Game Theory Fall 2003

Problem Set 1

[1] In this problem (see FT Ex. 1.1) you are asked to play with arbitrary 2×2 games just to get used to the idea of equilibrium computation. Specifically, consider the following game:

	L	R
U	a, b	c, d
D	e, f	g,h

To test whether a pair, say (U, L), is a pure strategy Nash equilibrium is a trivial exercise. For instance, you would simply have to check whether or not the inequalities $a \ge e$ and $b \ge d$ hold. The point of the exercise, however, is to draw reaction curves for understanding *mixed* strategy equilibria. To this end, let p be the probability that player 1 plays U and q the probability that player 2 plays L, and do the following with important locations on the axes properly computed and labeled:

(i) Plot each person's best response correspondence as a function of the other's mixed strategy (as summarized by the randomization probability). Pay special attention to the points at which the player is indifferent between her own two pure strategies.

(ii) For which parameters does each player have a strictly dominant strategy?

(iii) Show that if neither player has a strictly dominant strategy and the game has a unique equilibrium, the equilibrium *must* be in mixed strategies. Make sure you understand why both premises are assumed for the result.

[2] Find all equilibria of the following games:

	L	R
U	0, 0	2, 1
D	1, 2	0, 0

Battle of the Sexes

	L	R
U	-1, -1	2, 1
D	1, 2	0, 0

"Hawk-Dove" or "Chicken"

	L	R
U	3, 3	0, 0
D	0, 0	1, 1

Coordination

	L	R
U	10, 10	0,9
D	9,0	7,7

Risky Coordination

	L	R
U	1, -1	3,0
D	4, 2	0, -1

FT Ex. 1.1

 $\mathbf{2}$

	L	M	R
U	1, -2	-2, 1	0, 0
N	-2, 1	1, -2	0, 0
D	0, 0	0,0	1, 1

FT Ex. 1.6

[3] (i)Prove that if a pure strategy s_i is dominated under the definition:

There is $\sigma_i \in \Sigma_i$ such that $f(\sigma_i, s_{-i}) > f(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$,

then it is also dominated under the stronger requirement

There is $\sigma_i \in \Sigma_i$ such that $f(\sigma_i, \sigma_{-i}) > f(s_i, \sigma_{-i})$ for all $\sigma_{-i} \in \Sigma_{-i}$.

(ii) Are these two definitions equivalent? Standard one: A pure strategy s_i is dominated if

There is $\sigma_i \in \Sigma_i$ such that $f(\sigma_i, s_{-i}) > f(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

Alternative one: A pure strategy s_i is dominated if

There is $s'_i \in S_i$ such that $f(s'_i, s_{-i}) > f(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

[4] Prove that a mixed strategy is dominated if it assigns positive probability to some pure strategy that is dominated.

Why is the converse false?

[5] Consider the definition of strict iterated definition used in class. Set $S_i^0 \equiv S_i$ for each i, and then recursively, having defined S_j^k for all j, for some k, define \mathcal{M}

 $S_i^{k+1} \equiv \{s_i \in S_i^k | \not\exists \sigma_i \in \mathcal{M}(S_i^k) \text{ s.t. } f(\sigma_i, s_{-i}) > f(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k\},\$

where $\mathcal{M}(X)$ is the set of all probabilities over the set X. Prove that this recursion gives the same result as

$$S_i^{k+1} \equiv \{ s_i \in S_i | \not\exists \sigma_i \in \Sigma_i \text{ s.t. } f(\sigma_i, s_{-i}) > f(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k \}.$$

[6] Is a game with a unique pure-strategy Nash equilibrium solvable by iterated dominance?

[7] A strategy s_i is weakly dominated if

There is $\sigma_i \in \Sigma_i$ such that $f(\sigma_i, s_{-i}) \ge f(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

with strict inequality for some $s_{-i} \in S_{-i}$. It seems natural to eliminate weakly dominated actions, but notice that here we are not on very firm ground: a weakly dominated strategy is a best response to some belief.

[8] If you do insist on eliminating weakly dominated strategies, notice that the procedure of iterated elimination is not as robust as in the case of strict domination. In the following game (OR Fig. 63.1):

	L	R
U	1, 1	0, 0
N	1, 1	2, 1
D	0, 0	2, 1

show that the order in which weakly dominated strategies are eliminated affects the equilibrium that remains. Reflect on why this does not happen when strictly dominated strategies are eliminated.

[9] Maxmin and Zero-Sum Games.

Consider a two-person game.

A strategy σ_i^* is maxmin if it yields the highest possible payoff for player *i* under the assumption that player *j* is choosing her strategy to minimize *i*'s payoff. That is, σ_i^* maximizes

$$\min_{\sigma_j} f_i(\sigma_i, \sigma_j),$$

or equivalently,

$$\sigma_i^* \in \arg\max_{\sigma_i} \min_{\sigma_j} f_i(\sigma_i, \sigma_j).$$

(i) Does it matter whether we restrict σ_i to be a pure strategy?

(ii) Prove that $\max_{\sigma_i} \min_{\sigma_j} f_i(\sigma_i, \sigma_j) \leq \min_{\sigma_j} \max_{\sigma_i} f_i(\sigma_i, \sigma_j)$, and give an example in which the inequality is strict.

A two-person game is zero-sum if $f_1(s) + f_2(s) = 0$ for all $s \in S$ (so that $f_1(\sigma) + f_2(\sigma) = 0$ for all $\sigma \in \Sigma$).

(iii) Prove that if (σ_1^*, σ_2^*) is a mixed-strategy Nash equilibrium for a zero-sum game, then σ_i^* is maxmin for each *i*.

(iv) As a corollary, show that in this case the inequality in part (ii) holds with equality:

$$\max_{\sigma_i} \min_{\sigma_j} f_i(\sigma_i, \sigma_j) = \min_{\sigma_j} \max_{\sigma_i} f_i(\sigma_i, \sigma_j) = f_i(\sigma_i^*, \sigma_j^*)$$

and so, in particular, all Nash equilibria of a zero-sum game have exactly the same payoffs.

To be sure, we can prove existence for zero-sum games with an infinite number of pure strategies just as we did in class. The strategy sets can be compact convex subsets of \mathbb{R}^n

and the payoff functions continuous and quasi-compact in own strategies. Then versions of all the above hold for pure strategies. We will use this in an interesting way in the next problem.

[10] Supporting Hyperplane Theorem.

Let X be a compact convex set in \mathbb{R}^n_+ , and suppose that $\hat{x} \in X$ has the property that there is no $x \in X$ with $x \gg \hat{x}$. Then there exists a vector \hat{p} in the nonnegative unit simplex (the set of all vectors p with nonnegative coordinates and summing to unity over the components) such that

$$\hat{p}\hat{x} \ge \hat{p}x$$

for all $x \in X$.

This is a famous theorem which you've surely seen. We can derive this theorem from zerosum games. Define a two player game by setting the first player's strategy set to be X and the second player's strategy set to be Y. Let $f_1(x, y) = x \cdot y - \hat{x} \cdot y$ and $f_2(x, y) = -[x \cdot y - \hat{x} \cdot y]$. Then this is a zero-sum game. Use the previous problem to argue that a solution (x^*, y^*) exists in pure strategies. Now prove that the solution must have $f_1(x^*, y^*) = 0$. Finally, use the previous problem again to argue that this establishes the separating hyperplane theorem.

The problems that follow are a few applications where you learn to play with the idea of a Nash equilibrium.

[11] A Model of Political Competition.

There are *n* candidates running for political office. Each announces a *position* on [0, 1]. Citizens have preferences over positions, with ideal points distributed continuously (with no atoms) over [0, 1]. Faced with a choice between the candidates, a citizen votes for the one who is nearest to his idea point (and randomizes over the favorites if there are more than one). The candidate with the most votes wins. Set up this model precisely as a game, and prove that there is a pure-strategy Nash equilibrium when n = 2. What happens if n > 2?

[12] Joint production of a private good with varying property rights.

A group of *n* agents are engaged in joint production: output $Y = F(\mathbf{e})$, $\mathbf{e} = (e_1, \ldots, e_n)$ is a (nonnegative) vector of *efforts*. Each agent *i* seeks to maximize $c_i - e_i$, and supplies effort independently and noncooperatively.

A sharing rule for output is given by a nonnegative vector of shares $\ell \equiv (\lambda_1, \lambda_2, \lambda_n)$, which sum to unity.

[A] Describe precisely (without solving) an equilibrium of this joint activity.

An effort vector $\hat{\mathbf{e}}$ is *efficient* if it maximizes the expression

$$\hat{S} \equiv F(\mathbf{e}) - \sum_{i=1}^{n} e_i$$

over all possible effort vectors. Assume that appropriate end-point conditions hold so that the maximization problem above is well-defined. The surplus associated with any equilibrium \mathbf{e}^* is, likewise, $S^* \equiv F(\mathbf{e}^*) - \sum_{i=1}^n e_i^*$. Take $\hat{S} - S^*$ to be a measure of the inefficiency of an equilibrium.

[B] Suppose that F is an increasing, concave function of the sum of efforts: $F(\mathbf{e}) = f(\sum_{i=1}^{n} e_i)$ for some increasing differentiable concave f satisfying the Inada endpoint conditions. For any sharing rule ℓ , let $M(\ell) \equiv \max_i \lambda_i$. Show that if $M(\ell) > M(\ell')$, then the inefficiency under ℓ is lower than that under ℓ' , and that equilibrium inefficiency converges to zero as ℓ converges to any of the unit vectors.

[C] Suppose that F is an increasing concave function of the scale of activity, where scale is determined by *equi-proportional* contribution of efforts: $F(\mathbf{e}) = f(\min_i e_i)$ for some increasing differentiable concave f satisfying the Inada endpoint conditions. Let $m(\ell) \equiv \min_i \lambda_i$. Prove that there is a continuum of equilibria for each possible sharing rule ℓ , with $m(\ell) > 0$, and describe the one with the lowest degree of inefficiency.

[D] Focus on the least inefficient of the equilibria described in part [B]. Show that if $m(\ell) > m(\ell')$, then the inefficiency under ℓ is lower than that under ℓ' , but that inefficiency is never equal to zero.

[13] Work Points.

A family farm with n members produces a joint output using an increasing, smooth, strictly concave production function f(R), where R denotes the sum of individual efforts r_i . Each individual has a utility function u(c) - v(r), where c is his consumption and r is his effort. Assume that u and v are increasing and smooth, and that u is strictly concave while v is strictly convex.

The purpose of this question is to investigate two different reward systems: work points and equal sharing, and how they perform relative to the first best.

[A] Describe precisely the solution to the social planner's problem (in which all utilities are added up). Prove that it must involve equal effort and equal sharing of the output.

[B] Now suppose that r_i is chosen independently by agent *i*, under the assumption that the total output will be equally divided. write down the conditions characterizing a symmetric (interior) equilibrium. Compare these values with the first-best, and explain why they are different (and in which direction).

[C] Find out what happens to total contributions in this model when n is reduced (go ahead, take derivatives even though n is an integer). Under what conditions might it go up? Provide a verbal explanation.

Imagine, now, that effort decisions are taken selfishly. The following incentive scheme is in place. A fraction β of the output will be divided equally, and the remainder $1 - \beta$ allocated according to "work points". That is, individual *i* gets a share $\frac{e_i}{E}$. In what follows, continue to look only at symmetric Nash equilibria, where everybody puts in the same effort.

[D] Under this sharing rule, describe what happens (relative to the first best) as β varies from 0 to 1, and provide intuition. Find a value of β such that the Nash equilibrium of the game coincides with the first best.

[E] Note that in equilibrium, all players share the output equally anyway regardless of the value of β . Explain intuitively why it is that we get different results for different values of β , despite this fact.

[14] Voluntary Contributions to a Pure Public Good.

A community of n individuals produces two goods. Individual i has resources w_i , which must be divided between a private good c_i and contributions to a public good r_i . The contributions together produce a pure public good g, according to the production function g(r), where ris the *sum* of all the individual contributions r_i .

Each person has an identical utility function u(c) + v(g), where these functions have all the usual properties (smoothness, strict concavity, unbounded steepness at zero).

Contributions are made selfishly: each individual takes as given the sum of all other contributions and maximizes with respect to his own.

[A] Prove that every individual who makes a positive contribution must get *the same* utility in equilibrium, regardless of wealth. Explain this result verbally.

[B] Now suppose that there are only two individuals. Take both u and v to be logarithmic, and $g(r) = \sqrt{r}$. Find a critical ratio of wealth levels such that if individual wealths are less dispersed than this ratio, then the conditions of [A] hold.

[C] Mancur Olson has argued that higher inequality may sometimes be better for the provision of public goods, because a greater portion of the marginal gains from the public good is internalized by the rich. Assuming two individuals and using parts [A] and [B], evaluate the Olson argument.

[D] Discuss whether the results in part [C] are robust with respect to dropping the *additive* specification of inputs in the production function for g.