

Game Theory Fall 2003

Problem Set 3

[1] Recall problem [12] in problem set 1, but restrict to the case of two agents and a particular production function:

Two agents are engaged in joint production: output $Y = F(e_1 + e_2)$, where e_i represents the resources put in by agent i . Agent 1 gets a share λ of the output and agent 2 gets the rest; these shares are consumed. Suppose that agent 1 can move first, committing to e_1 , after which agent 2 chooses e_2 . Each agent seeks to maximize $c_i - e_i$, where c_i stands for consumption.

Describe the subgame perfect equilibrium efforts of this game as λ varies between 0 and 1. Contrast with the simultaneous-move version.

[2] (From FT and Rabin (1988)): Here is an informal description of a sequential game. There are three players. First player 1 moves A or B . If she moves A , the game is over with payoffs (6,0,6). If she chooses B , player 2 gets to move, choosing C or D . If she chooses C the game is over with payoffs (8,6,8). If she chooses D players 1 and 3 play a simultaneous-move coordination game (payoffs given are for all three players, of course, even though 2 is not active in this subgame):

	E	F
G	7, 10, 7	0,0,0
H	0,0,0	7, 10, 7

- (a) Draw a game tree to represent this game.
- (b) Prove that in every subgame perfect equilibrium of this game (including those in which behavior strategies are used), player 1 plays B at the beginning.
- (c) Describe a situation in which players 1 and 2 both predict Nash equilibria in subgames, *but different ones*, which justifies player 1 choosing A . What sort of assumption rules this out in equilibrium?

[3] Consider the following situation. A risk-neutral individual with income y may decide to pay taxes t or evade them. Assume that if the individual evades taxes and is audited, he pays a fixed fine F . An auditor can choose whether or not to audit the individual. Set up game trees for three situations: (i) the auditor can precommit whether or not he will audit, and the individual knows this before making his evasion decision; (ii) the auditor can precommit a *probability* of audit, and the individual knows this probability before making his evasion decision; (iii) no precommitment is possible. In each case be very careful in defining the strategies.

Assume that the cost of auditing with probability p is $c(p)$. [Define $c(1)$ to be the cost of auditing for sure.] Solve for the subgame perfect equilibria in the three cases.

[4] Draw the game tree for a three-period version of the centipede problem described in class, in which each player is either a rational type or is a “passing type”.

[5] Describe the set of all Nash equilibria in the centipede game studied in class, and prove that they all yield the same payoffs.

[6] Consider the following simple model of entry. A monopolist faces a sequence of N entrants. At each stage an entrant can choose to enter or not enter a market. At the end of that stage the entrant dies, to be replaced by another potential entrant who has seen all that has gone before.

Payoffs in each period are as follows. If an entrant decides to stay out, the monopolist earns a monopoly profit of 2, and the entrant earns nothing. If an entrant decides to enter, the monopolist can *fight* the entrant, or *accommodate* her. If the former, payoff is 0 to the monopolist and -1 to the entrant. If the latter, payoffs are 1 each.

(a) Set this up as an extensive form game and describe the subgame perfect equilibrium of the game.

(b) Now do what we did with the centipede game in class. Suppose that with probability $\epsilon > 0$, the monopolist is a “fighter”: she will fight any entrant. What does the equilibrium look like now? You should examine the outcome for large values of N just as we did for the centipede.

[7] [Technical but good practice.] Provide an example of an extensive game failing perfect recall in which no Nash equilibrium exists in behavior strategies.

[8] [OR Exercise 101.3.] “Armies 1 and 2 are fighting over an island initially held by a battalion of Army 2. Army 1 has K battalions and army 2 has L . Whenever the island is occupied by one army the opposing army can launch an attack. The outcome of the attack is that the occupying battalion and one of the attacking battalions are destroyed; the attacking army wins and, so long as it has battalions left, occupies the island with one battalion. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth more than one battalion but less than two. (If, after an attack, neither army has any battalions left, then the payoff of each commander is 0.) Analyze this situation as an extensive game and, using the notion of subgame perfect equilibrium, predict the winner as a function of K and L .”

[9] In the Cournot duopoly, it pays for one of the two agents to move first. Discuss the truth of this assertion for general two-player simultaneous move games.

[10] Suppose that the following two-person prisoners’ dilemma

	L	R
U	2,2	0,3
D	3,0	1,1

is repeated N times, with total payoffs equal to the sum of the payoffs received in each period.

- (a) Describe the set of all (pure strategy) Nash and subgame perfect equilibria of this game.
 (b) Do the same with the following variation ...

	L	R	X
U	2,2	0,3	-1, -5
D	3,0	1,1	-1, -5
Y	-5, -1	-5, -1	-5, -5

- (c) ... and the same with this variation:

	L	R	X
U	2,2	0,3	-10, -5
D	3,0	1,1	-10, -5
Y	-5, -10	-5, -10	-5, -5

[11] Two players A and B take turns at proposing a division of a cake of unit size. First A makes an offer. If B accepts, we are done. Otherwise B rejects, and a unit of time passes, during which both players discount their utility. This process continues for N periods. If an agreement (a, b) is reached at date t (with $a = b = 1$, presumably), then player A 's payoff is $\alpha^t a$ (α is player A 's discount factor), and player B 's payoff is $\beta^t b$ (β is player B 's discount factor).

- [a] Assuming that both α and β are less than one, find the subgame perfect equilibria of this game and discuss what happens as $N \rightarrow \infty$.
 [b] Explain what happens to this limit result when $\alpha = \beta = 1$.

[12] [Holdup.]

- (a) A sharecropper puts in effort e on the land and produces output using smooth, differentiable, concave $F(e)$. After the effort is sunk, and the output is produced landlord and sharecropper bargain over the output (there is no ex-ante contract), eventually sharing it 50-50. The sharecropper's payoff function is output minus effort. Prove that the subgame-perfect equilibrium of this game is not Pareto-optimal.
 (b) A union makes a take-it or leave-it offer to firm concerning wage rates. The firm produces output $F(L)$ (same assumption as in (a)) employing labor L . Given the wage rate, the firm chooses employment levels. The firm can always reject the wage offer and earn an "outside option" of π using nonunion labor. Describe the subgame perfect equilibrium of this game, and show once again that it creates Pareto inefficiency.
 (c) What are the sources of inefficiency in these examples?

[13] Osborne’s Conjecture.¹ There is a set of positions, the unit interval $[0, 1]$, over which a uniformly distributed continuum of citizens have ideal points. Each of candidates $1, \dots, n$ chooses a member of the set $[0, 1] \cup \text{OUT}$ (i.e. either chooses a “position” or opts out). The choices are made sequentially (starting with player 1), and every player is perfectly informed at all times. The outcome of the game is determined as follows. After all players have chosen their actions, each player who has chosen a position receives votes; the player who receives the most votes wins. A player who chooses the same position y as $k - 1$ other players obtains the fraction $1/k$ of the votes of all the citizens whose ideal points are closer to y than to any other chosen position. (In particular, voting is “sincere”.) Each player obtains the payoff 0 if she chooses OUT, the payoff $1/k$ if she is among the k players who receive the maximal fraction of votes, and -1 otherwise. (That is, each player wants to enter if and only if she has some chance of winning.)

In every subgame perfect equilibrium clearly every player who enters (chooses a position) obtains the same fraction of votes (if a player loses outright, then she could have done better by not entering).

Conjecture: The game has a unique subgame perfect equilibrium outcome, in which players 1 and n choose the median position $1/2$ and all other players choose OUT.

This conjecture is consistent with “Duverger’s law”, which states in part that plurality rule elections foster a two-party system.

(a) Examine this result when $n = 2$ and $n = 3$.

(b) [Optional!] Go ahead, make my day. Establish the conjecture or prove it wrong by example.

¹See <http://www.chass.utoronto.ca/~osborne/research/CONJECT.HTM>, which is where I’ve taken this from.