \\ (1 - \delta_{i+1}^t)(1 - \gamma_i^t + \gamma_{i+1}^t) + \delta_i^t(1 - \gamma_{i-1}^t + \gamma_i^t) = 1, \qquad 2 \leqslant i \leqslant n - 1 \\ \delta_n^t(1 - \gamma_{n-1}^t + \gamma_n^t) + (1 - \gamma_n^t) = 1.\end{gathered} \tag{8}$$

These equations are easy to solve recursively for $\{\delta_i^t\}$ in terms of $\{\gamma_i^t\}$. The transition probabilities at time t from state i to states $i - 1$, i, $i + 1$ are

$$\gamma_i^t(1 - \delta_i^t),\ \gamma_i^t\delta_i^t + (1 - \gamma_i^t)(1 - \delta_{i+1}^t),\ (1 - \gamma_i^t)\,\delta_{i+1}^t,$$

respectively, where $\delta_1^t \equiv 1$ and $\delta_{n+1}^t \equiv 0$. By this construction, we have ensured that the steady-state probabilities π_i in equations (1)-(5) are identically $1/n$, and that (1) is an accounting identity.

Under C1 or C2, for each t, as $K \to \infty$, γ_i^t approaches a limiting value γ_i (independent of t) with probability 1 by the weak law of large numbers. Since, according to C1, a player yields to an opponent of higher state, and yields with probability p_i to one of his own state, we find that in C1

$$\gamma_i = \frac{n - i}{n} + \frac{p_i}{n} \tag{9}$$

while in C2,

$$\gamma_i = \frac{i-1}{n} + \frac{p_i}{n} \tag{10}$$

We denote by $\{\delta_i\}$ the limiting solutions to (8). From the player's viewpoint only the limiting $\{\gamma_i\}$ and $\{\delta_i\}$ matter, and these enable us to express the entries $\{q_{ij}\}$ of the limiting (as $K \to \infty$) transition matrix Q in terms of n, i, j, and the $\{p_i\}$ (Lemmas 1 and 2 in Section 4). Similarly, since $\pi = (1/n, \ldots, 1/n)$, (6) determines r_i in terms of n, i, and p_i. Finally, so that each player is willing to play according to the custom, (5) takes the following form (where v_0 and v_{n+1} are arbitrary constants)

$$\begin{gathered} 3p_i + \sigma(\delta_{i+1} v_{i+1} + (1 - \delta_{i+1})v_i) = \quad + p_i + \sigma(\delta_i v_i \quad (1 \quad \delta_i)v_{i-1}) \\ \text{for} \quad 0 \leqslant p_i \leqslant 1, \qquad i = 1, \ldots, n. \end{gathered} \tag{11}$$

This condition calls for a few words of explanation. The quantity v_i, as previously defined in (4), is the discounted value of the game to a player starting with reputation i. For such a player, the right-hand side of (11) represents the expected value of yielding to an opponent of the same reputation (who is assumed to play the mixed strategy of choosing Y and $\bar{Y}$ with probabilities p_i and $1 - p_i$, respectively), since the terms being added there consist of the expected immediate payoff and the discounted future value of landing in the state $(i - 1/2)$. Similarly, the left-hand side of (11) measures the value of not yielding. In order that the player himself be willing to follow the mixed strategy in this situation, as the custom dictates, with $0 < p_i < 1$, the consequences for him of playing Y and $\bar{Y}$ must be the same, for otherwise he would choose one or the other exclusively. Equation (11) asserts precisely this. It is possible to verify that (11) likewise implies the inequalities which guarantee that players will respond according to the rules of the prevalent custom (C1 or C2) when facing an opponent of a different reputation and when $p_i = 0$ or 1. We omit the tedious details. When $p_n = 0$ under C1 and when $p_1 = 0$ under C2 it is only necessary that the left side of (11) be greater than or equal to its right side; and when $p_n = 1$ under C2 and when $p_1 = 1$ under C1 it is only necessary that the right side of (11) be greater than or equal to the left side. Since it can be shown that no C1 equilibria can occur with $p_1 = 1$ or $p_n = 0$ and no C2 equilibria can occur with $p_1 = 0$ or $p_n = 1$, we shall impose (11) rather than a slightly weaker but more complicated variant of it.

Thus the equilibrium criteria (1), (2), (3), and (5), reduce to the problem of determining $\{p_i\}$ so that

$$0 \leqslant p_i \leqslant 1, \qquad \leqslant i \leqslant n,$$

and so that (4) and (11) are satisfied. We show that this can be done for each discount factor σ, providing only that n is sufficiently large ($n \to \infty$ as $\sigma \to 1$).

Theorem 1. *For each $\sigma \in [0, 1)$ there exists $n_1(\sigma)$ such that for all $n \geqslant n_1(\sigma)$, there is a* C1 *societal equilibrium. As $\sigma \to 1$, $n_1(\sigma) \to \infty$.*

THEOREM 2. *For each* $\sigma \in [0, 1)$ *there exists* $n_2(\sigma)$ *such that for all* $n \geqslant n_2(\sigma)$, *there is a* C2 *societal equilibrium. As* $\sigma \to 1$, $n_2(\sigma) \to \infty$.

We should keep in mind that the individual's assumption that his environment is Markovian is a reasonable hypothesis only when the number of individuals in the society is sufficiently large relative to some fixed specification of the model (i.e., fixed σ, n and rules of reputation formation). In addition, we should recognize that the possibility of multiple C1 or C2 societal equilibria has not been ruled out.

We are also able to rank C1, C2, and C3 by their social values.

THEOREM. *For given* n *and* $\sigma > 0$, *if* C1 *and* C2 *are societal equilibria, the social value* $\pi \cdot v$ *for* C2 *exceeds that for* C1. *For* $\sigma = 0$ *the two are equal. Whenever* C2 *is a societal equilibrium, its social value exceeds that of* C3. *The same is true of* C1 *if* $n \geqslant 5$.

To explain this result, we note that, since π is uniform, social value is determined solely by the equilibrium values of p in C1 and C2, i.e., by the equilibrium probabilities with which players yield to opponents of the same reputation as their own. It is intuitively plausible that it pays to have a high reputation in C1 and a low reputation in C2, so that we may expect p_i to be larger in C2 than in C1. Thus the frequency of deadlock should be lower in C2, making the social value greater.

4. PROOFS FROM SECTION 3

Recursively from (8),

$$\delta_i(1 - \gamma_{i-1} + \gamma_i) = \gamma_i, \qquad 2 \leqslant i \leqslant n.$$

Consequently (8) has the solution

$$\delta_{i+1} = \frac{\gamma_{i+1}}{1 - \gamma_i + \gamma_{i+1}} \qquad 1 \leqslant i \leqslant n -$$

With (9), (10) and Fig. we then obtain the following evaluations for the transition probabilities.

LEMMA *Under C*1,

$$q_{i,i-1} = \left(\frac{n - i + p_i}{n}\right)\left(\frac{i - 1 - p_{i-1}}{n - 1 + p_i - p_{i-1}}\right) \qquad \text{for} \quad i = 2, \ldots, n,$$

$$q_{i,i+1} = \left(\frac{i - p_i}{n}\right)\left(\frac{n - i - 1 + p_{i+1}}{n - 1 + p_{i+1} - p_i}\right) \qquad \text{for} \quad i = 1, \ldots, (n - 1),$$

$$q_{i,i} = \begin{cases} 1 - q_{i,i-1} - q_{i,i+1} & \text{for} \quad i = 2, \ldots, (n-1) \\ 1 - q_{n,n-1} & \text{for} \quad i = n \\ 1 - q_{1,2} & \text{for} \quad i = 1, \end{cases}$$

$$q_{ij} = 0 \qquad \text{for} \quad j \neq (i - 1), i, (i + 1).$$

Lemma 2. *Under* C2,

$$q_{i,i-1} = \left(\frac{i-1+p_i}{n}\right)\left(\frac{n-i+2-p_{i-1}}{n+1+p_i-p_{i-1}}\right) \qquad \text{for} \quad i = 2,\dots, n,$$

$$q_{i,i+1} = \left(\frac{n-i+1-p_i}{n}\right)\left(\frac{i+p_{i+1}}{n+1+p_{i+1}-p_i}\right) \qquad \text{for} \quad i = 1,\dots, (n-1),$$

$$q_{i,i} = \begin{cases} 1 - q_{i,i-1} - q_{i,i+1} & \text{for} \quad i = 2,\dots, (n-1) \\ 1 - q_{n,n-1} & \text{for} \quad i = n \\ 1 - q_{1,2} & \text{for} \quad i = 1, \end{cases}$$

$$q_{ij} = 0 \qquad \text{for} \quad j \neq (i-1), i, (i+1).$$

Proof of Theorem 1. For any p in the unit n-cube, a unique matrix Q is defined by Lemma 1. This generates a vector v through the equation

$$v = r + \sum_{j=1}^{\infty} \sigma^j Q^j r$$

The vectors p and v generate, in turn, a new vector p' through the equations

$$p_i' = \frac{1}{2} + \frac{1}{2}\sigma\left[\frac{(i-1-p_{i-1})(v_{i-1}-v_i)}{n-1+p_i-p_{i-1}} + \frac{(n-i-1+p_{i+1})(v_i-v_{i+1})}{n-1+p_{i+1}-p_i}\right]$$
$$i = 1,\dots, n \qquad (12)$$

(where $p_0 \equiv 0$, $p_{n+1} \equiv 1$, and v_0 and v_{n+1} are arbitrary constants). Note that if $p_i' = p_i$ for all i, then the system defines a societal equilibrium (in this case (12) and (11) are the same). Now the function which produces the vector p' from the vector p is continuous. To see this, note that every step in the process (the generation of r, Q, v and finally p') is continuous. Brouwer's celebrated theorem therefore guarantees the existence of a societal equilibrium as long as p_i' in (12) is forced to lie between zero and one whenever $p_1, \dots, p_n$ do. A sufficient condition for $0 \leqslant p_i' \leqslant 1$ in (12) is that $| v_{i-1} - v_i | \leqslant \sigma^{-1}/2$ and $| v_i - v_{i+1} | \leqslant \sigma^{-1}/2$. We shall show that when n is large enough, this sufficient condition is satisfied. From (6),

$$r_{i-1} - r_i = \frac{1}{n}(-2 + 4p_{i-1} - 2p_{i-1}^2 - 4p_i + 2p_i^2),$$

hence $| r_{i-1} - r_i | \leqslant 4/n$ for $i = 2,\dots, n$. Denoting by Q_i^j the ith row of the matrix Q^j, we observe from the form of the matrix Q that

$$| Q_{i-1}^j r - Q_i^j r | \leqslant | r_{i-1-j} - r_{i+j} | \leqslant \frac{(2j+1)4}{n}$$

with the convention that $r_{n+j} = r_n$ and $r_{1-j} = r_1$ for $j \geqslant$

$$v_{i-1} - v_i \,| \leqslant \frac{4}{n} \sum_{j=0}^{\infty} \sigma^j (2j+1) \leqslant \frac{1}{2} \sigma^{-1}$$

for n large enough.

To prove that $n_1(\sigma) \to \infty$ as $\sigma \to 1$, we argue by contradiction. Accordingly, suppose that n is fixed and that a solution $p(\sigma)$, $v(\sigma)$ to (4) and (11) exists for a sequence of values σ which approach 1. Let us write (11) in the form

$$\begin{aligned} -2p_1(\sigma) &= \sigma\delta_2(\sigma)[v_2(\sigma) - v_1(\sigma)] \\ -2p_i(\sigma) &= [1 - \delta_i(\sigma)][v_i(\sigma) - v_{i-1}(\sigma)] + \delta_{i+1}(\sigma)[v_{i+1}(\sigma) - v_i(\sigma)], \\ & \qquad 2 \leqslant i \leqslant n-1 \\ -2p_n(\sigma) &= [1 - \delta_n(\sigma)][v_n(\sigma) - v_{n-1}(\sigma)]. \end{aligned} \tag{11a}$$

By multiplying both sides of the first equation by $(1 - \sigma)$, we see that

$$\delta_2(\sigma)(1 - \sigma)[v_2(\sigma) - v_1(\sigma)] \to 0$$

as $\sigma \to 1$. Now $\delta_2(\sigma) = \gamma_2(\sigma)/(1 + \gamma_2(\sigma) - \gamma_1(\sigma))$ cannot approach 0 as $\sigma \to 1$, since by (9) $\gamma_2(\sigma) \geqslant (n-1)/n \not\to 0$. Consequently $\lim_{\sigma\to1}(1 - \sigma)v_2(\sigma) = \lim_{\sigma\to1}(1 - \sigma)v_1(\sigma)$. The same reasoning shows that for $i = 1,\ldots, n$, $\{(1 - \sigma)v_i(\sigma)\}$ have a common limit as $\sigma \to 1$. But now from the first equation of (3) we find

$$v_1 = r_1 + \sigma(q_{11}v_1 + q_{12}v_2),$$

$$\begin{aligned} q_{11} &= \gamma_1(\sigma) + (1 - \gamma_1(\sigma))(1 - \delta_2(\sigma)) \\ q_{12} &= (1 - \gamma_1(\sigma))\delta_2(\sigma), \end{aligned}$$

so that

$$(1 - \sigma)v_1(\sigma) = r_1(\sigma) + \sigma(1 - \gamma_1(\sigma))\delta_2(\sigma)[v_2(\sigma) - v_1(\sigma)].$$

Using the first equation of (11a),

$$(1 - \sigma)v_1(\sigma) = r_1(\sigma) + (1 - \gamma_1(\sigma))(1 - 2p_1(\sigma));$$

whence, by (6) and (9),

$$\begin{aligned} (1 - \sigma)\, v_1(\sigma) &= 1 - \frac{1}{n} + \frac{1}{n}(4p_1(\sigma) - 2p_1^2(\sigma)) + (1 - 2p_1(\sigma))\,\frac{1 - p_1(\sigma)}{n} \\ &= 1 + \frac{p_1(\sigma)}{n}. \end{aligned}$$

Arguing similarly using the last equations of (3) and (11a) we find

$$(1 - \sigma)\, v_n(\sigma) = 3 - \frac{3}{n}[1 - p_n(\sigma)].$$

Since $0 \leqslant p_i(\sigma) \leqslant 1$, these values cannot approach a common limit as $\sigma \to 1$ for $n > 4$. This contradiction shows tha $n_1(\sigma)$ cannot remain bounded as $\sigma \to 1$.

Proof of Theorem 2. Existence of equilibrium follows analogously to the proof of Theorem 1. To show that $n_2(\sigma) \to \infty$ as $\sigma \to 1$ we can proceed initially as in Theorem 1. We find

$$(1-\sigma)\, v_1(\sigma) = 2 + p_1(\sigma)\left(2 - \frac{3}{n}\right)$$

$$(1-\sigma)\, v_n(\sigma) = (1 - p_n(\sigma))\left(2 + \frac{1}{n}\right),$$

hence the reasoning establishes only that $p_1(\sigma)$ must be near 0. However, the second equations of (3) and (11a), combined to eliminate $v_1(\sigma)$ and $v_2(\sigma)$, then show that $p_2(\sigma)$ cannot remain in its required range, and so produce the desired contradiction. We omit the details.

Proof of Theorem 3. From the formulas (6) and (7) we see that, whatever the values of $\{p_i\}$, the immediate payoff r_i is monotonic in i, nondecreasing for C1 and nonincreasing for C2. We now show that such monotonicity is preserved by the matrix Q, i.e., that for a vector x with components x_i monotonic in i, the components of Qx are likewise monotonic. As we have seen, when $1 < i < n$ the ith row of Q has the form

$$\begin{aligned} q_{i,i-1} &= \gamma_i(1-\delta_i), \\ q_{i,i} &= \gamma_i\delta_i + (1-\gamma_i)(1-\delta_{i+1}), \\ q_{i,i+1} &= (1-\gamma_i)\delta_{i+1} \end{aligned}$$

with the other entries vanishing. Hence the ith component of Qx is given by

$$\begin{aligned} &\gamma_i(1-\delta_i)x_{i-1} + [\gamma_i\delta_i + (1-\gamma_i)(1-\delta_{i+1})]x_i + (1-\gamma_i)\delta_{i+1}x_{i+1} \\ &\quad = \gamma_i[(1-\delta_i)x_{i-1} + \delta_i x_i] + (1-\gamma_i)[(1-\delta_{i+1})x_i + \delta_{i+1}x_{i+1}]. \end{aligned}$$

Now if the x_i are monotonically nondecreasing, say, we see that convex combinations of adjacent x_i are similarly ordered; i.e.,

$$(1-\delta_i)x_{i-1} + \delta_i x_i \leqslant (1-\delta_{i+1})x_i + \delta_{i+1}x_{i+1}\,.$$

The same reasoning now applies to the successive elements of Qx. A similar argument extends matters to the first and last components of Qx.

Formula (4) now shows that $\{v_i\}$ is nondecreasing in C1 and nonincreasing in C2, since $Q^j r = Q \cdot Q^{j-1} r$. Observing that in equilibrium p_i is given by (12), the monotonicity just established guarantees that $p_i \leqslant 1/2$ for C1 and $p_i \geqslant 1/2$ for C2 for all i. Consequently, the social loss $\sum_{i=1}^{n} \pi_i^2(1-p_i)^2 = (1/n^2)\sum_{i=1}^{n}(1-p_i)^2$ is smaller for C2 than for C1. When $\sigma = 0$, all p_i are 1/2 under both C1 and C2; but when $\sigma > 0$ this is no longer true and ordering of social loss can be seen to hold strictly. The comparisons of social loss for C2 and C3 and for C1 and C3 when $n \geqslant 5$ are immediate from this observation as well.

5. Second Variant of the Model

In the second variant of the model, the description of how reputations are transformed is much simpler than in the first variant. Whenever an individual in state i plays $\bar{Y}$, his next state is $(i + 1)$ when $i < n$ and n when $i = n$. Whenever an individual in state i plays Y, his next state is $(i - 1)$ when $i > 1$ and 1 when $i = 1$. For this variant, the transition probabilities for C1 and C2 can be written down immediately as functions of π and p.

For C1,

$$\begin{aligned}
q_{1,1} &= \quad -(1-p_1)\,\pi_1\,, & q_{1,2} &= (1-p_1)\,\pi_1\,; \\
q_{i,i-1} &= p_i\pi_i + \sum_{j>i} \pi_j\,, & q_{i,i+1} &= (1-p_i)\,\pi_i + \sum_{j<i} \pi_j\,, \quad \text{for } i = 2,\ldots, n-1; \\
q_{n,n-1} &= p_n\pi_n\,, & q_{nn} &= 1 - p_n\pi_n\,; \\
q_{ij} &= 0 & &\text{otherwise.}
\end{aligned}$$

For C2,

$$\begin{aligned}
q_{1,1} &= p_1\pi_1\,, & q_{1,2} &= \quad -p_1\pi_1\,; \\
q_{i,i-1} &= p_i\pi_i + \sum_{j<i} \pi_j\,, & q_{i,i+1} &= (1-p_i)\,\pi_i + \sum_{j>i} \pi_j\,, \quad \text{for } i = 2,\ldots, n-1; \\
q_{n,n-1} &= 1-(1-p_n)\,\pi_n\,, & q_{n,n} &= (1-p_n)\,\pi_n\,; \\
q_{ij} &= 0 & &\text{otherwise.}
\end{aligned}$$

With these respective transition matrices, we can state the conditions which must hold for a C1 or C2 societal equilibrium. There must be nonnegative vectors π, p, and v satisfying:

$$\begin{aligned}
&\sum_{i=1}^{n} \pi_i = 1, \\
&p_i \leqslant 1, \qquad i = \ ,\ldots, n, \\
&\pi = \pi Q, \\
&v = r + \sigma Q v,
\end{aligned}$$

where Q is as defined above for the two cases, respectively, and r is given by (6) and (7), respectively.

Lastly, we require

$$\begin{aligned}
&3p_1 + \sigma v_2 = 1 + p_1 + \sigma v_1\,, \qquad 0 \leqslant p_1 \leqslant 1 \\
&3p_i + \sigma v_{i+1} = 1 + p_i + \sigma v_{i-1}\,, \ 0 \leqslant p_i \leqslant 1, \quad i = 2,\ldots, n - \\
&3p_n + \sigma v_n = 1 + p_n + \sigma v_{n-1}\,, \ 0 \leqslant p_n \leqslant 1.
\end{aligned} \tag{13}$$

These conditions are completely analogous to (11). The same explanation and the same caveat apply. Since the transition rules do not impose fixed values for the vector π, it appears as an unknown here, together with the equation

$$\pi = \pi Q.$$

For this system, the analogs of Theorems and 2 have a somewhat different form. The value of n does not play a role.

THEOREM 4. *If* $\sigma \leqslant 1/5$ *there is a* C1 *societal equilibrium.*

THEOREM 5. *If* $\sigma \leqslant 1/4$ *there is a* C2 *societal equilibrium.*

Finally, the only comparison of the social values of the different equilibria we have for these transition rules is the following.

THEOREM 6. *For* $n \geqslant 6$ *and even and* σ *sufficiently small, the social value* $\pi \cdot v$ *for any* C1 *equilibrium exceeds that for any* C2 *equilibrium.* (*Both exceed the social value of the* C3 *equilibrium.*)

(We have proved the Theorem only for even n, but we believe that the result remains true for all n.)

By way of explanation here, since for σ sufficiently small the p_i approach 1/2, the social value depends only on how often two opponents have the same reputation. Although it is not intuitively obvious from the rules, it turns out that π, the equilibrium distribution of reputations, is more nearly uniform for C1 than for C2. Thus the chance for deadlock is lower in C1 than in C2, hence the social value is greater.

Although the results of this Section are strikingly incomplete, the ranking of C1 and C2 is reversed from what it had been for the same values of n and σ in Section 3. Thus the conclusions about what is best for society are shown to depend critically on the specification of the rules for reputation transitions.

6. PROOFS FROM SECTION 5

Proof of Theorem 4. Given any triple (π, p, v), where π is in the unit n-simplex, p is in the unit n-cube and v satisfies $0 \leqslant v_i \leqslant 3(1-\sigma)^{-1}$ for $i = 1,\ldots, n$, $|v_1 - v_2| \leqslant \sigma^{-1}$, $|v_{i-1} - v_{i+1}| \leqslant \sigma^{-1}$, $i = 2,\ldots, n-1$, and $|v_{n-1} - v_n| \leqslant \sigma^{-1}$, consider the transformation into the triple given below.

$\pi' = \pi Q$ (where Q is defined relative to π and p)

$v' = r + \sigma Q v$ (where Q and r are defined relative to π and p)

$$\begin{aligned} p_1' &= \tfrac{1}{2} + \tfrac{1}{2}\sigma(v_1 - v_2) \\ p_i' &= \tfrac{1}{2} + \tfrac{1}{2}\sigma(v_{i-1} - v_{i+1}), \qquad i = 2,\ldots, n - \\ p_n' &= \tfrac{1}{2} + \tfrac{1}{2}\sigma(v_{n-1} - v_n). \end{aligned} \tag{14}$$

Note that if $(\pi', p', v') = (\pi, p, v)$, then the system defines a societal equilibrium. Continuity of this transformation is obvious. In order to employ Brouwer's theorem to guarantee existence, we must show that π' is in the unit n-simplex, p' is in the unit n-cube,

$$0 \leqslant v_i' \leqslant 3(1-\sigma)^{-1} \qquad \text{for} \quad i = 1,\dots, n,$$

$$v_1' - v_2' \mid \leqslant \sigma^{-1}, \qquad v_{i-1}' - v_{i+1}' \mid \leqslant \sigma^{-1}, \qquad i = 2,\dots, n -$$

and

$$v_{n-1}' - v_n' \mid \leqslant \sigma^{-1}$$

Nonnegativity of π' is immediate. To see that $\sum_{i=1}^{n} \pi_i' = 1$, simply add the equations making up $\pi' = \pi Q$, obtaining

$$\sum_{i=1}^{n} \pi_i' = \left(\sum_{i=1}^{n} \pi_i\right) = 1$$

That $0 \leqslant v_i' \leqslant 3(1-\sigma)^{-1}$ is immediate from $v' = r + \sigma Q v$, the hypothesis that $0 \leqslant v_i \leqslant 3(1-\sigma)^{-1}$ all i, and the condition $0 \leqslant r_i \leqslant 3$. The inequalities in the absolute values of the v_i's guarantees that p' lies in the n-cube. There remain only the inequalities on the absolute values of the $\{v_i'\}$.

For $2 \leqslant i \leqslant n - 1$,

$$\begin{aligned} v_{i-1}' - v_{i+1}' = {} & 3 \sum_{j<i-1} \pi_j + \sum_{j>i-1} \pi_j + \pi_{i-1}(4p_{i-1} - 2p_{i-1}^2) + \sigma(q_{i-1.i}v_i + q_{i-1,i-2}v_{i-2}) \\ & - 3 \sum_{j<i+1} \pi_j - \sum_{j>i+1} \pi_j - \pi_{i+1}(4p_{i+1} - 2p_{i+1}^2) \\ & - \sigma(q_{i+1.i+2}v_{i+2} + q_{i+1.i}v_i), \end{aligned}$$

(where $q_{1.0} \equiv q_{1.1}$, $v_0 \equiv v_1$, $q_{n.n+1} \equiv q_{n.n}$ and $v_{n+1} \equiv v_n$

$$\begin{aligned} v_{i-1}' - v_{i+1}' = {} & \sigma(v_{i-2} - v_i) \sum_{j>i+1} \pi_j + \sigma(v_i - v_{i-2}) \sum_{j<i-1} \pi_j \\ & + \pi_{i-1}(\sigma p_{i-1} v_{i-2} - \sigma v_{i+2} + \sigma(1 - p_{i-1})\, v_i - 3 + 4p_{i-1} - 2p_{i-1}^2) \\ & + \pi_i(\sigma v_{i-2} - \sigma v_{i+2} - 3 + 1) \\ & + \pi_{i+1}(\sigma v_{i-2} - \sigma p_{i+1} v_i - \sigma(1 - p_{i+1})\, v_{i+2} + \quad - 4p_{i+1} + 2p_{i+1}^2). \end{aligned}$$

Hence, $| v_{i-1}' - v_{i+1}' | \leqslant 1 - \pi_{i-1} - \pi_i - \pi_{i+1} + \pi_{i-1}(5) + \pi_i(4) + \pi_{i+1}(3) \leqslant 5 \leqslant \sigma^{-1}$, since $\sigma \leqslant 1/5$. The arguments which show that $| v_1' - v_2' | \leqslant \sigma^{-1}$ and $| v_{n-1}' - v_n' | \leqslant \sigma^{-1}$ are completely analogous and will be omitted.

Proof of Theorem 5. The argument is analogous to the preceding one, the differences being in the form of Q and r. We shall verify below only the step $| v_{i-1}' - v_{i+1}' | \leqslant \sigma^{-1}$.

$$v'_{i-1} \quad v'_{i+1} = 3 \sum_{j>i-1} \pi_j + \sum_{j<i-1} \pi_j + \pi_{i-1}(4p_{i-1} - 2p_{i-1}^2) + \sigma(q_{i-1,i}v_i + q_{i-1,i-2}v_{i-2})$$

$$- 3 \sum_{j>i+1} \pi_j - \sum_{j<i+1} \pi_j - \pi_{i+1}(4p_{i+1} - 2p_{i+1}^2)$$

$$\sigma(q_{i+1,i+2}v_{i+2} + q_{i+1,i}v_i)$$

$$v_{i-1}' \quad v_{i+1}' \quad \sigma(v_{i-2} - v_i) \sum_{j<i-1} \pi_j + \sigma(v_i - v_{i+2}) \sum_{j>i+1} \pi_j$$

$$+ \pi_{i-1}(-\sigma v_i + \sigma p_{i-1}v_{i-2} + \sigma(1 - p_{i-1})\, v_i - \quad + 4p_{i-1} \quad 2p_{i-1}^2)$$

$$+ \pi_i(-\sigma v_i \quad \sigma v_i + 3 \quad 1)$$

$$+ \pi_{i+1}(-\sigma p_{i+1}v_i + \sigma v_i \quad \sigma(1 - p_{i+1})\, v_{i+2} + 3 \quad 4p_{i+1} + 2p_{i+1}^2).$$

Hence $| v'_{i-1} - v'_{i+1} | \leqslant \quad - \pi_{i-1} - \pi_i - \pi_{i+1} + \pi_{i-1}(2) + \pi_i(2) + \pi_{i+1}(4) \leqslant 4 \leqslant \sigma^{-1}$.

Proof of Theorem 6. We first consider the limiting case $\sigma = 0$. When $\sigma = 0$, $p_i = 1/2$ for each i by (13), and the equilibrium conditions reduce to the single equation (1). For C1 this becomes

$$\pi_1 = \pi_1(\tfrac{1}{2}\pi_1 + 1 - \pi_1) + \pi_2(\tfrac{1}{2}\pi_2 + 1 - \pi_1 - \pi_2)$$

$$\pi_i = \pi_{i-1}\left(\tfrac{1}{2}\pi_{i-1} + \sum_{j=1}^{i-2} \pi_j\right) + \pi_{i+1}\left(\tfrac{1}{2}\pi_{i+1} + 1 - \sum_{j=1}^{i+1} \pi_j\right),$$

$$\leqslant i \leqslant n - \tag{15}$$

$$\pi_n = \pi_{n-1}\left(\tfrac{1}{2}\pi_{n-1} + \sum_{j=1}^{n-2} \pi_j\right) + \pi_n(\tfrac{1}{2}\pi_n + \quad - \pi_n),$$

with

$$\sum^{n} \pi_j =$$

Since n is even, let $n = 2m$. The equations are then unchanged under the substitution $\pi_i = \pi_{n+1-i}$. Consequently we seek a symmetric solution by solving for the first m variables in the above system, subject to the requirement that

$$\sum_{1}^{m} \pi_j = 1/2. \tag{16}$$

By adding the first $(m - 1)$ of these equations we see that in the presence of (16) the m-th is automatically satisfied. We shall solve by adopting π_1 as a parameter, using the first equation to determine π_2, the second to determine π_3, and so on successively until the $(m - 1)$-st determines π_m; π_1 will then have to be chosen so that each $\pi_j \geqslant 0$ and (16) is satisfied.

Let us now introduce variables,

$$T_i = \sum_{j=1}^{i} \pi_j, \qquad i = 1, \ldots, m, \tag{17}$$

in terms of which the first $(m - 1)$ equations from (15) become

$$\begin{aligned}
-\pi_2 &= -T_1^2/2 - (T_2 - T_1)\left(\frac{T_2 + T_1}{2}\right) \\
\pi_2 - \pi_3 &= T_1^2/2 - (T_3 - T_2)\left(\frac{T_3 + T_2}{2}\right) \\
\pi_i - \pi_{i+1} &= (T_{i-1} - T_{i-2})\left(\frac{T_{i-1} + T_{i-2}}{2}\right) - (T_{i+1} - T_i)\left(\frac{T_{i+1} + T_i}{2}\right), \\
&\qquad 3 \leqslant i \leqslant m - 1.
\end{aligned}$$

Adding the first i of these equations, $i = 1, 2, \ldots, m -$ then yields

$$\begin{aligned}
2\pi_2 &= T_2^2 \\
2\pi_i &= T_i^2 - T_{i-2}^2, \qquad 3 \leqslant i \leqslant m.
\end{aligned}$$

It follows that π_i can be found from the quadratic equation

$$\pi_i^2 - 2\pi_i(1 - T_{i-1}) + T_{i-1}^2 - T_{i-2}^2 = 0, \qquad i = 2, \ldots, m,$$

where $T_0 \equiv 0$.

If $1/2 > T_{i-1} > T_{i-2}$, there are two positive roots; but the greater, $\pi_i = (1 - T_{i-1}) + (1 - 2T_{i-1} + T_{i-2}^2)^{1/2}$, is too large, since it yields $T_i = \pi_i + T_{i-1} > 1$. Thus the relevant solution is $\pi_i = 1 - T_{i-1} - (1 - 2T_{i-1} + T_{i-2}^2)^{1/2}$, and we conclude that the values of all π_i generated by (18) are in the range $0 < \pi_i < 1$. Moreover from (18)

$$\begin{aligned}
2\pi_i &= (T_i - T_{i-2})(T_i + T_{i-2}) \\
&= (\pi_{i-1} + \pi_i)(T_i + T_{i-2}),
\end{aligned}$$

so that

$$\begin{aligned}
\pi_i &= \pi_{i-1}\left(\frac{T_i + T_{i-2}}{2 - (T_i + T_{i-2})}\right), \qquad 3 \leqslant i \leqslant m \\
\pi_2 &= \frac{\pi_1 T_2}{2 - T_2}
\end{aligned} \tag{19}$$

Since, for $i < m$, $T_i < T_m = 1/2$, the factor $(T_i + T_{i-2})/(2 - (T_i + T_{i-2})) < 1$, hence, with π_1 fixed, the π_i decrease successively until π_m.

Next, let us consider the behavior of the solution of (18) as π_1 varies. Adding the first i of the equations of (18) gives

$$\begin{aligned} 2(T_2 - T_1) &= T_2^2 \\ 2(T_i - T_1) &= T_i^2 + T_{i-1}^2 - T_1^2, \qquad 3 \leqslant i \leqslant m, \end{aligned} \tag{20}$$

so that

$$\begin{aligned} T_2(2 - T_2) &= 2T_1 = 2\pi_1, \\ T_i(2 - T_i) &= T_{i-1}^2 + 2T_1 - T_1^2 = T_{i-1}^2 + \pi_1(2 - \pi_1), \qquad 3 \leqslant i \leqslant m \end{aligned}$$

Since $T(2 - T)$ is an increasing function of T for $T < 1$, we see from the first of these equations that T_2 increases with π_1, hence from the second T_3 likewise increases, and, successively, all the T_i increase. It then follows from (19) that each π_i likewise increases monotonically with π_1. Now as π_1 approaches zero each π_i generated by (18) evidently approaches zero, so that, with m fixed, T_m is small when π_1 is small. Consequently, by the monotonicity of T_m, there exists a value of π_1 for which $T_m = 1/2$. (We note in passing that (20) is valid also for $m < i \leqslant 2m = n$, so that the consequent monotonicity of T_n in π_1 also shows that there is only one choice of π_1 for which $T_n = 1$. Thus the solution of (15) is unique, and the assumed symmetry represents no loss of generality.) Finally, we can determine the limiting value of π_1 when $m \to \infty$. By (20), $2(T_m - T_1) = T_m^2 + T_{m-1}^2 - T_1^2$, and as $m \to \infty$, $T_{m-1} \to T_m = 1/2$, so that

$$-2\pi_1 \to 1/2 - \pi_1^2,$$

whence $\pi_1 \to 1 - (\sqrt{2}/2)$.

We have seen that the social loss generated is given by $\sum_{j=1}^{n} \pi_j^2(1 - p_j)^2 = \frac{1}{2} \sum_{j=1}^{m} \pi_j^2$. The monotonicity properties just established readily yield an upper bound for this quantity. Let Π_i be the limiting value of π_i as $m \to \infty$. With m fixed, let I be an integer, $2 \leqslant I < m$, and let us denote by τ_i, $1 \leqslant i \leqslant I$, the solution of (18) for which $T_I = 1/2$. Now from the monotonicity of the π_i, $\pi_i > \Pi_i$, $1 \leqslant i \leqslant m$ so that $\sum_{j=I+1}^{m} \pi_j = 1/2 - \sum_{j=1}^{I} \pi_j \leqslant \frac{1}{2} - \sum_{j=1}^{I} \Pi_j$. Since $\pi_j > 0$, each π_j separately must satisfy this bound, so that

$$\sum_{j=I+1}^{m} \pi_j^2 \leqslant \Big(\sup_{I+1 \leqslant j \leqslant m} \pi_j \Big) \sum_{j=I+1}^{m} \pi_j \leqslant \left(\frac{1}{2} - \sum_{j=1}^{I} \Pi_j \right)^2.$$

Again by monotonicity, $\pi_j \leqslant \tau_j$, $\leqslant j \leqslant I$. Consequently, for C1

$$\sum_{j=1}^{m} \pi_j^2 \leqslant \sum_{j=1}^{I} \tau_j^2 + \left(\frac{1}{2} - \sum_{j=1}^{I} \Pi_j \right)^2. \tag{21}$$

For C2 we analyze the model along similar lines. The equation (1) to be solved now reads

$$\begin{aligned}
\pi_1 &= \pi_1\left(\frac{\pi_1}{2}\right) + \pi_2\left(\frac{\pi_2}{2} + \pi_1\right) \\
\pi_i &= \pi_{i-1}\left(\frac{\pi_{i-1}}{2} + \sum_{j=i}^{n} \pi_j\right) + \pi_{i+1}\left(\frac{\pi_{i+1}}{2} + \sum_{j=1}^{i} \pi_j\right), \qquad 2 \leqslant i \leqslant n - \qquad (22) \\
\pi_n &= \pi_{n-1}\left(\frac{\pi_{n-1}}{2} + \pi_n\right) + \pi_n\left(\frac{\pi_n}{2}\right)
\end{aligned}$$

with

$$\sum_{j=1}^{n} \pi_j =$$

Letting $n = 2m$, we again invoke symmetry to deal with half the system. This represents no loss of generality as the solution is again unique. We omit the argument. Renumbering the variables to run from the middle out, that is, making the substitution

$$\pi_{m+1-j} = \omega_j \,,$$

we obtain the equations

$$\begin{aligned}
\omega_1 &= \omega_1\left(\frac{\omega_1}{2} + \frac{1}{2}\right) + \omega_2\left(\frac{\omega_2}{2} + \frac{1}{2} + \omega_1\right), \\
\omega_i &= \omega_{i-1}\left(\frac{\omega_{i-1}}{2} \quad \frac{1}{2} - \sum_{j=1}^{i-1} \omega_j\right) + \omega_{i+1}\left(\frac{\omega_{i+1}}{2} + \frac{1}{2} + \sum_{j=1}^{i} \omega_j\right), \\
&2 \leqslant i \leqslant \\
\omega_m &= \omega_{m-1}\left(\frac{\omega_{m-1}}{2} + \frac{1}{2} - \sum_{j=1}^{m-1} \omega_j\right) + \omega_m\left(\frac{\omega_m}{2}\right),
\end{aligned}$$

and

$$\sum_{j=1}^{m} \omega_j = 1/2.$$

Again, in the presence of this last condition, we may discard the last equation of (23) as redundant. Next let us introduce the variables

$$\begin{aligned}
S_0 &= 1/2 \\
S_i &= 1/2 + \sum_{j=1}^{i} \omega_j \,, \qquad \leqslant i \leqslant m,
\end{aligned}$$

in terms of which (23) and (24) become

$$\omega_1 = (S_1 - S_0)\left(\frac{S_1 + S_0}{2}\right) + (S_2 - S_1)\left(\frac{S_2 + S_1}{2}\right),$$

$$\omega_i - \omega_{i-1} \quad -(S_{i-1} - S_{i-2})\left(\frac{S_{i-1} + S_{i-2}}{2}\right) + (S_{i+1} \quad S_i)\left(\frac{S_{i+1} + S_i}{2}\right), \quad 2 \leqslant i \leqslant m - 1,$$

$$S_m$$

By adding successively the first i of these equations, $\leqslant i \leqslant m$, we now find

$$2\omega_1 = S_2^2 - S_0^2$$

$$2\omega_i = S_{i+1}^2 - S_{i-1}^2, \qquad 2 \leqslant i \leqslant m$$

Adding successively once more yields

$$S_2^2 = \tfrac{1}{4} + 2\omega_1$$

$$2(S_i - 1/2) = -S_0^2 \quad S_1^2 + S_i^2 + S_{i+1}^2, \qquad 2 \leqslant i \leqslant m -$$

We see from (25) that S_2 increases with the chosen ω_1. Since

$$S_{i+1}^2 = S_i(2 - S_i) + S_1^2 \quad 3/4, \qquad 2 \leqslant i \leqslant m -$$

and since $S(2 - S)$ is an increasing function of S for $S < 1$, we see that each S_i likewise increases with ω_1, providing only that $S_{i-1} < 1$. Moreover, from the first equation of (25),

$$\omega_2 = -\omega_1 + ((8\omega_1 + 1)^{1/2} - 1)/2.$$

So long as $\omega_1 + \omega_2 < 1/2$, we see from this expression that $(8\omega_1 + 1)^{1/2} < 2$, whereupon $d\omega_2/d\omega_1 > 0$. Hence ω_2 likewise increases with ω_1. (We remark that this monotonicity does not hold for the remaining ω_i.) We also note from (25) that

$$\omega_{i+1} = \omega_i \frac{(2 - (S_i + S_{i-2}))}{S_i + S_{i-2}}$$

so that, with ω_1 fixed, the ω_i decrease rapidly as $S_i \to 1$. Finally, we can again solve for the limiting value of ω_1 as $m \to \infty$; for, by (26), if $S_{m-1} \to S_m \to 1$ we obtain $\omega_1 \to (\sqrt{3} - 1)/2$.

We can convert these considrations into a lower bound for $\sum_{j=1}^{m} \omega_j^2$. We know that each S_i generated from the starting value $(\sqrt{3} - 1)/2$ will be smaller than the limit $S_\infty = 1$. Thus if $\omega_1, \ldots, \omega_m$ solve (23), we necessarily have $\omega_1 > (\sqrt{3} - 1)/2 > .366$, whence by (27) and the monotonicity of ω_2 as a function of ω_1 we find $\omega_2 > .124$. Consequently

$$\sum_{j=1}^{m} \omega_j^2 > (.366)^2 + \quad 124)^2 > .149. \qquad (28$$

To conclude our comparisons of social loss for C1 and C2 we use (21) and (28). We find $\Pi_1 \cong .2929$, $\Pi_2 \cong .0635$, $\Pi_3 \cong .0329$, and solving (15) directly for $m = 3$, $\pi_1 = .3390$, $\pi_2 \cong .0935$, $\pi_3 \cong .0676$. Thus

$$\sum_{j=1}^{3} \pi_j^2 < .1283. \tag{29}$$

Then, by (21),

$$\sum_{j=1}^{m} \pi_j^2 < .1283 + (.5 - .3893)^2 < .1405$$

This bound and (28) show that C1 has a lower social loss than C2 for all $m > 3$. By (29) this is true also for $m = 3$.

We have thus established Theorem 6 for the limiting case $\sigma = 0$. However, since the model and all its parameters are continuous in σ as $\sigma \to 0$, the Theorem remains valid for σ in some neighborhood of zero as well.

7. DISCUSSION

This section is devoted to some remarks about the model.

The principal goal of this study was to examine the role of reputation in the context of a dynamic model of bargaining, with specific attention to the trade-offs between present-period returns and effect on reputation. Modeling reputation as an integer-valued variable seemed to be the simplest way to achieve an ordering of reputations from high to low. In the second variant of the model, fixing the number of reputation states at some finite number may be seen as artificial, but it is necessary in order to derive any steady-state behavior for C1. A simple alternative approach would be to allow for an infinite state space but to have individuals die with some fixed probability, independent of their state, at each time, and be replaced in the next period by new-born individuals, all new-borns beginning in the same initial reputation state. Such an alternative permits steady-state analysis, and it is a relatively simple matter to show that if the death probability is sufficiently high, both C1 and C2 equilibria exist. Here, raising the death probability plays a role analogous to lowering the discount factor in an individual's decision problem.

Another interesting variant of the model arises with state space $\{0, \ldots, n\}$ and the following transition rules. If an individual's state at t is i and he plays $\bar{Y}$, then his state at $(t + 1)$ becomes $(i + 1)$ with probability $(n - i)/n$ and i with probability i/n. If he plays Y his state becomes $(i - 1)$ with probability i/n and i with probability $(n - i)/n$. For this variant one may think of a state as being a rough measure of how many times out of the last n games an individual played $\bar{Y}$. The new choice of Y or $\bar{Y}$ replaces the one which occurred $(n + 1)$ time units in the past, and with probability i/n (in the absence of any other information) the play being replaced was $\bar{Y}$. (Of course, keeping track of exactly how many times in the last n an individual played $\bar{Y}$ would lead to a much more complicated model.)

Given that the game in Section 1 is an extremely simple version of a bargaining situation, one might wish to consider the effects of various kinds of changes in it. One possibility is to posit a larger set of strategies, admitting a richer set of bargaining compromises and a richer class of interesting customs which could be candidates for equilibria. Another possibility is to vary the parameters in the 2 × 2 game. If one wishes to maintain the feature that the sum of the payoffs in three of the cells is 4 and the symmetry of the game, one might wish to explore the game below

$2, 2$	$1 - z, 3 + z$
$3 + z, 1 - z$	$0, 0$

where $-1 < z < 1$.

We have not pursued any of the possibilities raised above. What limited results we have obtained for the two variants of the model studied seem to indicate that all results are likely to be sensitive to changes in the model; and it does not seem to us that we know of a single "right" version of the model.

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