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ECON-UA 323

Development Economics

Outline of Answers to Problem Set 7

[1] This problem illustrates adverse selection in credit markets. The land of NoGood has a large number of producers of different types, all indexed by their probability p of successful production. Producer p produces quantity $y(p) = 110/p$ if successful, and with probability $1 - p$ their output is zero. No producer has collateral.

(a) Show that every type of producer has the same value of *expected* output.

Expected production for a type p producer is given by

$$py(p) + (1 - p) \times 0 = py(p) = 110.$$

Suppose that a loan of 100 is needed to start production. It is offered at interest rate r. (In what follows, we are going to move r around.)

(b) If only producers with *strictly positive* profits apply for a loan, describe the set of producers who will want to take on the loan of 100 at an interest rate r.

Note that in the absence of collateral, there is no repayment on failure. So if the interest rate is r, then the set of producers who make strictly positive expected profit are those for whom

$$y(p) = \frac{110}{p} \geq 100(1 + r),$$

from which we must conclude that

$$p \leq \frac{11}{10(1 + r)}.$$

Notice that if the interest rate is less than 10%, then the above inequality imposes no restriction at all, because $p \leq 1$ anyway. Everyone will apply. Once the interest rate climbs above 10%, then the loan applicants get riskier and riskier as p falls further below 1.

(c) If producers have their probabilities distributed uniformly on $[0, 1]$, calculate the expected return to the bank from choosing the interest rate r, and show that on average, the bank can never recover the loan.

From the previous part, we have $p \leq 1/(1 + r)$. Because p is uniformly distributed over $[0, 1]$, it follows that the conditional expectation of p must be

$$\mathbf{E}\left(p|p \leq \frac{11}{10(1 + r)}\right) = \begin{cases} \frac{1}{2} & \text{for } 0 \leq r \leq \frac{1}{10} \\ \frac{11}{20(1+r)} & \text{for } r > \frac{1}{10}. \end{cases}$$

Therefore the expected return on a loan of size 100 and carrying an interest rate r is given by

$$\frac{1}{2}100(1+r) = 50(1+r) \text{ when } r \leq \frac{1}{10} \text{ and } \frac{11}{20(1+r)}100(1+r) \text{ when } r > \frac{1}{10},$$

which proves that on average, even the principal cannot be recovered.

(d) Discuss this in the context of adverse selection more generally, and understand just how the market breaks down in this case.

In the adverse selection argument discussed in class (I hope I remembered to do this!), the second hand price of a car was determined by its average quality conditional on being on the market. But owners of cars at the upper limit of quality would withdraw their cars from the market, leading to further spiral downwards in quality that could have no end, thereby destroying the market. Something similar happens here. If the loan is priced at r, then only a certain set of risky types enter the market, turning that price into a disaster. If the price responds by rising (that is, if r goes up), the market becomes riskier still. Sometimes there can be no end to the process and the market collapses.

(2) Consider a moneylender who faces two types of potential customers: call them the *safe type* and the *risky type*...

This problem is fully solved in the second Appendix of the Chapter on Credit Markets.

(3) Here is an extension of the previous question. Suppose that a moneylender is lending one dollar to a risky borrower. There is a probability of default (through some involuntary or strategic source that we do not model here). If that happens, assume that the entire loan is lost, principal and interest.

(a) If the interest rate is r, and the repayment probability is p, write down the net expected payoff to the moneylender from making the loan.

The net expected payoff is given by

$$p(1+r) - (1+i)$$

where i is the opportunity rate of interest on the next best option to the lender.

Now we suppose that p comes down with the lender's choice of i. Specifically, assume that $p = 1 - ar$ for some constant $a > 0$. Of course, the above formula only makes sense if r is no bigger than $1/a$, after which p just stays at zero.

(b) Solve for the interest rate i that maximizes the lender's payoff. Interpret the results when a is very close to 1 or very close to zero.

If $p = 1 - ar$ for some constant $a > 0$, then using the expression in part (a), we see that expected return is given by

$$(1-ar)(1+r) - (1+i),$$

as long as r is no bigger than $1/a$, and is zero otherwise. So, *assuming that the lender will lend*, the interest rate r will be chosen to maximize

$$(1-ar)(1+r) = 1 + (1-a)r - ar^2 - (1+i)$$

If $a \geq 1$ it is obvious that the best thing to do is set $r = 0$. Otherwise, if $a < 1$, the interest rate r solves the first order condition:

$$(1-a) - 2ar = 0, \text{ or } r = \frac{1-a}{2a}.$$

Notice how the interest rate r is independent of i. It is not just a simple percentage markup over i, but rather a number that is determined by the extent of risky default in the credit market.

(c) If i is the safe rate of interest on some other option, find a formula for i (in terms of a) above which the lender will never lend to the risky borrower.

Note that if $a \geq 1$ then (assuming $i > 0$), the lender will never lend to the borrower. Otherwise, if $a < 1$, then we have r as given by the formula in part (b), so that the expected payoff is given by

$$(1-ar)(1+r) = \frac{(1+a)^2}{4a}.$$

For the safe option to be better, we must therefore have

$$1 + i \geq \frac{(1+a)^2}{4a},$$

or

$$i \geq \frac{(1+a)^2}{4a} - 1 = \frac{(1-a)^2}{4a}.$$

At any safe rate exceeding this level, the risky market will shut down altogether.

(4) A loan L is made every period, and repayment R charged. The loan produces output $F(L)$ for the borrower. The borrower decides to default or repay at the end of the period. If she defaults, she keeps the R but is banned by the lender and can only get some outside option v every period thereafter. She also loses her collateral C.

(a) Just as in class, derive the no-default constraint.

If default, get $F(L)$ today, lose collateral C, and v per period starting tomorrow.

$$F(L) - C + \delta \frac{v}{1-\delta}.$$

If repay, get $[F(L) - R]/(1-\delta)$. So the no-default constraint is given by:

$$\frac{F(L) - R}{1-\delta} \geq F(L) - C + \delta \frac{v}{1-\delta};$$

which simplifies to

$$\delta[F(L) - v] + (1-\delta)C \geq R.$$

Notice how the presence of collateral makes it easier to advance loans to the borrower. Not only will the lender get to keep something in the event of a default — see below — but the borrower is more willing to repay. As C becomes *larger*, it becomes *easier* to satisfy the constraint above.

Next, assume that lenders must recover their opportunity rate of interest i on loans.

(b) Following the same method as in class, describe what happens to loan size and the interest rate on loans as the collateral offered by the borrower goes up. You can assume throughout that the no-default condition binds and holds with equality, so that the problem of potential default is always present.

It would be great practice if you could redo the diagrams we did in class for the case of collateral. Let me accompany that with some simple algebra. If the lenders need to be assured of their opportunity rate of interest i, a couple of conditions must hold.

First, it must be that

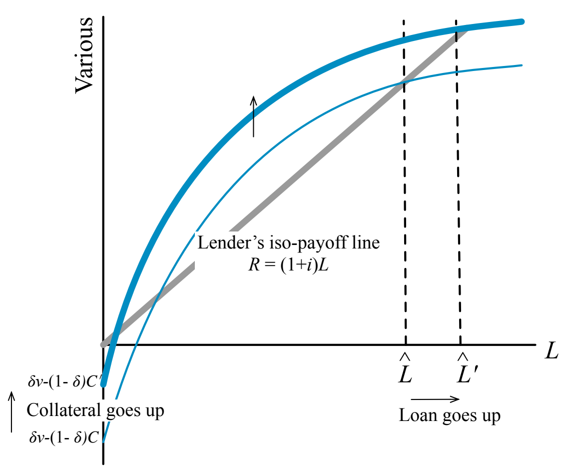
$$R = (1 + i)L,$$

Using this in the no-default constraint, we see that

$$\delta[F(L) - v] + (1 - \delta)C \geq (1 + i)L.$$

We are assuming that the no-default constraint binds, so this condition holds with equality. Rewriting it by moving a couple of terms around, we have:

$$\delta F(L) - (1 + i)L = \delta v - (1 - \delta)C.$$



This is the rightmost intersection of the iso-profit line for the lenders, which stays unchanged throughout, with the left-hand side of the no-default constraint given by $\delta[F(L)-v]+(1-\delta)C$. It is immediate from the diagram that the intersection moves up as the value of C goes up, leading to greater loan sizes and greater social efficiency.

(5) Now for a small variation on the previous question in which we unpack the outside option.

A borrower takes a loan of L in every period and produces output AL^{α}, where $A > 0$ and $0 < \alpha < 1$. He makes a repayment of R. He has no collateral. At any date, he can default on the repayment. Thereafter, he gains access to new terms forever after on a secondary market; the loan size there is L^* and the repayment required is R^*. Assume that he cannot default at all on the secondary market. He has a discount factor of δ between 0 and 1.

(a) Derive the borrower's no-default constraint for the current loan contract (L, R).

The borrower's per-period return on the secondary market is given by $F(L^*) - R^*$, and this is her value of v. Now we know from class that her no-default constraint from the contract (L, R) in the primary market is given by

$$\frac{F(L) - R}{1 - \delta} \geq F(L) + \delta \frac{v}{1 - \delta}.$$

So using the expression for v, we see that the constraint is

$$\frac{F(L) - R}{1 - \delta} \geq F(L) + \delta \frac{F(L^*) - R^*}{1 - \delta}.$$

(b) Show that if the borrower chooses not to default, she must enjoy a higher payoff from (L, R) than she does from (L^*, R^*). Prove this using the no-default constraint and also explain your answer intuitively.

Intuitively, the net return from the primary market must exceed the net return from the secondary market is because the latter acts as a punishment in case the borrower defaults, and the punishment value *must* be lower than the going value — because there is a temporary gain to default. Algebraically, using the condition from part (a) and rewriting slightly, we have

$$F(L) - R \geq (1 - \delta)F(L) + \delta[F(L^*) - R^*],$$

which means that the terms on the left is a convex combination of the two terms on the right (weighted by δ and $1 - \delta$). Because the first of these terms $F(L)$ exceeds $F(L) - R$, the second term must fall short of it.

(c) Suppose that after a default, the borrower has to wait several periods before access to *any* other market (getting a payoff of zero meanwhile), and then gets access to the secondary market at (L^*, R^*). Would part (b) still be true?

Not necessarily. It depends on how many periods he will have to wait before regaining access and how big the gain from default is. Intuitively, the wait acts as a punishment — because of discounting — and so it can serve to deter defaults even if (for instance) the borrower were to return to the primary market. Say the borrower has to wait until period T after defaulting. Then the no-default condition is

$$\frac{F(L) - R}{1 - \delta} \geq F(L) + \delta^{T+1} \frac{F(L^*) - R^*}{1 - \delta}.$$

Now it's different. You should check that if T is large enough and δ is close enough to 1, then the above inequality holds even if $F(L^*) - R^* \geq F(L) - R$.

(6) Another variation. There is a large population of borrowers and lenders. Each active borrower takes a loan L from a lender at every date, produces $F(L)$ and repays R. If a borrower defaults on a loan, the lender tries to spread the word that the borrower has defaulted. Just how successful he is in doing so is described as follows:

When a defaulting borrower appears to a new lender, the lender tries to remember if this borrower has been complained about. But with probability q between 0 and 1 he doesn't know the old lender, so the new lender may not have heard about the default. In that case the lender starts up a relationship with the borrower, and the past slate is wiped clean.

If the lender has heard about this borrower's past behavior, then he does not lend to the borrower. The borrower waits one period, getting zero, and then meets a lender again the day after, with exactly the same story repeating itself.

[a] If every borrower borrows on the same terms from every lender, show that the lifetime value to a borrower who has just defaulted — but does not plan to default again — is given by

$$V = \frac{q[F(L) - R]}{(1-\delta)[1-\delta(1-q)]}.$$

[Hint: Remember how we derived a similar equation in the last problem set.]

If V is the lifetime value to a defaulting borrower, then it must solve the equation:

$$V = q\frac{F(L) - R}{1-\delta} + (1-q)\delta V,$$

where the first term on the right hand side happens when the new lender has not heard about the default and so takes the borrower on board with a new lifetime contract, and the second term happens when the new lender *has* heard about the past default, so that the borrower gets zero today and then starts getting V again from next period onwards. That "second V" is calculated exactly like the "first V," so the "two V's" coincide. Solving this equation, we have the desired expression for V.

[b] Write down the no-default constraint using the solution from part (a), and show that after some simplification, it is given by

$$F(L) - R \geq [1-\delta(1-q)]F(L).$$

The no-default condition is given by:

$$\frac{F(L) - R}{1-\delta} \geq F(L) + \delta V = F(L) + \delta\frac{q[F(L) - R]}{(1-\delta)[1-\delta(1-q)]}.$$

Bring the second term on the right hand side over to the left hand side and simplify to get the answer.

[c] Using this formula, prove that if q is close to 1, then the market breaks down because no lender will be unable to recover his loan.

If q is close to 1, then the no-default condition reduces to

$$F(L) - R \geq F(L),$$

which can *never* be met for any positive value of repayment. The market dies. The reason is simple. If q is close to 1, then the lender almost never realizes that the borrower is a defaulter, and always lends to him. But then the borrower can default "for free" from the previous relationship.

[d] What can you say about the possibility of an active market for credit if q is close to zero?
If q is close to 0, then the no-default condition reduces to

$$F(L) - R \geq (1 - \delta)F(L),$$

and now we can get some action. For instance, it is possible for R to exceed L and for the above inequality still to hold, provided that δ is close enough to 1. In that case it is possible to have a flourishing credit market with positive payoffs to both lender and borrower. Intuitively, when $q \simeq 0$, a defaulting borrower can always be suitably tracked and isolated from the credit market.

[e] What does q mean? Explain using three examples: (i) a modern credit market in a developed country, (ii) a credit market in a very traditional rural society where everyone knows everyone, and (iii) a credit market in a fast-changing developing society without modern computer systems.

The probability q stands for the difficulty of information transmission. In both very traditional and very modern societies, the value of q can be close to zero, leading to efficient transmission. In traditional societies everyone knows everyone. In modern societies there are computers tracking your credit rating; so q can be close to zero in both cases. It is only in intermediate societies, not developed enough to have full computer networks tracing your credit rating, yet developed enough so that there is a lot of mobility, that *both* sources of information transmission can fail.

(7) Indebtia is a country in big trouble. It owes a huge amount of money to the ROW (the rest of the world), which is putting pressure on Indebtia to repay. ROW wants the Government of Indebtia to invest in a new project that will pay off the debt. The project will yield an amount 100 with probability p and nothing with probability $1 - p$. The government can put in effort $e \geq 0$ to influence this probability, and we suppose that

$$p(e) = e/200.$$

The cost of the effort to the government is $(1/400)e^2$.

The debt is massive, bigger than 100. ROW is trying to figure out how best to collect some part of it. That is, if R is the repayment asked for, and if Indebtia can pay only in the event of success, ROW is trying to maximize $p(e)R$. The problem is that e will depend on how much R is demanded. (And of course, R cannot exceed 100.)

(a) What is the net payoff to the Government of Indebtia if it incurs effort e and the repayment demanded is R?

It is given by

$$p(e)[Q - R] - (1/400)e^2 = \frac{(100 - R)e}{200} - (1/400)e^2.$$

(b) Maximize this payoff with respect to e to show that

$$e = 100 - R.$$

Maximize, and write down the first order conditions. The answer should fall out.

(c) Knowing this, solve for the repayment that ROW should demand from Indebtia.

ROW wants to maximize
$$p(e)R = \frac{100-R}{200}R.$$
This is maximized at $R = 50$. Clearly show your work, first-order condition etc.

(d) ROW can invest an additional amount x in Indebtia and double the size of the project from 100 to 200 (with the same probabilities of success as before). If x could be invested safely elsewhere by ROW at an interest of r, calculate the maximum value of x for which ROW will invest in Indebtia, knowing that it can adjust its repayment demand if it makes the investment.

If project size is doubled, then by the same logic as above, $e = (200 - R)/200$, and so ROW now maximizes
$$\frac{200-R}{200}R,$$
which by the above logic yields $R = 100$. So the expected return goes from
$$\frac{100-R}{200}R.$$
evaluated at $R = 50$, or 12.5, to
$$\frac{200-R}{200}R.$$
evaluated at $R = 100$, or 50. This is a gain of 37.5. So x will be invested if
$$37.5 > x(1+r), \text{ or if } x < \frac{37.5}{1+r}.$$

Answers to extra problems.

[12] Are the following statements true or false? It is *not* enough to just guess one or the other. You need to provide an argument for or against, and only then will any credit be awarded.

[**a**] Default risks can be adequately covered by the choice of a suitably high interest rate compared to the safe rate.

False. When the interest rate goes up, so does the probability of default, and so it may be impossible to fully cover the default risk. As shown in class, the formula for the interest rate i is
$$i = \frac{1+r}{p} - 1,$$
where p is the probability that the loan will be repaid. The problem is that p is generally a function of i, both in the adverse selection and in the moral hazard settings. At this stage you should be ready to give me an example from class: (i) adverse selection: higher interest rates create a riskier mix of borrowers, (ii) moral hazard with risk: higher interest rates creates a riskier *choice* of project, (iii) moral hazard with effort: a higher interest rate reduces effort

because of the debt overhang, or (iv) strategic default: a higher interest rate can create a bigger tendency to intentionally default on an existing loan. Any one will do.

[b] Borrowers and lenders that perfectly agree on maximizing expected returns might still disagree on the choice of project.

True. In general, because of limited liability, borrowers will be biased towards high risk projects, while lenders will be biased towards low-risk projects, even though both are interested in maximizing their expected return. Giving an example of some kind will be helpful here. In class, we used the following: Project A yields 120,000 for sure. Project B pays 220,000 with prob 1/2 and nothing with remaining probability. Interest rate is 10%. There is no collateral and limited liability. But then the lender would prefer Project A, while the borrower prefers to invest in Project B.

[c] Individuals with less collateral will want to take on riskier projects, all other things being equal.

True. This is right out of the section on moral hazard and risky project choice. You will need to give as an illustration the revealed preference argument we did for risky project choice, where we proved that a higher value of C must lead to a higher value of $p(\theta)$ in the choice of project θ.

[d] Collateral that's valuable for the borrower but completely valueless to the lender cannot be useful in enforcing the repayment of a debt.

False. It is true that if the collateral is valueless to the lender then he does not get anything if the debt goes un-repaid, but it is still true that the collateral affects the incentives of the borrower. For instance, amend the moral hazard model of effort to have the borrower still maximize:

$$p(e)[Q - R] - [1 - p(e)]C - e$$

but have the lender's payoff only given by $p(e)R$. That is, the collateral is not something that the lender values, so the payoff is different, but nevertheless as C goes up, the borrower will put in greater e, which increases the probability of repayment.

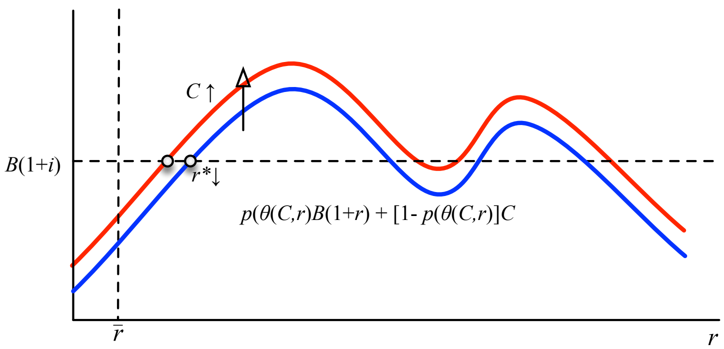
[e] In the moral hazard borrowing model, as we move down the incentive constraint for which lower repayment is demanded and higher effort is put into the project, the borrower's payoff is affected in an ambiguous way.

False. The borrower's payoff must rise. Here is a quick algebraic summary of what I said in class. Let $R_1 > R_2$ be two repayment demands and let e_1 and e_2 be the corresponding effort choices. We already know from class that $e_2 > e_1$. But more than that, by a revealed preference argument,

$$\begin{aligned} p(e_2)[Q - R_2] - [1 - p(e_2)]C - e_2 &> p(e_1)[Q - R_2] - [1 - p(e_1)]C - e_1 \\ &> p(e_1)[Q - R_1] - [1 - p(e_1)]C - e_1, \end{aligned}$$

where the first inequality is just revealed preference, and the second inequality follows from the fact that $R_1 > R_2$.

[f] In the model of moral hazard and risk-taking by borrowers, a higher collateral must improve the terms of credit to a borrower.



False. This is true when credit markets are competitive, but not necessarily if they are monopolistic. The accompanying diagram shows that when markets are competitive, the equilibrium settles at the left-most intersection of the profit curve with the horizontal line showing the opportunity cost of credit. When collateral improves, this curve shifts up because default rates go down and so does the amount of risk-taking. In the new equilibrium, the rate of interest falls. But with monopolistic lending, the monopolist will choose an interest rate that maximizes payoffs to him. When the curve goes up, the interest rate could move to the left or the right.

(9) A t-shirt exporter is trying to break into the market of a developed country. He can produce either high quality or low quality t-shirts. Production of the low quality shirt is costless, but production of the high quality t-shirt entails a cost of \$c per shirt.

The low quality has no value on the market, so the retailer tests the exporter by asking him to provide a sample (s units) for free. If the sample is bad, the exporter must start again with another retailer. If the sample is good, the retailer offers an indefinite contract for a high-quality order of $q > s$ units, for which he pays p per shirt *in advance*, where $p > c$. And it remains that way period after period, unless the exporter deviates by sending in low quality.

Every retailer behaves the same way, an exporter can meet a new retailer every period, and no retailer knows about the history of a new exporter who approaches him.

(a) Derive the discounted lifetime value of profits to the exporter starting from the sampling phase. (Of course, in what follows, you can assume this amount is positive.)

There is a sampling phase followed by an indefinite contract phase, so the present value is

$$-cs + +\delta\frac{(p-c)q}{1-\delta}.$$

(b) Show that the self-enforcement constraint for the exporter during the contract phase is given by

$$\delta(pq + s) \geq q(1 + \delta).$$

Once the exporter has the contract, his lifetime payoff is

$$\frac{(p-c)q}{1-\delta}$$

from compliance. If he deviates, then he supplies the low-quality merchandise, and thereafter needs to go through the trial process again with the next retailer he meets, so this yields a payoff of

$$pq - \delta cs + \delta^2 \frac{(p-c)q}{1-\delta},$$

where the first term is the immediate payoff from the deviation (he saves on production cost but gets paid in advance), the second term is the loss from the sampling phase, and the third term is the discounted return to the contract. The constraint is, therefore,

$$\frac{(p-c)q}{1-\delta} \geq pq - \delta cs + \delta^2 \frac{(p-c)q}{1-\delta}.$$

Rearranging and simplifying, we have

$$\delta(pq + cs) \geq cq(1+\delta).$$

(c) Prove that the self-enforcement contract can *never* hold when δ is close to zero, and that it does hold when δ is close enough to 1 provided that $pq + cs > 2cq$. Give some intuition for these results.

When δ is close to zero, then the future matters very little, so that the exporter can make a short-term gain today without caring much about the future. So the self-enforcement constraint will fail. We can easily see this from Part (b), where the inequality

$$\delta(pq + s) \geq q(1+\delta)$$

simply cannot hold if δ is close to zero. On the other hand, when δ is close to 1, then the above inequality holds provided $pq + cs > 2cq$.

When δ is close to 1, then for the exporter it becomes a question of supplying high and making two periods of honest profit equal to $2(p-c)q$, or deviate and earn pq today, sacrificing cs tomorrow. For the self-enforcement constraint to hold, it must be the case that the former option is better than the latter. That is,

$$2(p-c)q > pq - cs.$$

Rearranging, we have our condition $pq + cs > 2cq$.

(d) There is a second self-enforcement constraint: this is during the sampling phase. Show that this constraint is automatically satisfied as long as the lifetime value in part (a) is positive.

The second self-enforcement constraint comes up when the sample is being provided. Not providing the sample properly gets the exporter zero, thereby postponing the profits to the future. Because the profits are positive, there is no point in postponing them.

Now for the algebra. Compliance during sample phase yields

$$-cs + \delta \frac{(p-c)q}{1-\delta}$$

Deviation means that you get 0 today, followed by the same process starting up again, but discounted:

$$\delta \left[-cs + \delta \frac{(p-c)q}{1-\delta} \right]$$

Because $\delta < 1$, the first expression must be larger than the second.

(10) A borrower is about to take a loan of L, on which interest of r will be charged. He has collateral C, which he puts up for the loan. There is no strategic default, but there is moral hazard. The borrower puts in effort e at a cost $c(e)$ to generate a success probability of $p(e)$. Of course, $p(e)$ and $c(e)$ are increasing in e.

If there is success, then output is given by $Q = F(L)$ and the loan is repaid. If there is failure, output is zero and the borrower loses the collateral C.

As in class, e is chosen by the borrower in response to the terms of the loan contract and you can assume that there is a unique such optimal choice for every loan contract.

(a) Write down an expression for the net expected payoff of the borrower.

Net expected payoff is given by

$$p(e)[Q - L(1+r)] - [1 - p(e)]C - c(e).$$

(b) Prove that if the repayment r rises, or the collateral C falls, the optimally chosen effort put in by the borrower must come down.

The twist in this question is that we are not making any assumptions on the functions $p(e)$ and $c(e)$, except that a unique optimal choice exists for the borrower. So the analysis, which borrows the same techniques as the project choice model in class, can be much more general.

First define $Z = L(1+r) - C$. Then it is easy to see that the borrower chooses effort to maximize

$$p(e)[Q - Z] - c(e).$$

Consider two values of Z, call them Z_1 and Z_2, and let the corresponding unique effort choices be e_1 and e_2. By optimality,

$$p(e_1)[Q - Z_1] - c(e_1) > p(e_2)[Q - Z_1] - c(e_2),$$

and

$$p(e_2)[Q - Z_2] - c(e_2) > p(e_1)[Q - Z_2] - c(e_1).$$

Adding these two inequalities and canceling the common terms involving $c(e)$, we must conclude that

$$p(e_1)[Q - Z_1] + p(e_2)[Q - Z_2] > p(e_2)[Q - Z_1] + p(e_1)[Q - Z_2],$$

and rearranging and canceling some more common terms, we see that

$$[p(e_1) - p(e_2)][Z_2 - Z_1] > 0,$$

which is just another way of observing that e moves negatively with Z. Now simply unpack Z to get the desired result.

There is a second way to do this, if you assume that $p(e)$ is concave and $c(e)$ is convex or linear. This gets you a bit less credit because you have not been asked to assume these things. But anyway, you could then draw the benefit and cost curves from choosing e and show how these swivel when C or r changes, also getting the desired result.

(11) SelfHelp is a newly formed credit cooperative. Members of SelfHelp can deposit savings with the cooperative and they can also turn to SelfHelp for a loan if they need one. We summarize the payoff gain from all these activities as $S > 0$ per member, per period of time.

Without access to SelfHelp, payoff is just s per period, and we assume that $S > s$.

If a member misbehaves or defaults on a SelfHelp loan he is punished with a penalty of value F, and excluded from all future dealings with SelfHelp.

(a) If each member has a discount factor of δ between 0 and 1, write down the lifetime discounted value of having access to SelfHelp.

For this question, we assume that SelfHelp lasts forever, so that the lifetime discounted value is given by $S/(1-\delta)$.

(b) Suppose there is no telling whether SelfHelp will survive at any date; that is, for any date at which SelfHelp is currently alive, denote its conditional probability of survival in the *next* period by p. Now write down the *expected* lifetime discounted value of having access to SelfHelp, conditional on SelfHelp being alive today.

[Hint. Let V be this value. Then

$$V = S + \delta p V + \delta(1-p)\frac{s}{1-\delta}.$$

Understand why this equation is true, and then use it.]

Why is this equation true? If the lifetime value is given by V, it is made up of three components: (i) S today, (ii) V again starting tomorrow, discounted, if SelfHelp survives, and (iii) $s/(1-\delta)$ starting tomorrow, if SelfHelp fails. The beauty of this is that V appears both on the left and right hand sides, and helps us to solve the problem really compactly. Rearranging,

$$V = \frac{S}{1-\delta p} + \delta(1-p)\frac{s}{(1-\delta)(1-\delta p)}.$$

(c) Assume we are in the world of part (b). If a member needs to repay R to SelfHelp, write down the condition describing her incentives to repay.

Let's start from the instant the member needs to repay, so that the benefit from SelfHelp has already been enjoyed today. With that interpretation, if the repayment is made today, the return is zero, and from tomorrow the present discounted value is

$$\delta\left[pV + (1-p)\frac{s}{1-\delta}\right].$$

where V has been solved for in part (b). If the repayment is not made, the extra return today is R, followed by sure expulsion from SelfHelp, so that the lifetime discounted payoff is

$$R + \delta\frac{s}{1-\delta}.$$

Therefore the member will repay if

$$\delta\left[pV + (1-p)\frac{s}{1-\delta}\right] \geq R + \delta\frac{s}{1-\delta}.$$

or equivalently, if

$$R \leq \delta p\left[V - \frac{s}{1-\delta}\right].$$

Now use the formula for V from part (b) to see that

$$R \leq \delta p\frac{S-s}{1-\delta p}.$$

(d) Assume that p is a function of the percentage n of members who repay. Explain why this might be the case. Write down any one functional form that could link n to p.

How about $p = n$ or more generally, $p = n^{\alpha}$ for any $\alpha > 0$?

(e) Using the functional form you wrote down in part (d), and also using parts (b) and (c), show that there can be two outcomes or equilibria for the *same* parameters: (i) there is no (voluntary or strategic) default and SelfHelp survives with little or no government assistance or (ii) there are high rates of default and SelfHelp survives with low probability.

Recall the formula for repayment from part (c):

$$R \leq \delta p\frac{S-s}{1-\delta p}.$$

If no-one expects anyone to repay, then expected $n = 0$ and consequently, for any of the formulae in part (d), we also have $p = 0$. Therefore the right hand side of the above inequality is zero, and every positive R *will* be defaulted upon. Therefore $n = 0$ is always an equilibrium.

On the other hand, what if we expect everyone to repay? This is not a done deal, for the above inequality reduces to

$$R \leq \delta\frac{S-s}{1-\delta},$$

and this is not automatically satisfied unless δ close enough to 1, or if R is small enough, or if $S - s$ is large enough. But if it is, we have multiple equilibria.

(12) A *risk-lover* is an individual who prefers a lottery with some expected value to a fixed amount of money with the same value.

(a) Let $u(x)$ be the utility function of a risk-lover. Write down a specific example of $u(x)$ that exhibits risk-loving behavior.

A risk-loving person typically has a convex, not concave utility function. The best way to see it is to draw a picture of a convex function and then feed it with two-outcome risky lotteries. As you move the outcomes apart, keeping the means the same, the expected utility of the lottery will go up. An example of such a function is $u(x) = x^2$.

(b) Show that with limited liability, a risk-neutral or even a risk-averse borrower may behave *as if* he is a risk-lover when considering the projects in which he would like to invest. Explain how this feature (of limited liability) drives a wedge between projects that the lender considers profitable and those that the borrower considers profitable.

Exactly as done in class.