Game Theory Fall 2003

Answers to Problem Set 3

[1] If agent 1 chooses e_1, agent 2 will choose e to maximize

$$F(e_1 + e) - e.$$

This means that if agent 2 chooses positive effort e_2, the *sum* of effort will remain unchanged at the value e^* that solves $(1 - \lambda)F'(e^*) = 1$. In this case agent 1 might as well pull back her contribution to $e_1 = 0$. So one return that agent 1 can achieve is $\lambda F(e^*)$. You should check that this proves right away that if $\lambda \leq 1/2$, agent 1's optimal choice is never to put in any positive resources.

However, if $\lambda > 1/2$, agent 1 has a dilemma. She can choose e', the solution that maxes $\lambda F(e') - e'$. In that case agent 2 will put in nothing (why?). If λ is such that $\lambda F(e') - e' > \lambda F(e^*)$, she will decide to put in e' itself. If λ is close to one, this will be the optimal choice (why?).

[2] See FT, p. 100.

[3] Version (i): auditor's strategy space is { Audit, Not Audit }, individual's space is all maps from from the above space to { Evade, Not Evade }. Version (ii): auditor's strategy space is $[0, 1]$, where $p \in [0, 1]$ represents the probability of audit, individual's space is all maps from from the above space to { Evade, Not Evade }. Version (iii): auditor's strategy space { Audit, Not Audit }, individual's space is { Evade, Not Evade }.

Solving these games is very easy. The important point is to note the different interpretations of a probability of audit: in (ii), it is a pure strategy, in (iii) it is a mixed strategy.

[4] Follow class notes.

[5] Pick any Nash equilibrium and consider the first move at which a player plays T. [There must be such a first stage, why?] Let this be the mth move. If $m > 1$, then by the best response property, the mover just before m must also play T, a contradiction to m being the first such move. Therefore $m = 1$ in any Nash equilibrium, so all of them yield the same payoffs. [But note that there are several Nash equilibria in general.]

[6] (a) Unique subgame perfect equilibrium in which at every entrant node, the entrant enters and at every incumbent node, the incumbent accommodates.

(b) This part works the same way as it did for the game studied in class, but the calculations are not immediate. I would not worry about this particular problem as far as the exam is concerned (though of course you should know about the type model I did in class). If you are interested, read the classic paper by Paul Milgrom and John Roberts, available online at

`http://www.dklevine.com/archive/milrob.pdf`

[7] Verbal description: Players 1 and 2 move "simultaneously", i.e., in ignorance of each other. Player 2 chooses one of two actions, say U and D. Player 1 first chooses between L and R, then forgets what he chose, then chooses a or b. Written out in normal form, the game looks like this:

	U	D
L, a	$0, 2$	$2, 0$
L, b	$0, 0$	$0, 0$
R, a	$0, 0$	$0, 0$
R, b	$2, 0$	$0, 2$

There is a unique equilibrium of this normal form game, and it is in mixed strategies. Player 1 chooses between (L, a) and (R, b) each with probability $1/2$, while player 2 chooses between U and D with probability $1/2$. The point is that this equilibrium necessitates correlation between the play of L or R, and the later play of a or b. If perfect recall is violated, as it is here, there is no way in which this equilibrium can be implemented with behavior strategies, because these force independence between the $L - R$ and the $a - b$ choices.

[8] [OR Exercise 101.3.] Notice that if Army 1 has strictly more battalions than Army 2, then this will never change over the course of the game no matter what they do. In this case the unique equilibrium of the game is for Army 1 to attack when it can (in these cases $K \geq 2$) and for 2 never to attack. In this equilibrium Army 1's payoff is $K - 1 + x$, where $x > 1$ is the payoff from occupation. By deviating it can get only K. For Army 2 it suffices to consider a one-shot deviation. If it is in an attacking position, then for a profitable attack it must be that $L \geq 2$ (otherwise it cannot occupy the island). If it attacks, then it loses a battalion. If $L \geq 2$ then it loses the island next round for good (applying the strategies thereafter), so this deviation is not profitable.

So the only case to consider is where they have the same number of battalions; $K = L = M$. If $M = 1$, the attacker does not attack (it wins but cannot occupy). So if $M = 2$, an attack with permanent occupation thereafter will occur. This means that if $M = 3$ an attack will not occur. And so on. It follows that if M is odd an attack will not occur, while if M is even and positive, it will occur.

[9] Not true for matching pennies.

[10] In this problem I don't really want you to find *all* the Nash and subgame perfect equilibria, as it would entail getting into folk-theorem-like arguments, which I haven't yet talked about. It would suffice to show that in part (a) there is a unique Nash equilibrium, in part (b) there are several Nash equilibria involving cooperation, but none of these is subgame perfect, and in part (c) there are several possible subgame perfect equilibria. These last equilibria are credibly sustained by the threat of moving to the stage equilibrium (X, Y) if there is any deviation from the cooperative path (U, L). To be sure, such equilibria must have the property that on the equilibrium path, (D, R) is played at the last date.

[11] Call the last period 0. Count backwards at A's offer points and label these points $t = \{0, 1, 2, \ldots\}$. Let A's payoff at point t be a_t.

Now in the period just previous to t, B proposes and can obviously get $1 - \alpha a_t$ by offering A the amount αa_t. Therefore what a can get at stage $t+1$, which is just prior to this proposal by B, must be given by

$$a_{t+1} = 1 - \beta(1 - \alpha a_t) = (1 - \beta) + \alpha\beta a_t.$$

This is a simple difference equation starting from a_0 and if there are T offer points, the solution a_T, which is A's *first offer* (in real time) is given by

$$a_T = (1 - \beta)[1 + \alpha\beta + (\alpha\beta)^2 + \ldots + (\alpha\beta)^{T-1}] + (\alpha\beta)^T a_0$$

Because a_0 is surely bounded, its exact value is irrelevant as long as $\alpha\beta$ is less than one. The initial offer converges to

$$a^* \equiv \frac{1 - \beta}{1 - \alpha\beta}$$

as $T \to \infty$, which is exactly the Rubinstein bargaining solution. By the same token, B's initial offer must converge to

$$b^* \equiv \frac{1 - \alpha}{1 - \alpha\beta}.$$

When $\alpha = \beta = 1$, this convergence result breaks down. It can be verified that whenever A is the last person to make an offer, her first period payoff a_T equals 1, otherwise it equals 0 (likewise for B). Such a sequence has no limit as $T \to \infty$.

[12] We'll do part (b), part (a) is pretty trivial. Let v be the oppportunity cost to union labor of working in the firm. The way to look at this problem is first to calculate the maximum surplus that can be generated by the firm-union relationship. This is found by maximizing firm profit, $\max_L f(L) - vL$, call this surplus S, and then bargaining over a split of the surplus between the firm and the union. But if the union cannot control the *amount* of labor hiring by the firm, it can only make the firm generate this surplus by setting the offer wage w equal to v! But this gives no surplus at all to the union. So the point of the exercise is that the union must settle for a positive fraction of an *inefficient* pie rather than no fraction at all of an efficient pie. Specifically, the union chooses the offer w to maximize

$$(w - v)L(w)$$

but subject to the constraint that $L(w)$ will maximize the firm's profit evaluated at w; i.e., $L(w) = \arg\max F(L) - wL$.

In the equilibrium offer, w *must exceed* v. But then $L(w)$ is not set at the surplus-maximizing value as defined earlier, and the overall outcome must be inefficient.