Game Theory Fall 2003

Answers to Problem Set 4

[1] (a) Consider a game in which there is a single player who must repeatedly choose a number in $\{0, 1\}$. If he chooses 1 infinitely many times, he receives a strictly positive payoff; otherwise he gets nothing. The strategy, "choose 0 at each stage" is obviously suboptimal; however, it is unimprovable. No one-shot deviation can create an improvement.

It is easy to check that the continuity condition [A.2] in my notes is not satisfied.

(b) Consider any repeated game with discounting. If the action space is finite, there is for each player i $m_i < \infty$ such that $|f_i(a)| < m_i$ for every conceivable action profile A.

Now pick $\epsilon > 0$ and choose an integer N such that $\delta_i^N (1-\delta)^{-1} m_i < \epsilon/2$ for every player i (where δ_i is the discount factor of player i). Now consider any $T \geq N$, and suppose that two paths of action profiles $\{a_t\}$ and $\{a'_t\}$ coincide for the first T periods. Then for any player i,

$$\begin{aligned}
|\sum_{t=0}^{\infty} \delta_i^t [f_i(a_t) - f_i(a'_t]| &= |\sum_{t=0}^{T} \delta_i^t [f_i(a_t) - f_i(a'_t]| + |\sum_{t=T+1}^{\infty} \delta_i^t [f_i(a_t) - f_i(a'_t]| \\
&= |\sum_{t=T+1}^{\infty} \delta_i^t [f_i(a_t) - f_i(a'_t]| \\
&\leq 2 \sum_{t=T+1}^{\infty} \delta_i^t m_i \\
&= 2\delta_i^{T+1}(1-\delta)^{-1} m_i < \epsilon.
\end{aligned}$$

(c) Outputs across time are linked by the production function $y_{t+1} = f(k_t)$, where f is some increasing, smooth, concave function with $f'(\infty) < 1$. This means that there is a maximal sustainable output Y: for all $k > Y$, $f(k) < k$ (the production function crosses the 45^0 line). This proves that no matter what path of outputs we consider, consumption at any point can never exceed $\max\{y_0, Y\}$, where y_0 is the historically given initial output. Now proceed in the same way as in part (b).

(d) We'll do a counterexample with just a one-player problem. The player has vector payoffs and the "game" is described as follows. First the player chooses a or b. If she chooses a, she gets a vector payoff of $(1, 1)$ and the game is over. If she chooses b, she moves again, choosing between L and R. If L, payoff is $(3, 0)$. If R, payoff is $(2, 2)$.

The strategy (a, L) is not a subgame perfect equilibrium. But it is not improvable by a one-shot deviation.

[2] Use same method as in class to support any positive payoff vector (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 < 1$.

[3] Let $(m_1, \ldots m_n)$ be the response vector in any stationary equilibrium. If i is the proposer, she will obtain $1 - \delta \sum_{j \neq i} m_j$. If not, she will get m_i. The former happens with probability

$1/n$ upon a rejection; the latter with probability $(n-1)/n$. This means that when i is the responder, she should accept any offer that gives her at least

$$\delta\frac{1}{n}[1-\sum_{j\neq i} m_j] + \delta\frac{n-1}{n}m_i.$$

In other words, m_i solves the equation

$$m_i = \delta\frac{1}{n}[1-\sum_{j\neq i} m_j] + \delta\frac{n-1}{n}m_i.$$

Define $\lambda \equiv (\delta/n)/[(1-\delta)+(\delta/n)]$. Then $0<\lambda<1$, and

$$m_i = \lambda[1-\sum_{j\neq i} m_j].$$

It is well known that such an equation system has a unique solution.

When I began to answer this problem, I realized that it was not easy. I conjecture that there may be a uniqueness theorem hidden here. I don't think the problem has been studied in the literature. Let me know if you are interested. [By the way, don't start thinking that exam questions are made up in this way; there are tested beforehand!]

[5] Suppose that v is balanced. For any integer $1 \leq s \leq n-1$, define a map $\delta(S)$ by $\delta(S)=0$ if the cardinality of S is different from s, and is equal to $1/\binom{n-1}{s-1}$ if the cardinality of S is s. beacsue every i belongs to precisely $\binom{n-1}{s-1}$ coalitions of size s, this is a balanced map, and so

$$\begin{aligned} v(N) &\geq \sum_S \delta(S)v(S) \\ &= \binom{n-1}{s-1}^{-1} v(s)\binom{n}{s}, \end{aligned}$$

where $v(s)$ just stands for the worth of *any* coalition of size s (all the same, by symmetry). But this means that $v(n) \geq (n/s)v(s)$.

Conversely, suppose that $v(n)/n \geq v(s)/s$. Let $\delta(S)$ be any balancing map. Let $a(s) \equiv v(s)/s$. Then

$$\sum_S \delta(S)v(S) = \sum_S \delta(S)a(|S|)|S| \leq a(n)\sum_S \delta(S)|S|.$$

Note that $\sum_S \delta(S)|S| = n$, which completes the proof. [Much easier though indirect way: show that the equal division allocation is in the core, which means that the game must be balanced, by Bondareva-Shapley.]

[6] To be completed.

[7] For any coalition S, we can construct $v(S)$ very easily:

$$v(S) = \sum_{i\in S} w_i + su(g_s) - c(g_s),$$

where s is the size of S and g_s maximizes the expression $su(g)-c(g)$. Similarly,

$$v(T) = \sum_{i\in T} w_i + tu(g_t) - c(g_t),$$

where t is the size of T and g_t maximizes the expression $tu(g) - c(g)$.

We will prove that this game is convex and therefore has nonempty core.

Let m be the size of $S \cap T$ and g_m the optimal choice for that coalition. Likewise, let M be the size of $S \cup T$ and g_M the optimal choice for that coalition. Assume without loss of generality that $g_t \leq g_s$. Now

$$\begin{aligned}
v(S) + v(T) &= \sum_{i \in S} w_i + su(g_s) - c(g_s) + \sum_{i \in T} w_i + tu(g_t) - c(g_t) \\
&= \sum_{i \in S \cup T} w_i + \sum_{i \in S \cap T} w_i + su(g_s) + (t-m)u(g_t) + mu(g_t) - c(g_s) - c(g_t) \\
&\leq \sum_{i \in S \cup T} w_i + \sum_{i \in S \cap T} w_i + Mu(g_s) - c(g_s) + mu(g_t) - c(g_t) \\
&\leq \sum_{i \in S \cup T} w_i + \sum_{i \in S \cap T} w_i + Mu(g_M) - c(g_M) + mu(g_m) - c(g_m) \\
&= v(S \cup T) + v(S \cap T),
\end{aligned}$$

where the first inequality uses $g_t \leq g_s$ and the fact that u is increasing, and the second inequality simply uses the fact that g_M and g_m are the relevant maximizers.

[8] See Osborne-Rubinstein, Proposition 264.2 and its proof.

[9] Consider the idea of the credible core. For any characteristic function V, this is a (possibly empty-valued) selection: $C^*(S) \subseteq V(S)$ for all coalitions S, with the property that for each coalition S, $C^*(S)$ is precisely the set of all allocations that are unblocked by any coalition T *using allocations in* $C^*(T)$ *alone.*

(a) This is not a formal definition because it is circular: it uses the credible core of T to define the credible core of S. However, notice that we only look at blocking by *subcoalitions*, so we can recursively define this concept starting from singletons and going "up": we will get precisely the notion introduced above (see Ray (1989, Int. J. Game Theory) for more on this).

(b) The point is that the credible core coincides with the usual core mapping, which we'll call $C(S)$ — the set of allocations in $V(S)$ which are unblocked by *any* allocation from $T \subset S$. because the core requirement is more stringent, it's obvious that $C(S) \subseteq C^*(S)$ for all S. Suppose that for some S, equality does not hold. Then there is $x \in C^*(S) \setminus C(S)$. Because $x \notin C(S)$, there is $T \subseteq S$ which blocks x with $y \in V(T)$. Because $x \in C^*(S)$, $y \notin C^*(T)$. So y is in turn blocked by some coalition W using an allocation z in $C^*(W)$. Now $W \subseteq S$. Moreover, it is easy to check that if (W, z) blocks (T, y) while (T, y) blocks (S, x), then (W, z) blocks (S, x). But this contradicts our presumption that $f \in C^*(S)$, and the proof is complete.

(c) This argument does not work if the number of players is infinite. The notion C^* has to be introduced as a solution concept (whose existence has to be established), not as a recursive definition. Here is a small example in case you're interested:

Let $N = \{1, 2, 3, \ldots\}$, and $V(S)$ be a TU characteristic function: $v(S) = 1$ if $S = \{t, t+1, t+2, \ldots\}$ for some $t \geq 1$, and is zero otherwise. The core of this game is empty. What is its credible core?

For more on the infinite case, see Einy and Shitovitz (1997, Int. J. Game Theory).

[10] Example of a non-superadditive game which satisfies condition [M]: Let $v(12) = 0$, and $v(i) = 1$ for $i = 1, 2$. This two-person game is not superadditive. Yet $m_i(i, \delta) = m_i(N, \delta) = \delta$ for each i, so that condition [M] is satisfied.

Example of a superadditive game which fails condition [M]: $N = \{1, 2, 3\}$, $v(1j) = 1$ for $j = 2, 3$, $v(23) = 2$, $v(i) = 0$ for all i, $v(123) = 2$. Now we shall check that condition [M] is not satisfied for player 1.

To see this, compute $m_1(S, \delta)$ for the coalition $S = \{12\}$. It is easy to see that $m_1(S, \delta) = \delta/(1 + \delta)$. However, for the grand coalition, it is easy to verify that

$$m_1(N, \delta) = \delta \left[2 - \frac{4\delta}{1 + \delta} \right],$$

the idea being that players 2 and 3 can always form their own coalition, so they have response numbers of $2\delta/(1 + \delta)$ each. The best player 1 can do is to offer each of them this and take the rest, but it is obvious that the value of this dwindles to zero for δ close to 1, while the value of $m_1(S, \delta)$ goes to 1/2. So condition [M] fails.

It is interesting to note that while [M] fails, there is a unique stationary equilibrium in this example and it is no-delay! That is, [M] is sufficient for no-delay but it is not necessary.

To construct a superadditive example in which [M] fails *and* there is no equilibrium without delay, use the "superadditive completion" of Example 1 in Chatterjee et al (1993, RES, p.466).

[11] See Chatterjee et al (1993, RES, p.473).