Game Theory Fall 2003

Selected Answers to Problem Set 1

[1] (iii) Suppose, on the contrary, that (U, L) is a pure strategy Nash equilibrium. Then $a \geq e$ and $b \geq d$.

Case 1: $b = d$. Then it must be that $g > c$, otherwise (U, R) would be another Nash equilibrium. But then $f > h$, otherwise (D, R) would be Nash. But this means that to avoid (D, L) being Nash, $a > e$.

So in this case, $a > e$ and $b = d$. Now let q be the probability of the Column player playing L. For *all* $q \in (0, 1)$ but close enough to 1,

$$qa + (1 - q)c > qe + (1 - q)g,$$

so playing U for Row is a strictly best response. And if so, Column is indifferent between L and R (since $b = d$), so all pairs (p, q) with $p = 1$ and q sufficiently close to 1 are also Nash, a contradiction.

Case 2. $b > d$. But then, because there are no strictly dominant strategies, $h \geq f$. But then, to avoid (D, R) being Nash, it must be that $c > g$. But there are no dominant strategies, so that $a = e$. Now apply Case 1 starting from $a = e$ (instead of $b = d$) to get a similar contradiction.

[2] I will do the last example.

	L	M	R
U	$1, -2$	$-2, 1$	$0, 0$
N	$-2, 1$	$1, -2$	$0, 0$
D	$0, 0$	$0, 0$	$1, 1$

Of course, (D, R) is a pure strategy Nash equilibrium and there are no other pure strategy Nash equilibria. To look for mixed Nash, let the column player use the mixed strategy (p, q, r) (where at least one of p and q is nonzero). Then the payoff to Row from L is $p - 2q$, from N is $q - 2p$, and from R is r. Note that player 1 cannot be indifferent between U and N *and* play these as best responses. For indifference means that $p - 2q = q - 2p$, but then $p = q$ and the payoff from each is negative, while from playing D it is nonegative.

So the only possibilities are that Row plays a mixture of U and D, or N and D, as best responses. For the first to happen note that $p - 2q = r$, so that in particular, p and r must both be strictly positive. But if Row puts no weight on N, L is dominated by R for Column. A similar argument holds if Row is hypothesized to play a mixture of N and D as best responses.

[3] and [4]: Done in class.

[5] Let

$$S_i^{k+1} \equiv \{s_i \in S_i^k | \nexists \sigma_i \in \mathcal{M}(S_i^k) \text{ s.t. } f(\sigma_i, s_{-i}) > f(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^k\},$$

where $\mathcal{M}(X)$ is the set of all probabilities over the set X.

First we show that this is equivalent to the recursion

$$T_i^{k+1} \equiv \{s_i \in S_i^k | \nexists \sigma_i \in \Sigma_i \text{ s.t. } f(\sigma_i, s_{-i}) > f(s_i, s_{-i}) \text{ for all } s_{-i} \in T_{-i}^k\},$$

Proceed inductively. True by assumption for $k = 0$ (we start from S_i in both cases). Suppose true for some k. Now let us pick any $\sigma_i \in \Sigma_- \setminus \mathcal{M}(S_i^k)$. Then there exists an index $s < k$ and $\sigma_i' \in \mathcal{M}(S_i^s)$ such that $f(\sigma_i', s_{-i}) > f(\sigma_i, s_{-i})$ for all $s_{-i} \in S_{-i}^s$ (because σ_i must be placing positive weight on some dominated pure strategy at some stage s). If this σ_i' happens to be in $\mathcal{M}(S_i^k)$, stop. Otherwise it is dominated in turn at some stage $s' > s$ ...finally we must get σ_i" that dominates σ_i' over S_{-i}^k. Use this to show that T_i^{k+1} must be equal to S_i^{k+1}.

With this in place, you should be easily able to complete the second step of the equivalence, which consists in showing that the restriction $s_i \in S_i^k$ can be removed.

[6] No. Look at the game in question [2] above.

[7]–[8] omitted.

[9] (i) Yes, it does matter. In matching pennies, the use of a pure strategy can at best assure a maxmin payoff of -1, while the use of a 50-50 mixed strategy will assure you 0.

[ii] Obviously,

$$\min_{\sigma_j} f_i(\sigma_i, \sigma_j) \le f_i(\sigma_i', \sigma_j')$$

for every pair (σ_i', σ_j'), so that

$$\min_{\sigma_j} f_i(\sigma_i', \sigma_j) \le \max_{\sigma_i} f_i(\sigma_i, \sigma_j')$$

for every (σ_i', σ_j'). But if it is true for every σ_j', it must be true for the "worst" such choice; i.e.,

$$\min_{\sigma_j} f_i(\sigma_i', \sigma_j) \le \min_{\sigma_j} \max_{\sigma_i} f_i(\sigma_i, \sigma_j)$$

for every σ_i'. But if it is true for every σ_i', it must be true for the "best" such choice; i.e.,

$$\max_{\sigma_i} \min_{\sigma_j} f_i(\sigma_i, \sigma_j) \le \min_{\sigma_j} \max_{\sigma_i} f_i(\sigma_i, \sigma_j).$$

The question about a strict inequality is ill-posed, sorry. What I meant to say that in general, a function of two variables will not have the property that the above inequality holds with equality. For instance, suppose: $f(x, y)$ is defined on two values of x — a_x and b_x — and two values of y — a_y and b_y: $f(a_x, a_y) = f(b_x, b_y) = 2$, $f(a_x, b_y) = 1$ and $f(b_x, a_y) = 0$. Then you can check that

$$\max_x \min_y f(x, y) < \min_y \max_x f(x, y).$$

For f defined as expected utilities on mixed strategies the equality always holds, as part (iii) below shows.

[iii] Let (σ_1^*, σ_2^*) be a mixed-strategy Nash equilibrium for a zero-sum game. Then

$$f_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1} f_1(\sigma_1, \sigma_2^*). \tag{1}$$

Of course, at the same time,

$$\max_{\sigma_2} f_2(\sigma_1, \sigma_2) \geq f_2(\sigma_1, \sigma_2^*)$$

for every σ_1, with equality holding when $\sigma_1 = \sigma_1^*$. Because the game is zero-sum, this means that

$$\min_{\sigma_2} f_2(\sigma_1, \sigma_2) \leq f_1(\sigma_1, \sigma_2^*), \tag{2}$$

for every σ_1, with equality holding when $\sigma_1 = \sigma_1^*$. Combining (1) and (2), we see that $f_1(\sigma_1^*, \sigma_2^*)$ is not only the maximum value of $f_1(\sigma_1, \sigma_2^*)$, but is also the maximum value of $\min_{\sigma_2} f_1(\sigma_1, \sigma_2)$. That is,

$$f_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1} \min_{\sigma_2} f_1(\sigma_1, \sigma_2). \tag{3}$$

By the same logic,

$$f_2(\sigma_1^*, \sigma_2^*) = \max_{\sigma_2} \min_{\sigma_1} f_2(\sigma_1, \sigma_2),$$

but because $f_1 = -f_2$, this simply means that

$$f_1(\sigma_1^*, \sigma_2^*) = \min_{\sigma_2} \max_{\sigma_1} f_1(\sigma_1, \sigma_2). \tag{4}$$

Now combine (3) and (4) to obtain the result.

[10] Define a two player game by setting the first player's strategy set to be X, the given compact convex set, and the second player's strategy set to be Y, which is the nonnegative unit simplex. Let $f_1(x, y) = x \cdot y - \hat{x} \cdot y$ and $f_2(x, y) = -[x \cdot y - \hat{x} \cdot y]$. Then this is a zero-sum game.

Notice that every game with payoff functions that are continuous and quasiconcave in one's own action admits a *pure strategy* Nash equilibrium (class lectures). All these assumptions are satisfied in our case. Therefore our zero-sum game admits a Nash equilibrium in pure strategies; call it (x^*, y^*).

Now we prove that the solution must have $f_1(x^*, y^*) = 0$. First notice that

$$\max_x \min_y f_1(x, y) \geq 0,$$

because player 1 can always choose $x = \hat{x}$. At the same time, strict inequality cannot hold, because for any choice of $x \neq \hat{x}$, it must be that $x_j \leq \hat{x}_j$ for some coordinate j. [This is because $\hat{x}$ is efficient in X.] And for such x, we can always take y to be the j^{th} unit vector. Therefore

$$\max_x \min_y f_1(x, y) = 0,$$

and by the earlier problem,

$$f_1(x^*, y^*) = 0.$$

Interpret y^* to be $\hat{p}$, the "supporting vector" we are looking for. Notice that x^* *must be equal to* $\hat{x}$. Now use the fact that x^* is a best response to y^* to complete the argument.

[11] This is a standard model. We know that for $n = 2$ there is a unique pure strategy equilibrium which involves both candidates choosing the median point. For $n = 3$ there is no pure strategy equilibrium, which is something very easy to check by exhausting different cases.

[12] [A] Equilibrium definition: standard.

[B] Suppose that F is an increasing, concave function of the sum of efforts: $F(\mathbf{e}) = f(\sum_{i=1}^n e_i)$ for some increasing differentiable concave f satisfying the Inada endpoint conditions.

The important point here is that you should not blindly write down first-order equality conditions describing a Nash equilibrium. Suppose you do that here: you will get the absurd result that

$$\lambda_i f'(\sum_{i=1}^n e_i) = 1,$$

which, of course, cannot hold simultaneously for all the different values of λ_i! Of course, once you see this you will understand right away that in any equilibrium $\mathbf{e}^*$, e_i^* is positive only if $\lambda_i = M(\ell)$, and consequently, if $E^* \equiv \sum_{i=1}^n e_i^*$, E^* must maximize $M(\ell)f(E) - E$ with respect to E. Thus inefficiency — call it $I(\ell)$ — is simply $\hat{S} - M(\ell)f(E^*) + E^*$. It is easy to check that $I(\ell)$ is monotonically decreasing in $M(\ell)$ and indeed, that $I(\ell) \to 0$ along any sequence such that $M(\ell) \to 1$.

[C] + [D] Suppose that F is an increasing concave function of the scale of activity, where scale is determined by equi-proportional contribution of efforts: $F(\mathbf{e}) = f(\min_i e_i)$ for some increasing differentiable concave f satisfying the Inada endpoint conditions. Let $\ell = (\lambda_1, \ldots, \lambda_n)$ and $\ell' = (\lambda'_1, \ldots, \lambda'_n)$ be two different access inequalities, with the property that $\min_i \lambda_i > \min_i \lambda'_i$. Then $I(\ell) < I(\ell')$.

To prove this, note first that in this case we have a continuum of equilibria for each possible level of access inequality ℓ. These are characterized as follows.

Define $m(\ell) \equiv \min_i \lambda_i$. Then for each ℓ, $\mathbf{e}^*$ is an equilibrium vector of efforts if and only if $e_1^* = e_2^* = \ldots = e_n^* = e$ (say), and $m(\ell)f'(e) \geq 1$. Thus

$$I(\ell) = \hat{S} - m(\ell)f(e) + ne.$$

It is easy to check that $I(\ell)$ is a decreasing function of $m(\ell)$.

You should be easily able to check that ineffiency never goes to zero in this scenario (unlike in Part [B]).

[13] [A] Consider the maximization problem:

$$\max \sum_{i=1}^n [u(c_i) - v(r_i)]$$

subject to

$$\sum_{i=1}^n c_i \leq f(\sum_{i=1}^n r_i).$$

Of course you can use Lagrangeans to do this, but a simpler way is to first note that all c_i's must be the same. For if not, transfer some from a larger c_i to a smaller c_j: by the strict concavity of u the maximand must go up. The argument that all the r_i's must be the same is just the same: again, proceed by contradiction and transfer some from larger r_i to smaller r_j. By the strict concavity of $-v$ the maximand goes up. Note in both cases that the constraint is unaffected.

So we have the problem:

$$\max u\left(\frac{f(nr)}{n}\right) - v(r)$$

which (for an interior solution) leads to the necessary and sufficient first-order condition

$$u'(c^*)f'(nr^*) = v'(r^*).$$

[B] The (symmetric) equilibrium values $\hat{c}$ and $\hat{r}$ will satisfy the FOC

$$(1/n)u'(\hat{c})f'(n\hat{r}) = v'(\hat{r}),$$

[We showed in class that there are no asymmetric equilibria.] It is easy to see that this leads to underproduction (and underconsumption) relative to the first best. For if (on the contrary) $n\hat{r} \geq nr^*$, then $\hat{c} \geq c^*$ also. But then by the curvature of the relevant functions, both sets of FOCs cannot simultaneously hold.

[C] First think it through intuitively. As n is reduced there should be a direct accounting effect: total effort should come down simply because there are less people. But then there is the incentive effect: each person puts in more effort because they will have to share the output with a smaller number of people. Now let's see this a bit more formally. Let $\hat{R}$ denote total equilibrium effort, and rewrite the FOC as

$$(1/n)u'(f(\hat{R})/n)f'(\hat{R}) - v'(\hat{R}/n) = 0.$$

Now we take derivatives. For ease in writing, we will write u', f'', etc., with the understanding that all these are evaluated at the appropriate equilibrium values. Doing this, we have

$$-\frac{1}{n^2}u'f' + \frac{1}{n}u''f'\left[-\frac{f}{n^2} + \frac{f'}{n}\frac{d\hat{R}}{dn}\right] + \frac{1}{n}u'f''\frac{d\hat{R}}{dn} - v''\left[\frac{1}{n}\frac{d\hat{R}}{dn} - \frac{\hat{R}}{n^2}\right] = 0,$$

and rearranging,

$$\frac{d\hat{R}}{dn} = \frac{\frac{1}{n^2}u'f' + \frac{1}{n^3}u''f'f - \frac{1}{n^2}v''\hat{R}}{\frac{1}{n^2}u''f'^2 + \frac{1}{n}u'f'' - \frac{1}{n}v''}.$$

The denominator is unambiguously negative. The numerator is ambiguous for the reasons discusssed informally above.

[D] Each person chooses r to maximize

$$u\left(\left[\beta(1/n) + (1-\beta)\frac{r}{r+R^-}\right]f(r+R^-)\right) - v(r)$$

where R^- denotes the sum of other efforts. Let (c, r) denote the best response. Write down the FOC (which are necessary and sufficient for a best response — why?):

$$u'(c)\left(\left[\beta(1/n) + (1-\beta)\frac{r}{r+R^-}\right] f'(r+R^-) + f(r+R^-)\frac{(1-\beta)R^-}{(r+R^-)^2}\right) = v'(r)$$

Now impose the symmetric equilibrium condition that $(c, r) = (\tilde{c}, tr)$ and $R^- = (n-1)\tilde{r}$. Using this in the FOC above, we get

$$u'(\tilde{c})\left[\frac{1}{n}f'(n\tilde{r}) + \frac{(1-\beta)(n-1)f(n\tilde{r})}{n^2\tilde{r}}\right] = v'(\tilde{r}).$$

Examine this for different values of β. In particular, at $\beta = 1$ we get the old equilibrium which is no surprise. The interesting case is when β is at zero (all output divided according to work points). Then you should be able to check that

$$u'(\tilde{c})f'(n\tilde{r}) < v'(\tilde{r})!$$

[Hint: To do this, use the strict concavity of f, in particular the inequality that $f(x) > xf'(x)$ for all $x > 0$.]

But the above inequality means that you have *overproduction relative to the first best.* To prove this, simply run the underproduction proof in reverse and use the same sort of logic.

You should also be able to calculate the β that gives you exactly the first best solution. Notice that it depends only on the production function and not on the utility function.

[E] Think about it!

[14] Given the contribution R_{-i} of all persons other than i, individual i will choose r_i to maximize

$$u(w_i - r_i) + g(r_i + R_{-i}).$$

In Nash equilibrium, therefore

$$u'(w_i - r_i) \geq g'(R) \tag{5}$$

with equality if $r_i > 0$ (where R is the aggregate contribution). This condition must simultaneously hold for all i.

[A] It follows from (5) that if two individuals i and j both make strictly positive contributions,

$$u'(w_i - r_i) = u'(w_j - r_j).$$

Since marginal utility is strictly decreasing, this *must* mean that

$$w_i - r_i = w_j - r_j$$

whenever r_i and r_j are both positive. Wealth inequalities cancel out.

[B] Under the parametric specifications, notice that (5) reduces to

$$w_i - r_i \leq 2\sqrt{r_1 + r_2} \tag{6}$$

for $i = 1, 2$, with equality if $r_i > 0$. Now let us keep total wealth constant: $w_1 + w_2 = W$, but move around the distribution. Just before we do that, notice that if both make positive contribuutions and we add up (6) over $i = 1$ and $i = 2$, we see that

$$W = R + 4\sqrt{R}$$

so that total contributions is pinned down as a function of total wealth. Note carefully that $R < W$ (as it should be).

Now return to moving around the distribution. Start with $w_1 = w_2$. Certainly both contributions here are positive and equal to $R/2$ in fact. Now start raising w_1 at the expense of w_2. *Notice that as long as $w_2 > R$, the system is uniquely solved by both making positive contributions that add up to R.*

The situation changes when w_2 hits R. At this point r_2 drops to zero and r_1 is at the value R. A further redistribution in favor of person 1 will now keep r_2 at zero, so that person 2's consumption of the private good is precisely w_2. Person 1 will now be the sole contributor. By using (6) for $i = 1$, we see that

$$w_1 = r_1 + 2\sqrt{r_1}$$

so that r_1 and therefore total contributions) rise with added inequality.