Professor Debraj Ray
19 West 4th Street, Room 608
email: debraj.ray@nyu.edu, homepage: https://debrajray.com/
Webpage for course: https://pages.nyu.edu/debraj/Courses/23UGDev/index.html

Econ-UA 323

*Development Economics*

# Answers to Problem Set 8

**(1)** A farming family owns land of size $a$ (acres), and farms it with labor $\ell$, using the production function

$$Y = 100\ell^{1/2}a^{1/2}.$$

The farm has access to 4 total units of labor (you can think of 4 as the family size), which it divides as finely as it wishes (fractionally if needed) between working *on* the farm and *off* the farm. Off-farm employment yields a wage of 100 per unit.

The farm can also hire *in* labor, again at the wage cost of 100 per unit. But unlike family labor, hired labor has to be supervised, and for this the farm has to hire a supervisor at a cost of 225. Once paid, the supervisor can costlessly supervise all hired labor.

(a) Prove that if the family has less than 16 acres of land ($a \leq 16$), it will devote family labor equal to $a/4$ to the farm, hire in no additional labor, and hire out the remainder $4 - (a/4)$ for off-farm employment.

The marginal product of family labor on the land of size $a$ is given by

$$\frac{\partial Y}{\partial \ell} = 50\ell^{-1/2}a^{1/2} = 50\sqrt{a/\ell}.$$

That means if all the family labor of 4 is assigned to the family land, the marginal product is $50\sqrt{a/4}$, which is lower than 100 as long as the land holdings are less than 16. So that proves that there is no point *hiring in* any labor, which costs 100 (plus supervision). In fact, if land size $a < 16$, the marginal product of family labor falls *strictly* below 100, which means that family income can be improved by cutting back on family labor assigned to family land, until the marginal product is *just* 100, and then hiring out rest of the family labor on off-farm employment.

(b) Prove that if the family has between 16 and 49 acres of land, it will continue to operate as a full family farm, with all its family members working full time on it, but will not hire in any labor. Above 49 acres, it hires a supervisor and at this threshold, its hiring of outside labor jumps up from 0 to slightly over 8 units of hired labor, and then keeps climbing as $a$ continues to rise.

Suppose that the fam has $a > 16$ acres of land. Option 1 is to continue to use *only* family labor. In that case the total income $Y_1$ is given by

$$Y_1 = 100\ell^{1/2}a^{1/2} = 100 \times 4^{1/2} \times a^{1/2} = 200a^{1/2}.$$

(You can subtract off the imputed cost of family labor but it won't make any difference to the argument.) Option 2 is to bring in hired labor, which means hiring a supervisor at 225. Then the net income $Y_2$ to the family after paying for hired labor and supervision is

$$Y_2 = 100\ell^{1/2}a^{1/2} - 100(\ell - 4) - 225$$

with $\ell$ chosen optimally to maximize this expression. The optimal choice of $\ell$ is given by the first order condition "marginal product equals marginal cost," or

$$\underbrace{50\sqrt{a/\ell}}_{\text{Marginal Product}} = \underbrace{100}_{\text{Marginal Cost}}.$$

which solves out to $\ell = a/4$ (some of this is family labor and the rest is hired). So we have, using this formula,

$$Y_2 = 100(a/4)^{1/2}a^{1/2} - 100((a/4) - 4) - 225 = 25a + 175.$$

You can check that $Y_1 > Y_2$ for $a < 49$, and the inequality flips when $a > 49$.

Now suppose that the family has the additional option of leasing out some or all of its land at a fixed rental rate of $R$ per unit. But assume that it cannot lease *in* any land.

(c) Calculate a threshold for $R$ such that above this threshold, the family *never* farms any land, no matter how much or how little land it owns, and leases it all out. [Hint: work out the implicit return to land on the family farm after subtracting the imputed costs of family labor.]

Calculate the implicit return per acre to having land when all labor is valued at 100 — family or hired. Never mind the supervisor. It is given by

$$r \equiv (Y - 100\ell)/a = (100\ell^{1/2}a^{1/2} - 100\ell)/a,$$

where again $\ell$ is chosen optimally as before, and so equals $a/4$. Using this information, we have

$$r = [100(a/4)^{1/2}a^{1/2} - 100(a/4)]/a = 25.$$

It follows that if the rental rate on land *exceeds* 25, it is never worth it for this family to farm its own land. It is better off leasing that land out.

(d) Can you work out what would happen for lower values of the land rental? For instance, can you show values of $R$ such that for small values of land, the family leases out nothing, then leases out some land, and then again goes back to leasing out no land as its holdings get large?

If the rental rate is smaller than 25, then there is absolutely no point in leasing out the land as long as $a < 16$. The reason is that the return per acre is precisely 25 in that case. But after that the return per acre starts to fall. For instance in the zone where $16 < a < 49$, the marginal product of land steadily drops as more and more land is applied to a fixed amount of family labor. The marginal product of land there is given by

$$50(\ell/a)^{1/2} = 100/\sqrt{a}$$

which drops from 25 when $a = 16$ all the way down to $100/7 \simeq 14.3$ when $a = 49$. If the outside rental rate is *still smaller* than this threshold, then land will never be rented out.

But if it is larger, say at $R_0$ (but still less than 25 of course), then some of the land will be rented out until the marginal product of the remaining farmed land rises back up to $R_0$.

Now, once the farm has enormous amounts of land and hires in labor, then the cost of the supervisor becomes tiny in comparison to overall net income. The implied return to land per acre almost becomes 25 again. So for any $R_0 < 25$, there is a threshold size of land large enough after which the supervisor will be hired and no land will be leased out.

(e) How would your answers to parts (b)-(d) change (if at all) if there were no fixed costs to supervision, and if hired labor costs 25 *per unit* to supervise instead?

If labor costs 25 per unit to supervise, this is just the same as saying that the cost of hiring in labor is 125 per worker. So back we go to the drawing board. Labor will be hired only when land is abundant enough so that the marginal product of labor rises above 125. The marginal product of labor is given by

$$50\sqrt{a/\ell} = 25\sqrt{a},$$

as long as no labor is hired in. At $a = 16$, this is precisely 100, and continues to climb as $a$ goes up. It is only when $a$ reaches 25 acres, that this marginal product hits 125. *After this point*, labor will be hired in. The total amount of labor will then be given by

$$50\sqrt{a/\ell} = 125,$$

or $\ell = 4a/25$. At $a = 25$, this is precisely 4, so that no fresh labor is hired in, and for all larger $a$ the amount of hired labor will be given by $(4a/25) - 4$. Contrast this with part (b), where no labor is hired until $a$ hits 49. However, you can check that at this value, there is a bigger jump in the amount of hired labor than in the current case. (Think about why.)

What about leasing out land to rent? Well, once labor starts getting hired at 125, the marginal return to owned land per acre is given by

$$50(\ell/a)^{1/2} = 50(4/25)^{1/2} = 20.$$

So this means that as long as $R$ is between 20 and 25, the land will be leased out rather than hire in labor, once the marginal product of land drops from 25 to $R$ (which will happen at some acreage between 16 and 25). If $R < 20$, however, land will never be leased out.

**(2)** Kumar, a small farmer, has his own plot of land (call it $A$) and leases another plot (call it $B$) from a large landowner, Malini. These are separate plots and he must farm them separately by allocating his endowment of one unit of effort to the two plots, in the form of $e_A$ and $e_B$. There is no cost of effort — the opportunity cost on one plot is just the cost of not using his effort on the other plot. The production functions on the two plots are

$$Y_A = A\sqrt{e_A} \text{ and } Y_B = B\sqrt{e_B},$$

so that total output is $Y_A + Y_B$, and of course, $e_A + e_B = 1$.

Kumar's utility is strictly concave in his own income — he is risk averse. If Kumar earns $x$, his utility is given by $\log(x)$.

(a) In the hypothetical case in which Kumar owns both plots of land, show that his effort allocation is given by

$$e_A = \frac{A^2}{A^2+B^2} \text{ and } e_B = \frac{B^2}{A^2+B^2},$$

with total output equal to $\sqrt{A^2+B^2}$.

Kumar must allocate labor across the two plots to maximize

$$A\sqrt{e_A} + B\sqrt{e_B},$$

subject to the constraint that $e_A + e_B = 1$. Clearly, he should equate marginal products across the two plots, otherwise he can always gain by transferring a bit of effort from one plot to the other. So the first order condition is given by

$$(A/2)e_A^{-1/2} = (B/2)e_B^{-1/2},$$

which means that $e_A/e_B = A^2/B^2$. Using this in the constraint $e_A + e_B = 1$, we must conclude that

$$\frac{A^2}{B^2}e_B + e_B = 1, \text{ or } \left[e_B = \frac{B^2}{A^2+B^2} \text{ and } e_A = \frac{A^2}{A^2+B^2}\right].$$

Substituting these solutions into the production functions for the two plots, we see that

$$\text{Total Output } = A\sqrt{e_A} + B\sqrt{e_B} = \frac{A^2+B^2}{\sqrt{A^2+B^2}} = \sqrt{A^2+B^2}.$$

Assume that if Kumar does not rent Malini's plot, he simply farms his own plot.

(b) With part (a) in mind, show that Malini can extract a total of $\sqrt{A^2+B^2} - A$ in rent, and demonstrate how she can do that using a fixed rent contract.

If Kumar farms only his own plot with his effort, his return is given by $Y_A = A\sqrt{e_A} = A\sqrt{1} = A$. So the *maximum* that Malini can extract from Kumar is

$$\sqrt{A^2+B^2} - A.$$

Malini can do this by asking for a fixed rent $R$ equal to the above amount. Now we are effectively in the world of part (a). Kumar is paying a "lump sum tax" of $R$ to Malini and so will make exactly the same decisions as in part (a), and after paying his rent, will be left with precisely $A$.

(c) Suppose that output is uncertain (we won't formally model this here, though see question 3) and that Malini can only take a *share* $\sigma$ of $Y_B$ as rent. Find an expression for the rent that Malini can get out of Kumar, expressed as a function of $\sigma$ and the other exogenous parameters $A$ and $B$ of the model. (You will need to solve out for Kumar's effort level on the plot for each $\sigma$, the answer will be similar to that in part (a).)

If Malini asks for a share $\sigma$ of the output on the rented plot, then Kumar will choose his effort allocation to maximize

$$A\sqrt{e_A} + (1-\sigma)B\sqrt{e_B}.$$

This is exactly the same problem as in part (a) where all we do is replace $B$ by a new constant $(1-\sigma)B$. So it must be that Kumar now chooses

$$e_B = \frac{(1-\sigma)^2 B^2}{A^2 + (1-\sigma)^2 B^2}.$$

That means that total output on the rented plot is

$$Y_B = B\sqrt{e_B} = B\sqrt{\frac{(1-\sigma)^2 B^2}{A^2 + (1-\sigma)^2 B^2}} = \frac{(1-\sigma)B^2}{\sqrt{A^2 + (1-\sigma)^2 B^2}},$$

and consequently Malini's total rent is

$$\sigma Y_B = \frac{\sigma(1-\sigma)B^2}{\sqrt{A^2 + (1-\sigma)^2 B^2}}.$$

(d) Without doing any further calculations, try and use your intuition to argue why Malini's rent must now be lower compared to what she gets in part (b).

Well, Kumar must still be given at least $A$. In fact you can show that he gets strictly more than $A$ under the sharecropping problem. You can prove this by simply looking at Kumar's maximization problem under sharecropping — he had the *option* to set $e_A = 1$ and pick up an income of $A$, but he chose to put in some effort on $B$ as well, so his income must strictly exceed $A$. Moreover, the total output is now distorted: we know from parts (a) and (b) that it is maximized at the value generated by the fixed rent problem; at the value $\sqrt{A^2 + B^2}$. Therefore under sharecropping, total output is *lower* than this maximum, *and* Kumar's income is higher than $A$, so Malini *must* get a lower rent than under fixed rent.

**(3)** Problem 2 might leave us wondering why on earth Malini would choose to sharecrop if fixed rent is better. We kind of waved our hands and said that otherwise the situation is too risky for Kumar. We are now going to try and formalize this argument in a very simple setting. In the previous problem, let us say that that when Kumar farms the land, $A = B = 1$ with probability $1/2$, or $A = B = \lambda > 1$ with probability $1/2$. (Good and bad outcomes are perfectly correlated across the two plots.) In other words,

$$Y_A^+ = \lambda\sqrt{e_A} \text{ and } Y_B^+ = \lambda\sqrt{e_B},$$

with probability $1/2$, while

$$Y_A^- = \sqrt{e_A} \text{ and } Y_B^- = \sqrt{e_B},$$

again with probability $1/2$, where I have used the signs "+" and "-" to distinguish good and bad outputs. Kumar's next-best alternative (if he does not rent) is just to farm his own plot. And total labor endowment equals 1, as before, and supplied at zero cost.

(a) Show that if Kumar only has his own land (plot $A$) and does not rent (plot $B$), his expected utility is given by $(1/2)\log(\lambda)$.

If Kumar has only his own plot, he puts in effort 1 into it, so that output is now 1 with probability $1/2$, and $\lambda$ with probability $1/2$. It follows that Kumar's *expected utility* is given by

$$\frac{1}{2}\log(1) + \frac{1}{2}\log(\lambda) = \frac{1}{2}\log(\lambda), \tag{1}$$

where we use the well known fact that $\log(1) = 0$.

(b) Show that if Kumar rents Plot $B$ on fixed rent tenancy with rent $R$, then he will put equal effort on both plots, with $e_A = e_B = 1/2$, so that his expected utility is given by

$$\frac{1}{2}\log(\sqrt{2} - R) + \frac{1}{2}\log(\lambda\sqrt{2} - R)$$

If Kumar rents plot $B$ on fixed rent $R$, and chooses an allocation $(e_A, e_B)$ of effort, his expected utility is given by

$$\frac{1}{2}\log\left(\sqrt{e_A} + \sqrt{B} - R\right) + \frac{1}{2}\log\left(\lambda[\sqrt{e_A} + \sqrt{B}] - R\right).$$

To maximize this, it is clearly necessary to maximize $\sqrt{e_A} + \sqrt{B}$. Again using the idea that marginal products have to be equated across the two plots, we must conclude that $e_A = e_B = 1/2$. Therefore Kumar's expected utility is given by

$$\frac{1}{2}\log\left(2\sqrt{1/2} - R\right) + \frac{1}{2}\log\left(2\lambda\sqrt{1/2} - R\right) = \frac{1}{2}\log\left(\sqrt{2} - R\right) + \frac{1}{2}\log\left(\lambda\sqrt{2} - R\right), \tag{2}$$

as claimed in the question.

(c) Using the answers to parts (a) and (b), write down an equation that describes how Malini would pick down the maximum rent that she can extract from fixed rent tenancy. You don't need to explicitly solve this equation (you can try it, though, it's not hard), but prove that $R$ cannot exceed $\sqrt{2}$. You will have to use some standard properties of logarithmic functions.

Malini can choose $R$ to bring down Kumar's expected payoff, as given in equation (2), down to Kumar's outside option utility, which is given by equation (1). Therefore $R$ should be chosen to solve the equation:

$$\frac{1}{2}\log\left(\sqrt{2} - R\right) + \frac{1}{2}\log\left(\lambda\sqrt{2} - R\right) = \frac{1}{2}\log(\lambda). \tag{3}$$

You can easily solve equation (3) by noting that it is equivalent to solving the equation

$$\left(\sqrt{2} - R\right)\left(\lambda\sqrt{2} - R\right) = \lambda$$

with each term in the product on the left restricted to be positive,[1] and solving out the resulting quadratic equation. Note that $R$ cannot become as high as $\sqrt{2}$, for then one of the terms in the product on the left approaches 0, so that the equation cannot hold.

(d) Now suppose that Malini offers a sharecropping contract with share $\sigma$ to herself and $1-\sigma$ to Kumar. Using a similar logic to that in the previous question, show that

$$e_B = \frac{(1-\sigma)^2}{1 + (1-\sigma)^2}.$$

[1] If you are algebraically-minded, you will see that there is a second mathematical solution in which $R > \lambda\sqrt{2}$, but this has no economic meaning, as Kumar would be paying out more than his output in both states.

so that the *expected rent* that Malini receives is given by

$$\text{Sharecropping rent } = \frac{1+\lambda}{2}\frac{\sigma(1-\sigma)}{\sqrt{1+(1-\sigma)^2}}.$$

If Malini offers a sharecropping contract with $\sigma$ to herself and $1-\sigma$ to Kumar, then Kumar's expected utility is given by

$$\frac{1}{2}\log\left(\sqrt{e_A}+(1-\sigma)\sqrt{e_B}\right)+\frac{1}{2}\log\left(\lambda[\sqrt{e_A}+(1-\sigma)\sqrt{e_B}]\right),$$

and to maximize this it is necessary to maximize $\sqrt{e_A}+(1-\sigma)\sqrt{e_B}$. This is just the same exercise as in Question 2(a), where we replace $A$ by 1 and $B$ by $1-\sigma$, so effort on the rented plot must be given by

$$e_B = \frac{(1-\sigma)^2}{1+(1-\sigma)^2},$$

as claimed. So Malini's expected rent is given by

$$\frac{1}{2}\frac{\sigma(1-\sigma)}{\sqrt{1+(1-\sigma)^2}}+\frac{1}{2}\frac{\sigma(1-\sigma)\lambda}{\sqrt{1+(1-\sigma)^2}} = \frac{1+\lambda}{2}\frac{\sigma(1-\sigma)}{\sqrt{1+(1-\sigma)^2}}, \tag{4}$$

as the problem asserts.

(e) Use part (c) and the formula in part (d) to show that if $\lambda$ is large enough, a risk-neutral Malini prefers sharecropping to fixed rent tenancy. Intuitively explain your answer.

All you have to do is show that this last calculated rent in equation (4) strictly exceeds $\sqrt{2}$ if $\lambda$ is large enough, because already know from part (c) that $R$ cannot exceed $\sqrt{2}$. But this is easy. Just fix the share $\sigma$ at any value — say $1/2$ — and note that the expression in (4) then grows linearly in $\lambda$, and so must become unboundedly high, higher than $\sqrt{2}$ in particular for sufficiently large $\lambda$.

**(4)** Miguel works on a tea plantation as a plucker. (If you are wondering how someone named Miguel could be working on a tea plantation, remember that Argentina is the ninth largest exporter of tea in the world!) He gets paid a basic wage — we will call it $b$ — and an extra incentive payment $s$ for every kilo of tea leaves that he plucks. So his total payment $w$ is given by

$$w = b + sy,$$

where $y$ is the number of kilos of tea that he plucks. Miguel has a "consumption utility function" given by $u(w) = 100\log w$, and his cost of plucking $y$ kilos of tea is given by $y$. So Miguel's net utility is given by

$$u(w) - y.$$

(a) For any given $b$ and $s$, show that Miguel will pluck $y$ kilos of tea, where

$$y = 100 - \frac{b}{s}$$

provided that $b < 100s$, otherwise Miguel won't pluck any tea at all!

Given $b$ and $s$, Miguel will choose output $y$ to maximize

$$u(w) - y = 100\log(b+sy) - y.$$

Writing down the first order condition to set marginal benefit equal to marginal cost, we have:

$$\frac{100s}{b+sy} = 1, \text{ or } y = 100 - \frac{b}{s}.$$

When $b \geq 100s$, this first-order condition cannot be met for any positive level of plucking — the marginal benefit of the additional consumption is too small relative to the marginal cost of effort.

Can you explain intuitively why $b$ and $s$ affect the amount of tea plucked in the way they do here? In particular, why does Miguel's effort drop to zero if $b \geq 100s$?

Notice that as $b$ goes up, Miguel lowers his output. The reason is that a higher baseline wage has an "income effect": it reduces the marginal utility of additional wages. So Miguel cuts back on the effort to pluck tea. In fact, in this case he exactly cuts back on income so it is just the same as before, and puts all the extra gains into "effort saving." (You might want to think about why this happens here, or read more about it below.)

When $b$ exceeds $100s$, then you cannot even get a first order condition that holds with equality, because the marginal benefit of even the first unit of effort is smaller than the marginal cost. At that stage Miguel just stops working ...

Some more comments:

(i) This result on $b$ cutting back effort to the point at which income stays constant is pretty general as far as $u$ goes. If he had an abstract (but strictly concave) utility function, his first order condition would read

$$su'(b+sy) = 1,$$

and you can see that $b+sy$ — which is his total income — does not change as you change $b$.

(ii) But it does depend on the linearity of effort cost. For instance, if his cost of effort is convex, then the first order condition is

$$su'(b+sy) = c'(y)$$

and now $b+sy$ *will* change as $b$ changes — can you see why, and in which direction?

On the other hand, an increase in $s$ *increases* Miguel's effort, because now he is incentivized by the piece rate. This is why plantation owners don't like paying a baseline wage $b$ whenever tasks are observable. Sometimes they have to be pushed to do it by government law. (And it is the same for many jobs in the United States — Uber won't hire drivers for a wage, and real-estate companies pay their employees on commission.)

(b) Assume that the minimum wage $b$ is fixed by the government at a strictly positive number, but smaller than 100. The tea plantation cannot tamper with it, but can freely choose the piece rate. The plantation wants to maximize profits from hiring Miguel, which are given by

$$y - w = y - [b + sy] = (1-s)y - b.$$

where we are setting the price of tea equal to 1. Prove that for any $b < 100$, the tea plantation will want to set

$$s = \frac{\sqrt{b}}{10}.$$

assuming that it hires Miguel to begin with. Notice how as $b$ goes up, $s$ goes up as well. Why?

Given $b$, the plantation choose the piece rate to maximize its profits, knowing that Miguel will behave the way he does as described in the previous part. In short, the plantation seeks to maximize

$$(1 - s)y - b$$

by choosing $s$, knowing that $y = 100 - (b/s)$. In other words, the plantation chooses $s$ to maximize

$$(1 - s)\left[100 - \frac{b}{s}\right] - b$$

where $b$ is given. Setting the derivative of this expression (with respect to $s$) equal to zero (using the product rule), we get the first order condition:

$$-\left[100 - \frac{b}{s}\right] + \frac{b(1 - s)}{s^2} = 0,$$

and simplifying this expression, we obtain:

$$s = \frac{\sqrt{b}}{10}.$$

The piece rate $s$ has to climb with the baseline wage $b$ because Miguel has a diminishing marginal utility of income. The richer Miguel is at baseline, the larger will have to be the piece rate in order to adequately incentivize him. We continue this line of reasoning in the next question.

(c) Prove that if the baseline wage $b$ goes above 25, the plantation would rather not hire Miguel to begin with.

Of course the above solution maximizes profits for the the plantation, but those profits may be negative — after all, it is being asked to pay Miguel a baseline wage and so whether or not it makes positive profits from Miguel depends on whether he plucks enough tea in response to $s$. To examine this, substitute the choice of $s$ from the previous question — which is the best that the plantation can do under the circumstances — into the plantation's expression for profit, yielding a profit of

$$(1-s)y-b = (1-s)\left[100 - \frac{b}{s}\right]-b = \frac{10-\sqrt{b}}{10}\left[100 - 10\sqrt{b}\right]-b = (10-\sqrt{b})^2-b = 100-20\sqrt{b},$$

which means that if $b > 25$, the plantation's profit from hiring Miguel is negative.

**(5)** An economy has a labor force of 100, and a production function that uses labor to produce output. Output price is fixed at 1, and the production function is given by

$$y = A\ell^{1/2}$$

with labor chosen to maximize profit at the going wage rate. If a worker is unemployed, she obtains the net monetary equivalent of \$30 per day, perhaps doing home tasks or working on the farm. If she is employed, she earns a wage of $w$ (to be determined), but has to work at a minimum pace, which incurs a personal cost of \$27. Each worker has a discount factor of $\delta = 9/10$.

(a) Assume that detection is certain if you work below the minimum required pace, and that no fired worker is ever hired again (which may sound unreasonable but let's do it for practice, and also for a reason that I will reveal below). Then, show that the minimum wage for self-enforcement is \$60.

Drawing on class notes, the self-enforcement constraint in this case is given by

$$x \leq \delta[w - s]$$

where $x$ is the effort cost (or equivalently, the gain from shirking) — here it is 27 — and $s$ is the payoff from staying at home (here it is 30). Using these numbers along with $\delta = 9/10$, we see that

$$27 \leq \frac{9}{10}[w - 30]$$

The lowest wage for self-enforcement is the one that meets this with equality, so $w = 60$.

(b) Remember there are 100 units of labor in the whole economy. Show that if the technological coefficient $A$ is smaller than 1200, then there is involuntary unemployment, and the market wage settles at 60. But also show that if $A$ exceeds 1200, the market wage rises *above* the minimum necessarily for self-enforcement, and there is full employment.

For any $w$, the demand for labor is given by the first order condition to profit maximization of:

$$y - w\ell = A\ell^{1/2} - w\ell.$$

Taking derivatives, we see that the first order condition is:

$$(A/2)\ell^{-1/2} = w, \text{ or equivalently, } \ell = \left(\frac{A}{2w}\right)^2.$$

Now first fix $w = 60$, so that for this case, we simply have

$$\ell = \left(\frac{A}{120}\right)^2.$$

This is just fine as long as total labor demand does not exceed the supply of 100. In other words, to make this work, we need:

$$\left(\frac{A}{120}\right)^2 \leq 100, \text{ or equivalently, } A \leq 1200.$$

Notice that for all $A < 1200$, demand is smaller than supply, but that's how things will remain. None of the unemployed can undercut the wage from 60, because that won't be a credible promise — employers will think that they will shirk on the job. So we have an equilibrium with involuntary unemployment. That answers the first part of this question.

What happens, now, once $A$ crosses 1200? Then the demand for labor exceeds its supply, and now the wage must be bid up from its minimum of 60 — there is no credibility problem on the part of workers when the wage is going *up*, because workers will continue not to shirk. So the wage has to rise until the demand just equals the supply, or in other words: the new wage, call it $w^*$, must solve

$$\left(\frac{A}{2w^*}\right)^2 = 100, \text{ or equivalently, } w^* = \frac{A}{20}.$$

Remember again that this case can only come into being when $A \geq 1200$. For still higher values of $A$, the wage will keep climbing and there will be full employment. The self-enforcement constraint is no longer binding, and the model looks very classical in this zone.

The market wage can *only* rise strictly above the self-enforcement constraint if there is full employment. Otherwise an unemployed person could credibly bid that wage down (slightly) and get employed — because the self-enforcement constraint would still hold.

(c) Now I want you to contrast this scenario with the realistic case, in which a fired worker might be re-employed. Let this re-employment probability *exactly equal* the fraction of people who are employed in the economy; i.e., it is equal to the employment rate $e$. Now show that the self-enforcement constraint is given by:

$$w \geq 30\frac{2-e}{1-e}. \tag{5}$$

Again we recall our class notes but now with the additional twist that a fired worker gains re-entry into employment with probability $q$ in each period. Now the self-enforcement constraint is given by

$$27 = x \leq \delta(1-q)[w-30] = \frac{9}{10}(1-e)[w-30].$$

Moving terms around and simplifying, we get the formula (5).

(d) This is very different from part (b). Show that there can *never* can be full employment in this scenario, unlike in part (b). In particular, *the self-enforcement constraint always holds with equality.* Carefully explain why things are so different now.

The proof is by contradiction. Suppose that there is full employment in any equilibrium. Then it must be the case that $e = 1$. But then

$$w \geq 30\frac{2-e}{1-e} = \infty,$$

and at an infinite wage, the demand for labor must fall to zero, contradicting the fact that labor demand is enough for full employment. So there is involuntary unemployment in any equilibrium no matter how massively large $A$ is, and the wage is pinned down by equation (5) throughout. The contrast can be explained by noting that as the labor market gets tighter, the shirking constraint also gets very expensive, because workers can find a job again with ease. So the wage rate has to climb along with $A$, which keeps the demand for labor always below 100. What a dramatic difference. In the first case we could blacklist a worker permanently; in the second we cannot.

(e) Now we will work to solve for the equilibrium in this labor market. First note that for any wage rate $w$, the demand for labor is given by

$$\ell = \left(\frac{A}{2w}\right)^2.$$

(you will surely have solved for this earlier, but if not, do so now). Combine this with the self-enforcement constraint (5) that we know to always hold with equality:

$$w = 30\frac{2-e}{1-e}$$

to show that in equilibrium, $e$ must solve the equation

$$600\sqrt{e}\left(\frac{2-e}{1-e}\right) = A,$$

where you will need to use the fact that the employment rate $e$ is the total employment $\ell$ divided by the total labor force, which is 100. (That is, $e = \ell/100$.)

We know that the demand for labor is given by

$$\ell = \left(\frac{A}{2w}\right)^2.$$

Combining this with the fact that

$$w = 30\frac{2-e}{1-e},$$

and remembering that $e = \ell/100$, we must conclude that

$$e = \frac{\ell}{100} = \frac{1}{100}\left(\frac{A}{60\left[\frac{2-e}{1-e}\right]}\right)^2.$$

Take square roots and move terms around now to get the solution to the problem. Examine this solution and notice that no matter how big or small $A$ is (as long as it is a positive), any solution to $e$ lies strictly between zero and 1. Unemployment is endemic no matter how productive the economy is.

**Solution to optional problem:**

**(6)** Consider a production cooperative with just two farmers. Each farmer chooses independently how much labor — $\ell_1$ and $\ell_2$ — to supply to the cooperative. The cooperative output is given by

$$Y = A(\ell_1 + \ell_2)^\alpha$$

where $A > 0$ and $\alpha$ lies between 0 and 1. Each unit of labor is supplied at an opportunity cost of $w$, so the total cost of effort supply is $w\ell_1$ for farmer 1 and $w\ell_2$ for farmer 2.

(a) Draw production and total cost as a function of labor input. Find (both diagrammatically and using first order conditions) the amount of labor input that maximizes farm surplus.

Drawing the production function as a function of $\ell_1 + \ell_2$ is a completely standard exercise. Note that farm surplus is given by

$$A(\ell_1 + \ell_2)^\alpha - w(\ell_1 + \ell_2)$$

and so we can find the *aggregate labor input* that maximizes surplus. It is given by some number $L^*$ which maximizes

$$AL^\alpha - wL$$

and is solved out by setting marginal product equal to marginal cost:

$$\alpha A L^{\alpha-1} = w, \text{ or transposing terms, } L^* = \left(\frac{\alpha A}{w}\right)^{1/(1-\alpha)}. \tag{6}$$

Notice that the division of this labor $L^*$ between the two farmers is irrelevant for the maximization of the surplus.

(b) Now suppose that each labor is supplied independently by each cooperative member in an effort to maximize her own net profit. Say that a pair of labor allocations $(\ell_1, \ell_2)$ forms an *equilibrium* if, given $e_2$, the choice of $e_1$ is optimal for farmer 1, and given $e_1$, the choice of $e_2$ is optimal for farmer 2.

Show that if total output is divided equally among the farmers, production *must fall short* of the answer in part (a).

If output is divided 50-50, and given farmer 2's input $\ell_2$, farmer 1 will want to maximize his own net return, which is

$$\frac{1}{2}A(\ell_1 + \ell_2)^\alpha - w\ell_1, \tag{7}$$

and similarly farmer 2 will choose $\ell_2$ (given $\ell_1$) to maximize

$$\frac{1}{2}A(\ell_1 + \ell_2)^\alpha - w\ell_2. \tag{8}$$

Here is how we solve this problem. First define $\hat{L}$ as the solution to

$$\frac{\alpha}{2}A\hat{L}^{\alpha-1} = w, \text{ or transposing terms, } \hat{L} = \left(\frac{\alpha A}{2w}\right)^{1/(1-\alpha)}. \tag{9}$$

This is just a mathematical definition, but now we give it economic meaning. Notice that if farmer 2 is anticipated to provide input $\ell_2$ smaller than $\hat{L}$, the maximization of farmer 1's payoff will lead him to just make up the difference between $\ell_2$ and $\hat{L}$; that is, he will set $\ell_1 = \hat{L} - \ell_2$. Likewise farmer 2 will do the same, given his anticipation of $\ell_1$. You can solve both these by going to the maximization problems in equations (7) and (8) and writing down the first order conditions for maximization.

Additionally, observe that if, say, $\ell_2 \geq \hat{L}$, then farmer 1's best input choice is just $\ell_1 = 0$, because the marginal product of *his* labor will be lower than his marginal cost right from the get-go. The same is true of farmer 2 of course.

It follows that there are many equilibria of this situation. These are *all the combinations* of $\ell_1$ and $\ell_2$ which add up to $\hat{L}$.

But notice, to complete this part, that $\hat{L}$ must be smaller than $L^*$. Compare the two solutions: clearly, $\hat{L}$ is smaller. The equilibrium, no matter which one it is, fails to maximize social surplus.

(c) Try to intuitively relate this exercise to the problem of inefficiency in sharecropping.

Omitted. left to you.

(d) Next, suppose that farmer 1 receives a share $s > 1/2$ of the total output, while farmer 2 gets $1-s$ (everything else is the same as before). Show that there is now a unique equilibrium labor allocation, and describe what it looks like.

Define two numbers, $L_1$ and $L_2$. The first sets the marginal product of labor, multiplied by the share accruing to farmer 1, equal to marginal cost:

$$\alpha s A {L_1}^{\alpha-1} = w, \text{ or transposing terms, } L_1 = \left(\frac{\alpha s A}{w}\right)^{1/(1-\alpha)}. \tag{10}$$

The second does the same using the share going to farmer 2:

$$\alpha(1-s) A {L_2}^{\alpha-1} = w, \text{ or transposing terms, } L_2 = \left(\frac{\alpha(1-s) A}{w}\right)^{1/(1-\alpha)}. \tag{11}$$

Because $s > 1-s$ ($s > 1/2$), it is easy to see that $L_1 > L_2$. Now observe that by the same logic as in part (b), farmer 1 wants *total* effort to add up to $L_1$, and will put in effort to make up any difference between $\ell_2$ and $L_1$, while farmer 2 wants effort to add up to $L_2$, and will put in effort to make up the difference between $\ell_1$ and $L_2$. If any farmer is putting in extra effort over the other farmer's total, the other farmer will put zero effort.

Combine these thoughts. You will see that the only equilibrium when $s > 1/2$ is that farmer 1 puts in all effort $\ell_1 = L_1$, and farmer 2 puts in nothing!

(e) Show that if if $s$ is slightly larger than 1/2, then farmer 1 — who gets the larger share — is actually *worse off* in terms of her *net* payoff.

So in particular, even if $s$ is a *tiny bit* bigger than 1/2, farmer 1 will put in all the effort. What are their payoffs? For farmer 1, it is $sAL_1^\alpha - wL_1$, and for farmer 2 it is $(1-s)AL_1^\alpha$, with no effort cost. Now you can see that their output shares are almost the same, but farmer 1 incurs all the cost. It follows that farmer 1 has to be worse off than farmer 2, even if he is getting a slightly larger share of total output! Crazy, huh? But this happens all the time in real life — think about it.

(f) Parametrically moving $s$ from 1/2 to 1, describe what happens to production and labor efforts. Show that the system maximizes overall social surplus when one share equals 1.

As farmer 1' share varies from 1/2 to 1, his choice of $L_1$ will also vary, so write this explicitly as a function of $s$. Simply rewriting (10), we see that

$$L_1(s) = \left(\frac{\alpha s A}{w}\right)^{1/(1-\alpha)},$$

for all such $s$. Meanwhile, we know the equilibrium for any $s > 1/2$: $\ell_1 = L_1(s)$, while $\ell_2 = 0$. Notice how total effort converges to the first-best surplus maximizing solution as $s \to 1$, because $L_1(s)$ converges to $L^*$ as defined in equation (6). The division of surplus between the two is unequal and ambiguous, but we do know that at $s = 1$ farmer 1 gets the whole surplus and farmer 2 gets nothing. Surplus maximization in this cooperative is associated with high inequality.

(g) The result in (f) is strange on a number of grounds! First, efficiency is reached when the system is highly unequal. Second, what happened to the double moral hazard problem we discussed in class? Shouldn't that place limits on one side being a residual claimant? Think about this intuitively and now move to the next part.

In the double moral hazard problem discussed in class, both inputs are needed separately and cannot be fully substituted for. The landlord supplied one kind of input and the tenant supplied another, so we need incentives for both of them. Here the inputs (efforts) are perfect substitutes: they enter into the production function *additively*, so there is no need to provide incentives on both sides.

(h) Change the problem by supposing that the joint production function is of the form

$$Y = A\ell_1^{1/3}\ell_2^{1/3}.$$

Notice that we have diminishing returns in the two inputs jointly $(1/3 + 1/3 < 1)$ because land is also an input and it is fixed.

Now which value of the share do you think maximizes social surplus? Explain why the answer is so different from the preceding answer.

Ah, now things are different, because now, as you see, the inputs are not perfect substitutes — the isoquants across these inputs are curved, and it would be great to incentivize both of them to put in effort. Let's see what the equilibrium looks like in this case for any share $s \geq 1/2$ (the case in which $s \leq 1/2$ is just a mirror image of the same problem and does not have to be solved out separately.

Now a pair $(\ell_1, \ell_2)$ is an equilibrium if, given $\ell_2$, $\ell_1$ maximizes farmer 1's return:

$$sA\ell_1^{1/3}\ell_2^{1/3} - w\ell_1,$$

while given $\ell_1$, $\ell_2$ maximizes farmer 2's return:

$$(1-s)A\ell_1^{1/3}\ell_2^{1/3} - w\ell_2.$$

It is easy to see that farmer 1's first order condition is given by

$$\frac{sA}{3}\ell_1^{-2/3}\ell_2^{1/3} = w, \text{ or } \frac{sA}{3}(\ell_1\ell_2)^{1/3} = w\ell_1 \tag{12}$$

while farmer 2's corresponding first order condition is given by

$$\frac{(1-s)A}{3}\ell_1^{1/3}\ell_2^{-2/3} = w, \text{ or } \frac{(1-s)A}{3}(\ell_1\ell_2)^{1/3} = w\ell_2 \tag{13}$$

Multiplying (12) and (13) together, we see that

$$\frac{A^2 s(1-s)}{9}(\ell_1\ell_2)^{2/3} = w\ell_1\ell_2, \text{ or } (\ell_1\ell_2)^{1/3} = \frac{A^2 s(1-s)}{9w} \tag{14}$$

Using this solution in (12) and (13), we must conclude that

$$\ell_1 = \frac{A^3 s^2(1-s)}{27w^2} \text{ and } \ell_2 = \frac{A^3 s(1-s)^2}{27w^2}. \tag{15}$$

Combining (14) and (15) with the formula for social surplus, which is

$$\text{SS} = A\ell_1^{1/3}\ell_2^{1/3} - w\ell_1 - w\ell_2,$$

we must conclude that

$$\text{SS} = \frac{A^3 s(1-s)}{9w} - w\left[\frac{A^3 s^2(1-s)}{27w^2} + \frac{A^3 s(1-s)^2}{27w^2}\right] = \frac{2A^3 s(1-s)}{27w}.$$

(Can you see how to get the last equality?) Now all the hard work is done, and you can see that to maximize social surplus, the best choice is $s = 1/2$! Why? Because the function $s(1-s)$ is maximized at $1/2$.