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ECON-UA 323

Answers to Examination 1, Fall 2026

(1) (30 points, 6 points per part, 5 parts) Are the following statements true or false? It is *not* enough to just guess one or the other. You need to provide an argument for or against, and only then will any credit be awarded.

[a] If the Cobb-Douglas production function $Y = AK^aL^b$ exhibits *decreasing* returns to scale, then it must exhibit diminishing returns to *each* of its two inputs.

True. Decreasing returns to scale means that $F(\lambda K, \lambda L) < \lambda F(K, L)$ for all $\lambda > 1$. For the Cobb-Douglas function, $A(\lambda K)^a(\lambda L)^b = \lambda^{a+b}AK^aL^b$, so decreasing returns to scale is the statement that $\lambda^{a+b} < \lambda$ for all $\lambda > 1$, which holds if and only if $a + b < 1$. Because each input is productive, $a > 0$ and $b > 0$, and so $a + b < 1$ forces $a < 1$ and $b < 1$: the marginal product of capital, $aAK^{a-1}L^b$, falls as K rises, and likewise for labor. That is diminishing returns to each input. (Note that the converse is false: diminishing returns to each input is compatible with constant or even increasing returns to scale, as in $a = b = 3/4$.)

[b] In the Solow model with no technical progress, and with the same production function, population growth and depreciation rate across countries, a country with a higher savings rate will have a lower output-capital ratio in its steady state.

True. The fundamental equation of the Solow model is

$$\frac{k(t+1)}{k(t)} = \frac{(1-\delta) + s\Gamma(k(t))}{1+n},$$

where $\Gamma(k) = f(k)/k$ is the output-capital ratio. In steady state $k(t+1) = k(t) = k^*$, so that

$$\Gamma(k^*) = \frac{n+\delta}{s}.$$

A higher savings rate therefore *lowers* the steady-state output-capital ratio. The intuition is that a higher savings rate builds up more capital per person, and by diminishing returns each unit of capital then produces less output on average.

[c] Conditional convergence means that the dispersion of cross-country incomes narrows, conditional on the passage of time.

False. Conditional convergence is not about conditioning on time at all. “Conditional” refers to conditioning on the parameters that determine each country’s steady state in the Solow model: the savings rate s , population growth n , depreciation δ , and technology (A and π). Each country converges to its *own* steady state path, and it grows faster the further below that path it currently is. So poor countries grow faster than rich ones only *after controlling for* these parameters, as in the Mankiw-Romer-Weil regressions.

This does not imply that the dispersion of incomes narrows. Countries with different parameters converge to different steady states, so their incomes stay apart permanently. Dispersion can even widen: two countries that start at the same income but have different savings rates will move apart over time, each towards its own steady state. A narrowing of cross-country dispersion is closer to what *unconditional* convergence would deliver, when all countries share the same steady state.

[d] A country that grows at 3.5% a year for 60 years, and then shrinks at 1% a year for the next 70 years, ends up approximately four times as rich (in per-capita income) as it started.

True. Use the rule of 70. At 3.5% per year, income doubles every $70/3.5 = 20$ years, so over 60 years it doubles three times: it is multiplied by 8. At -1% per year, income halves every $70/1 = 70$ years, so over the next 70 years it is multiplied by $1/2$. The net effect is a factor of $8 \times 1/2 = 4$. (The exact figure is $1.035^{60} \times 0.99^{70} \simeq 7.9 \times 0.50 \simeq 3.9$.)

[e] A country's per-capita income is generally higher when measured by PPP rather than by the exchange-rate method.

True, as a general rule, though not without exception. The exchange-rate method converts local incomes at market exchange rates, which are driven by the prices of internationally *traded* goods. But many goods and services are not traded: haircuts, local transport, domestic services, construction. These are cheap in poorer countries, because they are produced with cheap local labor. The PPP method values every country's output at a common set of international prices, so it corrects for this, and it raises the measured income of countries where nontraded goods are cheap.

Since the reference prices are essentially those of the United States, and most countries have lower price levels than the United States, most countries look richer under PPP. Slides 1 make the point in the aggregate: world per-capita income is about \$14,200 by the exchange-rate method but about \$25,500 by PPP (2025), and the share of world income going to low and lower-middle income countries rises from 6.8% to 16.1%. The effect is largest for the poorest countries.

The exceptions are rich countries with *higher* price levels than the United States, such as Switzerland or Norway: for them, PPP income lies *below* exchange-rate income. That is why the statement holds “generally” but not always.

Grading note. Because the statement says “generally,” the expected verdict is True. But a student who answered False and pointed to the exceptions (high-price countries such as Switzerland or Norway) has understood the same economics. Either verdict received full credit, as long as the argument explained *why* nontraded goods make PPP and exchange-rate incomes differ, and recognized both the general rule and the exception.

(2) (17 points) The country of Raoland has 8000 inhabitants and two sectors, a and m . *It is a closed economy:* every employed person must be fed a work subsistence of $z = 4$, defined in units of the a -good, and that food must be produced at home. The same skills are needed

in either sector and labor moves freely across them. The production functions are

$$Y_a = 40\sqrt{K_a L_a} \text{ and } Y_m = 10\sqrt{K_m L_m}.$$

The a -good is the numeraire, and the price p of the m -good is determined within Raoland. In both sectors, labor is chosen to maximize profits.

(a) [4 points] Write down the Kalecki-Rao constraint. Show that if $K_a = 100$, then total employment in market equilibrium is 5000, so that Raoland has unemployment, and find employment in each sector.

The Kalecki-Rao constraint states that domestic food production must feed every employed person at the rate $z = 4$:

$$Y_a = 4(L_a + L_m).$$

In the a -sector, producers choose L_a to maximize $40\sqrt{K_a L_a} - wL_a$. The first-order condition is $20\sqrt{K_a} L_a^{-1/2} = w$, so that

$$(1) \quad L_a = \frac{400K_a}{w^2} \quad \text{and therefore} \quad Y_a = 40\sqrt{K_a} \cdot \frac{20\sqrt{K_a}}{w} = \frac{800K_a}{w}.$$

Using this in the Kalecki-Rao constraint, total employment is

$$(2) \quad L_a + L_m = \frac{Y_a}{4} = \frac{200K_a}{w}.$$

With $K_a = 100$, total employment is $20000/w$. Because the wage cannot fall below work subsistence, $w \geq 4$, and so total employment cannot exceed 5000, which is below the labor force of 8000. There must be unemployment, and with unemployment the wage is driven down to $w = z = 4$ (any higher wage would be undercut by the unemployed). Total employment is then exactly 5000. From (1), $L_a = 400 \times 100/16 = 2500$, and therefore $L_m = 5000 - 2500 = 2500$ as well.

(b) [4 points] Continue with $K_a = 100$, and suppose that $K_m = 400$. Find the equilibrium price p . Now suppose that K_m doubles to 800. Show that employment in *both* sectors is unchanged, and find the new equilibrium price.

In the m -sector, producers choose L_m to maximize $p \cdot 10\sqrt{K_m L_m} - wL_m$. The first-order condition is $5p\sqrt{K_m} L_m^{-1/2} = w$, so that

$$(3) \quad L_m = \frac{25p^2 K_m}{w^2}.$$

The price p adjusts so that this profit-maximizing labor demand equals the employment that the Kalecki-Rao constraint permits, which from part (a) is $L_m = 2500$. With $K_m = 400$ and $w = 4$, equation (3) reads $25p^2 \times 400/16 = 625p^2 = 2500$, so $p^2 = 4$ and $p = 2$.

Now let K_m double to 800. Nothing in part (a) involved K_m : $L_a = 2500$ still follows from (1) with $w = 4$, and total employment is still 5000 from (2), so $L_m = 2500$ as before. Employment in both sectors is unchanged. What changes is the price: (3) now reads $25p^2 \times 800/16 = 1250p^2 = 2500$, so $p^2 = 2$ and $p = \sqrt{2} \simeq 1.41$. The doubling of modern capital raises modern labor demand at any given price, and the price of the modern good falls by the factor $1/\sqrt{2}$.

to exactly cancel that increase. This is jobless modernization: the terms of trade turn against the modern sector.

(c) [4 points] Find the minimum value of K_a that permits full employment in market equilibrium. Explain in words why it is K_a , and not K_m , that governs the employment of Raoland.

As long as there is unemployment, $w = 4$, and equation (2) gives total employment $L_a + L_m = 200K_a/4 = 50K_a$. Full employment requires $50K_a \geq 8000$, so the minimum value is $K_a = 160$.

In words: in a closed economy, every worker in either sector must be fed from domestic agricultural output. Employment is therefore limited by the size of the agrarian surplus (agricultural output net of what agricultural workers themselves consume), and that surplus is generated by agricultural capital K_a . Modern capital raises the *demand* for modern labor, but that demand cannot be met unless there is food to support the extra workers, so the price of the modern good simply falls until the extra demand disappears. Only more agricultural capital (or higher agricultural productivity) relaxes the constraint.

(d) [5 points] Now suppose that $K_a = 250$. Find the equilibrium wage and the employment levels in the two sectors. Explain why, even under full employment, an increase in K_m would change none of these three numbers, and describe what *does* change.

With $K_a = 250 > 160$ there is full employment, and the wage must rise above work subsistence to clear the labor market. Setting total employment in (2) equal to the labor force,

$$\frac{200 \times 250}{w} = 8000, \text{ so that } w = \frac{50000}{8000} = 6.25,$$

which is indeed above $z = 4$. Employment in agriculture follows from (1): $L_a = 400 \times 250/6.25^2 = 100000/39.0625 = 2560$, and therefore $L_m = 8000 - 2560 = 5440$. (Check: $Y_a = 800 \times 250/6.25 = 32000 = 4 \times 8000$, so the Kalecki-Rao constraint holds with equality.)

None of these three numbers involves K_m . The wage is pinned down by the requirement that the agrarian surplus feed the whole labor force, which depends only on K_a ; agricultural employment then follows from the wage and K_a ; and modern employment is the remainder. If K_m rises, the modern sector would like to hire more labor at the going wage and price, but there is no more labor to hire and no more food to feed it. What changes is the price p : it falls, by (3) in proportion to $1/\sqrt{K_m}$, until modern labor demand is back at 5440. The modern sector produces more output (its capital has risen), but it sells that output at a lower relative price, so the gains from modern accumulation are passed on to the rest of the economy through the terms of trade rather than showing up in employment or wages.

(3) (15 points) The country of Andante has a production function which depends on capital and labor, and at any date t , it is given by

$$Y = 4K^{1/2}[e(t)L]^{1/2},$$

where $e(t) = (1 + \pi)^t$ is the exogenous productivity of labor at time t , which grows at a constant rate π . Capital depreciates at the rate δ , population grows at the rate n , and the savings rate is s .

(a) [3 points] Find a formula for per-effective-capita production $\hat{y}$ as a function of the per-effective-capita capital stock $\hat{k}$, explaining the steps to get there.

Divide both sides of the production function by effective labor $e(t)L$:

$$\frac{Y}{e(t)L} = \frac{4K^{1/2}[e(t)L]^{1/2}}{e(t)L} = 4 \left[\frac{K}{e(t)L} \right]^{1/2},$$

so that, defining per-effective-capita magnitudes by $\hat{y} = Y/e(t)L$ and $\hat{k} = K/e(t)L$, we have

$$(4) \quad \hat{y} = 4\hat{k}^{1/2}.$$

(b) [4 points] Show that the steady state path of output per-capita is given by

$$(5) \quad y^*(t) = 16 \left(\frac{s}{n + \delta + \pi} \right) (1 + \pi)^t,$$

describing precisely all the steps that lead to this conclusion.

Start from the accumulation equation $K(t+1) = (1 - \delta)K(t) + sY(t)$ and divide through by $e(t)L(t)$:

$$\frac{K(t+1)}{e(t)L(t)} = (1 - \delta)\hat{k}(t) + s\hat{y}(t).$$

Because $e(t+1) = (1 + \pi)e(t)$ and $L(t+1) = (1 + n)L(t)$, the left-hand side equals $(1 + \pi)(1 + n)\hat{k}(t+1)$, and using (4),

$$(6) \quad (1 + \pi)(1 + n)\hat{k}(t+1) = (1 - \delta)\hat{k}(t) + 4s\hat{k}(t)^{1/2}.$$

In steady state $\hat{k}(t) = \hat{k}(t+1) = \hat{k}^*$, so that $[(1 + \pi)(1 + n) - (1 - \delta)]\hat{k}^* = 4s\hat{k}^{*1/2}$. Using the approximation $(1 + \pi)(1 + n) \simeq 1 + \pi + n$ (the term $n\pi$ is tiny), this is $(n + \delta + \pi)\hat{k}^{*1/2} = 4s$, or

$$\hat{k}^{*1/2} = \frac{4s}{n + \delta + \pi}, \quad \text{so that} \quad \hat{y}^* = 4\hat{k}^{*1/2} = \frac{16s}{n + \delta + \pi}.$$

Finally, per-capita income is $y(t) = Y/L = \hat{y}(t)e(t) = \hat{y}(t)(1 + \pi)^t$, so along the steady state path

$$y^*(t) = 16 \left(\frac{s}{n + \delta + \pi} \right) (1 + \pi)^t,$$

which establishes (5). Per-capita income grows at the rate π in steady state.

(c) [4 points] Consider two countries, Andante and Allegro, with identical productions functions as above, and the same values of n , δ and π , but Allegro saves twice as large a fraction of its income. What is the ratio of steady-state per-capita income in Allegro to that in Andante? Suppose that today Andante is nevertheless the richer of the two countries. Describe, with a diagram, how per-capita incomes in the two countries evolve over time, and explain why Allegro must eventually overtake Andante.

From (5), steady-state income is proportional to s (the exponent $a/(1 - a)$ equals 1 when $a = 1/2$), so with all other parameters equal the ratio of steady-state incomes is $s_{\text{Allegro}}/s_{\text{Andante}} =$

2: Allegro's steady state path lies at twice the height of Andante's, and both paths grow at the same rate π .

To see how the two countries move, divide (6) by $\hat{k}(t)$:

$$\frac{\hat{k}(t+1)}{\hat{k}(t)} = \frac{(1-\delta) + 4s\hat{k}(t)^{-1/2}}{(1+\pi)(1+n)}.$$

The right-hand side falls as $\hat{k}(t)$ rises, and equals 1 exactly at $\hat{k}^*$. So a country below its own steady state has $\hat{k}$ rising over time, which means that per-capita income grows *faster* than π ; a country above its own steady state has $\hat{k}$ falling and per-capita income growing more slowly than π (possibly even shrinking).

Draw the diagram with time on the horizontal axis and the logarithm of per-capita income on the vertical axis, so that the two steady state paths are parallel straight lines with slope $\ln(1+\pi)$, Allegro's line above Andante's. Today Andante is richer, so its income lies above Allegro's; but Allegro's steady state line is higher, so Allegro is farther below its own line than Andante is below (or above) its own. Allegro therefore grows faster: its income curves up toward the higher line while Andante's curves toward the lower line. Since Allegro's path converges to a line that lies above the line Andante's path converges to, the two income paths must cross, and from then on Allegro is the richer country. This is *conditional* convergence: each country converges to its own steady state, and steady states differ because the parameters do.

(d) [4 points] Mankiw, Romer and Weil ran a regression of log per-capita income across countries on $\ln s$ and $\ln(n + \delta + \pi)$. What coefficients on these two variables does the model of this question predict? Their estimated coefficients were roughly 1.4 and -2.0 . Explain briefly what this discrepancy suggests about the role of physical capital, and how the model might be amended to address it.

Take logarithms in (5):

$$\ln y^*(t) = \ln 16 + t \ln(1 + \pi) + \ln s - \ln(n + \delta + \pi).$$

So the model of this question predicts a coefficient of $+1$ on $\ln s$ and -1 on $\ln(n + \delta + \pi)$. (In general the coefficients are $\pm a/(1-a)$, where a is the exponent on capital, which is also capital's share of national income; here $a = 1/2$ gives ± 1 . With a more realistic capital share of a third, the prediction would be $\pm 1/2$.)

The estimated coefficients have the right signs but are far too large in magnitude: 1.4 and -2.0 against a prediction of ± 1 (or $\pm 1/2$). Large coefficients require $a/(1-a)$ to be large, that is, a capital share far above what physical capital actually earns. This is the heart of the difficulty with the Solow model: physical capital alone cannot generate the observed spread of incomes across countries. The natural amendment is to recognize that other inputs can be accumulated too, above all human capital. With a three-input production function $Y = AK^a U^b H^c$ and separate savings rates for physical and human capital, the steady state coefficients become $a/(1-a-c)$ on $\ln s_k$, $c/(1-a-c)$ on $\ln s_h$, and $-(a+c)/(1-a-c)$ on the population term. With $a = c = 1/3$ these are 1, 1 and -2 , which is much closer to what the regressions find.