Game Theory Fall 2003

Problem Set 5

[1] If δ is close enough to zero then every subgame perfect equilibrium must involve the play of a static Nash equilibrium after every t-history, if the number of actions is finite. To see this, fix any action profile a that is not a Nash equilibrium. Let $d(a) \equiv \max_i [d_i(a) - f_i(a)]$, where $d_i(a)$ is the maximum payoff to player i assuming all others are playing a_{-i}. Then $d(a) > 0$, because a isn't a Nash profile. Let $d \equiv \min d(a)$, where the minimum is taken over all profiles a that are *not* one-shot Nash. Because there are only finitely many action profiles, $d(a)$ must be positive. Let M and m be the maximum and minimum payoffs available to anyone in the game. Now pick δ close enough to zero such that

$$d > \frac{\delta}{1-\delta}(M - m).$$

It is easy to see that for such δ there is no other subgame perfect equilibrium than the play of a one-shot Nash in every period.

This conclusion may be false if there are infinitely many actions available to each player. Take this two-person game in which each $A_i = [0, 1]$, and the payoff to player i from (a_i, a_j) is $2a_j - a_i^2$. Then each i's dominant strategy is to set $a_i = 0$, but of course, it is easy to check that all symmetric actions (a, a) yield payoffs that are increasing in a (on $[0, 1]$). For any discount factor, some $a > 0$ can be supported if

$$2a(1 - \delta) \leq 2a - a^2.$$

It is easy to see that no matter how close δ is to zero, there is some positive value of a (depending on δ, of course), that can be supported.

[2] (a) Not exactly correct. If the worst punishment to player i does not entail her playing a best response (statically), then player i's continuation payoff must be strictly higher than the worst punishment. Let p^i denote the vector that punishes player i and let $(a, p', \hat{p}^1, \ldots, \hat{p}^n)$ be any supporter of p^i. Then by the condition of support,

$$p_i^i = (1 - \delta) f_i(a) + \delta p_i',$$

while

$$p_i^i \geq (1 - \delta) d_i(a) + \delta \hat{p}_i^i$$

Combining the two and using the fact that $d_i(a) > f_i(a)$, we see that $p_i' > \hat{p}_i^i \geq p_i^i$.

(b) The player can alway play a static best response to the action profile at any date, for every history. This will guarantee her at least her security level in every period and therefore over her lifetime.

[3] (a) Take any $p \in F^*$. Then there exist convex weights $\lambda_1, \ldots \lambda_m$ (where m is finite) and m action profiles $a^1, \ldots a^m$ such that

$$p = \sum_{k=1}^{m} \lambda_k f(a^k).$$

Because any system of weights can be approximated by a set of rational weights, for every $\epsilon' > 0$ there exists an integer N and numbers $s_1, \ldots, s_m$ adding to N such that

$$|\left(\frac{s_1}{N}, \ldots, \frac{s_m}{N}\right) - \lambda| < \epsilon'.$$

Consequently, for every $\epsilon'' > 0$ if I define

$$p'' = \sum_{k=1}^{m} \frac{s_k}{N} f(a^k),$$

then

$$(1) \qquad |p'' - p| < \epsilon''.$$

Now consider the repeated game, and play the action profiles $a^1, \ldots a^m$ in sequence, playing a^k s_k-many times, for a total of N plays. Repeat this cycle forever. For each δ, let $p(\delta)$ be the normalized lifetime payoff thus generated. It is easy to see that

$$(2) \qquad p(\delta) \to p'' \text{ as } \delta \to 1.$$

Combining (1) and (2), we are done.

[4] Properties of the "support mapping" ϕ.

[a] Let $p \in \phi(E)$, then it has supporter $(a, p', p^1, \ldots, p^n)$, where $(p', p^1, \ldots, p^n) \in E$. If $E \subset E'$. then $(p', p^1, \ldots, p^n)$ lie in E' as well. It follows that $p \in \phi(E')$.

[b] Let $p^{(m)}$ be a sequence of payoff vectors in $\phi(E)$ converging to p. We need to show that $p \in \phi(E)$. Attached to each $p^{(m)}$ is a supporter $(a^{(m)}, p'^{(m)}, p^{1(m)}, \ldots, p^{n(m)})$. All $a^{(m)}$'s lie in a compact set, and so does the rest of the supporter. Extract convergent subsequence such that $(a^{(m)}, p'^{(m)}, p^{1(m)}, \ldots, p^{n(m)}) \to (a, p', p^1, \ldots, p^n)$. By compactness of A and E, this last collection is itself a valid supporter. We have to show that it supports p. The only step to take care of here is the use of the maximum theorem, to argue that the "maximum deviation function" $d_i(a)$ is continuous in A.

[5] In a homogeneous Bertrand game the one-shot NE gives everyone their security level, so it must be the worst punishment. Can't say the same for Cournot.

[6] [a1] I will need to add a condition to Conditions 1 and 2:

Condition 3. For every symmetric action a for the others, $f(0, a)$ is bounded in a (for instance, can set $f(0, a) = 0$).

For each person, write the action set as $[0, \infty)$. Fix some symmetric action a for the other players and look at one player's best response. Condition 1 tells us that this player's payoff is quasiconcave in his own actions and condition 2 tells us that the best payoff is well-defined. By quasiconcavity, the set of best responses $A(a)$ to a is convex-valued. By continuity of payoffs, $A(a)$ is upperhemicontinuous.

Now I claim that for large a we have $a' < a$ for all $a' \in A(a)$. Suppose on the contrary that there is $a_m \to \infty$ and $a'_m \in A(a_m)$ for each a_m such that $a'_m \geq a_m$. So there is a sequence $\lambda_m \in (0, 1]$ such that $a_m = \lambda_m a'_m$ for all a_m. By quasiconcavity, $f(a_m) \geq$

$\min\{f(0,a_m), f(a'_m,a_m)\}$. But the former term is bounded by Condition 3, and the latter term is bounded below by condition 2. This contradicts the fact that $f(a_m) \to -\infty$ as $m \to \infty$.

This proves, by a slight variation on the intermediate value theorem, that there exists a^* such that $a^* \in A(a^*)$. Clearly, a^* is a strongly symmetric equilibrium, which proves [a1].

[a2] This means that the set of strongly symmetric perfect equilibrium payoffs V is nonempty. Simply repeat a^* regardless of history. Now define $d \equiv \inf_a d(a) > -\infty$ by assumption. d is like the strongly symmetric security level. Lifetime payoffs can't be pushed below this. Therefore the infimum of payoffs in V is at least as great as d.

Of course, the supremum is bounded because all one-shot payoffs are bounded by assumption, so in particular the symmetric equilibrium payoff is bounded above.

[a3] Let p^m be a sequence of strongly symmetric equilibrium payoffs in V converging down to the infimum payoff p. For each such p^m let $a^m(t)$ be an action path supporting p^m using strongly symmetric action profiles at any date. Let M be the maximum strongly symmetric payoff in the game. Then, because infimum payoffs in V are bounded below, say by d, it must be the case that

$$(1-\delta)f(a^m(t)) + \delta M \geq d$$

for every date t. But this means that there exists an upper bound $\bar{a}$ such that $a^m(t) \leq \bar{a}$ for every index m and every date t.

This bound allows us to extract a convergent subsequence of m —call it m' — such that for every t,

$$a^{m'}(t) \to \underline{a}(t).$$

It is very easy to show that that the simple strategy profile defined as follows:

"Start up $\{\underline{a}(t)\}$. If there are any deviations, start it up again,"

is a simple penal code that supports the infimum punishment.

[a4] Finally, prove the compactness of V. Take any payoff sequence p^m each of which lies in V, converging to some p. Each can be supported by some action path $a^m(t)$, with the threat of starting up the simple penal code of [a3] in case there is any deviation. Take a convergent subsequence of $a^m(t)$, call the pointwise limit path $a(t)$, and show that it supports p with the threat of retreating to $\underline{a}(t)$ if there is any deviation.

(b) Part [a]. Fix some strongly symmetric equilibrium $\hat{\sigma}$ with payoff v_*. Because the continuation payoff can be no more than v^*, the first period action along this equilibrium must satisfy

$$f(a) \geq \frac{-\delta v^* + v_*}{1-\delta}.$$

Using Condition 1, it is easy to see that there exists a_* such that $f(a_*) = \frac{-\delta v^* + v_*}{1-\delta}$. By Condition 2, it follows that $d(a_*) \leq d(a)$. Now, because $\hat{\sigma}$ is an equilibrium, it must be the case that

$$v_* \geq (1-\delta)d(a) + \delta v_* \geq (1-\delta)d(a_*) + \delta v_*,$$

so that the proposed strategy is immune to deviation in Phase I. If there are no deviations, we apply some SGPE creating v^*, so it follows that this entire strategy as described constitutes a SGPE.

Part [b]. Let $\tilde{\sigma}$ be a strongly symmetric equilibrium which attains the equilibrium payoff v^*. Let $\mathbf{a} \equiv \mathbf{a}(\tilde{\sigma})$ be the path generated. Then $\mathbf{a}$ has symmetric actions $a(t)$ at each date, and

$$v^* = (1-\delta)\sum_{t=0}^{\infty} \delta^t f(a_t).$$

Clearly, for the above equality to hold, there must exist some date T such that $f(a_T) \geq v^*$. Using Condition 1, pick $a^* \geq a_T$ such that $f(a^*) = v^*$. By Condition 2, $d(a^*) \leq d(a_T)$. Now consider the strategy profile that dictates the play of a^* forever, switching to Phase I if there are any deviations. Because $\tilde{\sigma}$ is an equilibrium, because v_* is the worst strongly symmetric continuation payoff, and because v^* is the largest continuation payoff along the equilibrium path at any date, we know that

$$v^* \geq (1-\delta)d(a_T) + \delta v_*.$$

Because $d(a_T) \geq d(a^*)$,

$$v^* \geq (1-\delta)d(a^*) + \delta v_*$$

as well, and we are done.

(c) We know that in the punishment phase,

$$v_* \geq (1-\delta)d(a_*) + \delta v_*, \tag{3}$$

while along the equilibrium path,

$$v_* = (1-\delta)f(a_*) + \delta v^*. \tag{4}$$

Suppose that strict inequality were to hold in (3), so that there exists a number $v < v_*$ such that

$$v \geq (1-\delta)d(a_*) + \delta v. \tag{5}$$

Using Condition 1, pick $a \geq a_*$ such that

$$v = (1-\delta)f(a) + \delta v^*. \tag{6}$$

[To see that this is possible, use Condition 1, (4), and the fact that $v < v_*$.] Note that $d(a) \leq d(a_*)$, by Condition 2. Using this information in (5), we may conclude that

$$v \geq (1-\delta)d(a) + \delta v. \tag{7}$$

Combining (6) and (7), we see from standard arguments (check) that v must be a strongly symmetric equilibrium payoff, which contradicts the definition of v_*.

[7] and [8] read the references!