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Econ-UA 323

Development Economics

Answers to Problem Set 5

(1) The economy of ShortLife has two kinds of jobs, which are the only sources of income for the people. One kind of job pays \$200, the other pays \$100. Individuals in this economy live for two years. In each year, only half the population can manage to get the high-paying job. The other half has to be content with the low-paying one. At the end of each year, everybody is fired from existing positions, and those people assigned to the high-paying job next year are chosen randomly. This means that at any date, each person, irrespective of past earnings, has probability 1/2 of being selected for the high-paying job.

(a) Calculate the coefficient of variation based on people's incomes in any one particular period. Now calculate each person's average per period *lifetime* (i.e., two-year) income and compute the coefficient of variation again based on *these* incomes. Does the latter measure suggest more or less inequality? Explain why.

In any one year, half the people have 100, and the other half have 200. The formula for the coefficient of variation is

$$C = \frac{1}{\mu}\sqrt{\sum_{j=1}^{m} n_j (y_j - \mu)^2},$$

where n is the total number of people, n_i the share of people in income class i, y_i is the income of class i, and μ is the mean income.

In our case, we have $(y_1, y_2; n_1, n_2)$ equal to $(100, 200; 1/2, 1/2)$ in any year. Mean income μ is therefore given by 150. Consequently, the coefficient of variation in this case is

$$\begin{aligned} C &= \frac{1}{150}\sqrt{\frac{1}{2}(100-150)^2 + \frac{1}{2}(200-150)^2} \\ &= 1/3. \end{aligned}$$

Now calculate lifetime income. The expected income in the second period is 150 for everybody (this is because there is a probability 1/2 of getting a high job and a probability 1/2 of getting the low job in period 2). Thus average income for someone currently holding a low job is $\frac{100+150}{2} = 125$, while for someone holding a high job it is $\frac{200+150}{2} = 175$. Note that there is a narrowing of average incomes relative to part (a), because of the mobility in the economy. You can calculate the coefficient of variation as above, and you will see that it is lower.

(b) Now change the scenario somewhat. Suppose that a person holding a job of one type has probability 3/4 of having the same kind of job next year. Calculate the expected lifetime income (per year average) of a person who currently has a high-paying job, and do the same for a person with a low-paying job. Compute the coefficient of variation based on these expected per period incomes and compare it with the measure obtained in case (a). Explain the difference you observe.

See below, part (c).

(c) Generalize this idea by assuming that with probability p you hold your current job, and with probability $1-p$ you change it. Find a formula for inequality as measured by the coefficient of variation for each p, examine how it changes with p, and explain your answer intuitively.

I will just do part (c) because it is a generalization of part (b) (but you should try part (b) separately). If you hold your current job with probability p, then for a low income person (today), the expected income tomorrow is $100p + 200(1-p)$. For a high income person (today) it is $100(1-p) + 200p$. Thus expected *average* incomes for the low income person and high income person are $50 + 50p + 100(1-p)$ and $100 + 100p + 50(1-p)$ respectively. The mean income in this society is still 150, as you can easily see by taking the 50-50 average of these two incomes. So the coefficient of variation, calculated just as in part (a), is

$$
\begin{aligned}
C &= \frac{1}{150}\sqrt{\frac{1}{2}(150 - 50 - 50p - 100(1-p))^2 + \frac{1}{4}(100 + 100p + 50(1-p) - 150)^2} \\
&= \frac{1}{150}\sqrt{\frac{1}{2}(50p)^2 + \frac{1}{4}(50p)^2} = \frac{p}{3}.
\end{aligned}
$$

Note that if $p = 1$, with full immobility, this gives you exactly the same answer as in part (a) for single-period inequality, while if $p = 1/2$, we get exactly the same answer as in part (a) for lifetime inequality. This is as it should be. If $p = 1$, there is no mobility at all (why?), so that the answer to overall inequality is the same as the answer to inequality *within* a single time period. In contrast, if $p = 1/2$, there is perfect mobility, which is the case studied for lifetime inequality in part (b). As p goes up from $1/2$ to 1, mobility becomes progressively lower and lower, and in response the coefficient of variation goes up, signaling greater inequality in the presence of lower mobility. When $p = 3/4$, you get the case in part (b).

It might also be fun — though not very realistic — to think about the case in which p is *smaller* than $1/2$. Now the fact of holding a high job means that you are *more* likely to hold a low job tomorrow. When $p = 0$ your high job today precisely cancels with your low job tomorrow, and lifetime inequality goes down to zero, as confirmed by $C = 0$.

(2) Draw various Lorenz curves.

Just apply the formulae and draw the Lorenz curves. This is good practice! Let me only comment on the statement at the end of the question: understanding the implicit transfers that are moving us from one distribution to the other. (i) $[a \to b]$: there are no transfers, the inequality should be the same by the relative income principle. (ii) $[b \to c]$: there should be no change in inequality by a double application of the relative income principle and the population principle. You should thus get the same Lorenz curves for distributions (a), (b), and (c).

For parts (d)–(f), there are implicit transfers that move you from one distribution to the other, but you should always *first* use the relative income principle to get total income to be the same across the two distributions. Then you will be able to visualize the transfers very well. You should try and explain whether the move can be accomplished by simply regressive transfers or by progressive transfers alone, or whether you need a mix of the two.

(3) The economy of Nintendo has ten people. Three of them live in the modern sector of Nintendo and earn \$3000 per month. The rest live in the traditional sector and earn only \$1000 per month. One day, two new modern sector jobs open up and two people from the traditional sector move to the modern sector.

(a) Show that the Lorenz curves of the income distributions before and after must cross. Do this in two ways: (i) by graphing the Lorenz curves and (ii) by first expressing both income distributions as divisions of a cake of size 1, and then showing that the two distributions are linked by "opposing" Dalton transfers.

The beginning income distribution is given by

$$(1000, 1000, 1000, 1000, 1000, 1000, 1000, 3000, 3000, 3000)$$

and the final distribution is

$$(1000, 1000, 1000, 1000, 1000, 3000, 3000, 3000, 3000, 3000)$$

Total income in the first case is 16,000, and in the second case is 20,000. Let's scale incomes in the second distribution by using the relative income principle so that the sum is 16,000. This means we multiply all incomes in the second distribution by 4/5. This gives us the following distribution (which has the same inequality as the second (by the relative income principle):

$$(800, 800, 800, 800, 800, 2400, 2400, 2400, 2400, 2400)$$

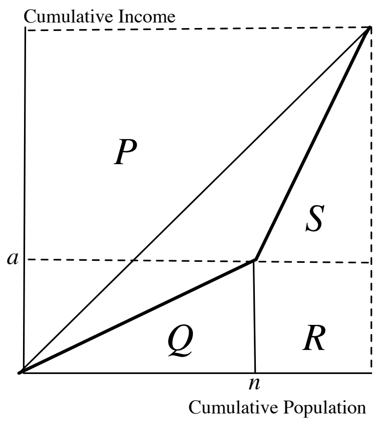
Now focus on individuals 1 through 5. They have lost money relative to the first distribution. Look at individuals 6 and 7. They have gained. Individuals 8 through 10 have lost, again in relative terms. So this is as if there has been a disequalizing transfer from 1–5 to 6–7, and an equalizing transfer from 8–10 to 6–7. These transfers run in opposite directions (and do not cancel each other out), so that the Lorenz curves must cross. You can also verify this directly by drawing the Lorenz curves.

(b) Calculate the Gini coefficients and the coefficients of variation of the two distributions.

Just apply the formulae.

(4) A beautiful property of the Gini coefficient is that it turns out to be proportional to the *area* between the Lorenz curve of an income distribution and the 45^0 line. (It is actually equal to the ratio of this area relative to the area of the triangle formed below the 45^0 line. Try and prove this for the case in which there are just two income classes, with a proportion n in the poorer class who earn a fraction a of total income, and the remaining richer fraction $1 - n$ earning the remaining fraction $1 - a$ of income.

We've drawn the Lorenz curve that pertains to the question. We are going to show that the Gini is the entire square minus the sum of P, Q, R, and S.



We know that the n poorer fraction earn a fraction a of income, so $a < n$. Because a fraction n people effectively earn a/n each, and $1-n$ people earn $(1-a)/(1-n)$ each, the Gini is given by

$$G = n(1-n)\left[\frac{1-a}{1-n} - \frac{a}{n}\right] = (1-a)n - a(1-n) = n - a.$$

Now we do some area calculations using the diagram. The desired area equals

$$\begin{aligned} 1 - P - Q - R - S &= 1 - \frac{1}{2} - \frac{an}{2} - (1-n)a - \frac{(1-n)(1-a)}{2} \\ &= \frac{1}{2}(n-a). \end{aligned}$$

So the Gini is the ratio of the desired area divided by the area of the triangle below the 45^0 line, which is $1/2$.

(5) Are the following statements true, false, or uncertain? In each case, back up your answer with a brief, but precise explanation.

(a) The mean absolute deviation satisfies the Dalton transfer principle.

False. Refer to discussion in class. The mean absolute deviation is unresponsive to transfers that occur on the same side of the mean.

(b) If the Lorenz curve of two situations cross, the Gini coefficient and the coefficient of variation must disagree.

False. They may agree or they may not. You should refer to class lectures for an example where they disagree, and try and construct an example for which the two measures agree even though the Lorenz curves are crossing. (Hint: take a look at Question 3b.) But what *is* definitely true is that the two measures can *only* disagree if the Lorenz curves cross and not otherwise, as they are both Lorenz-consistent.

(c) If a relatively poor person loses income to a relatively rich person, the Gini coefficient *must* rise.

True. See class slides or text.

(d) The Lorenz curve must *necessarily* be convex; that is, going from left to right on the Lorenz diagram, the slope of the Lorenz curve can never fall, and will generally rise.

True. The Lorenz curve has on its horizontal axis cumulative percentages of the population arrayed from poorest to richest. So each successive percentile (or decile, or quantile more generally) must add at least as much cumulative income to the cumulative income share than the previous percentile (or decile, or quantile). That means that the slope of the Lorenz curve can never decline and will generally increase as we move over progressively richer quantiles.

(e) The ethical principles of inequality measurement — anonymity, population, relative income, and transfers — are enough to compare any two income distributions in terms of relative inequality.

False. These principles do allow us to make judgements when Lorenz curves do not cross, or equivalently, when there is a sequence of Dalton transfers all going in the same direction. When opposing Dalton transfers occur, we need a way to compare the strengths of these opposing transfers. This is not possible with the ethical principles stated here. Thus, for instance, the Gini and the coefficient of variation, which are both consistent with all these principles, nevertheless might disagree with each other in their comparison of inequality across pairs of income distributions that have crossing Lorenz curves.

(f) If everybody's income decreases by a constant dollar amount, inequality *must* rise.

True. Imagine that the original income distribution is $(y_1, y_2, \ldots, y_n)$, written in ascending order. Now it is $(y_1 - 100, y_2 - 100, \ldots, y_n - 100)$. Using the relative income principle, rescale the new situation so that it has the same mean income as it did before: to do this, you simply multiply all new incomes by the fraction $\frac{\mu}{\mu - 100}$, where μ was the mean income of the original income distribution (why?). Therefore the rescaled income for person i is just

$$y_i' \equiv (y_i - 100)\frac{\mu}{\mu - 100}.$$

Note that y_i' is lower than y_i if y_i was less than mean income, is higher than y_i if y_i had exceeded mean income, and stays unchanged if y_i had been precisely *at* mean income. This means that we effectively have a set of disequalizing transfers from relatively poor (those below the mean in this example) to relatively rich (those above the mean in this example), which means that inequality must have gone up (by the transfers principle).

(6) Suppose that there are three individuals with three wealth levels in the economy. Denote the wealth levels by A, B, and C, and suppose that $A > B > C$.

(a) Suppose that the person with wealth level A earns an annual income of w_A, and consumes a fraction c_A of it, investing the rest. If the rate of interest on asset holdings is r, write down a formula for this person's wealth next year.

Wealth next year is

$$A' = (A + [1 - c_A]w_A)(1 + r).$$

(b) Show that if income earnings of each individual is proportional to wealth (that is, $w_B/w_A = B/A$ and $w_C/w_B = C/B$), and if the savings rate is the same across individuals, inequality of wealth next year must be the same as inequality this year, as measured by the Lorenz curve.

Denote the common consumption rate by s. Take any two persons, say the ones with wealth A and B, and suppose that $A = \lambda B$. Then it is also true that $w_A/w_B = A/B$, by the assumption of the question. Then the ratio of wealth next year is given by

$$\begin{aligned}\frac{A'}{B'} &= \frac{(A + sw_A)(1 + r)}{(B + sw_B)(1 + r)}\\ &= \frac{A + sw_A}{B + sw_B}\\ &= \frac{\lambda B + s(\lambda w_B)}{B + sw_B}\\ &= \lambda,\end{aligned}$$

which means that the ratio of the two wealths next year is the same as the ratio of the two wealths this year. But the Lorenz curve is insensitive to the uniform scaling of all wealths and must therefore be unchanged.

(c) Retain the same assumptions on income and wealth as in part (b) but now suppose that the savings rates satisfy $s_A < s_B < s_C$. Now how does the Lorenz curve for wealth next year compare with its counterpart for the current year?

Try the same logic as above. Take any two people, say those with wealth A and B with $A > B$. A similar chain to that above holds, but there is one inequality when we replace s_A by s_B:

$$\begin{aligned}\frac{A'}{B'} &= \frac{(A + s_A w_A)(1 + r)}{(B + s_B w_B)(1 + r)}\\ &= \frac{A + s_A w_A}{B + s_B w_B}\\ &< \frac{A + s_B w_A}{B + s_B w_B}\\ &= \frac{\lambda B + s_B \lambda w_B}{B + sw_B}\\ &= \lambda,\end{aligned}$$

which means that the ratio of A to B has come down if $A > B$ to begin with.

So the Lorenz curve must be more equal next period. You can draw a diagram and show that the poorest third and the poorest 2/3 own a larger fraction of total wealth in the next period than they did before.

(d) Carry out the same exercise as in part (c), but now assume that all wages are equal and so are savings rates.

We can again take the same approach of comparing ratios. Take any two wealths A and B with $A > B$. Write their common savings rate as s and their common income as y. Then

$$\begin{aligned}
\frac{A'}{B'} &= \frac{(A+sy)(1+r)}{(B+sy)(1+r)} \\
&= \frac{(A+sy)}{(B+sy)} \\
&= \frac{\lambda B + sy}{B + sy} \\
&< \frac{\lambda B}{B} \\
&= \lambda = \frac{A}{B}
\end{aligned}$$

where the above inequality follows from the fact that if you add the same positive number to the numerator and denominator of a fraction that is larger than 1, the value of the fraction must go down.

Now the Lorenz curve must be more *equal* next period. You can draw a diagram and show that the poorest third and the poorest 2/3 own a larger fraction of total wealth in the next period than they did to begin with.

(7) This problem tests your understanding of the effect of the income distribution on savings rates.

The economy of Sonrisa has people in three income categories: poor, middle class, and rich. The poor earn \$500 per year and have to spend it all to meet their consumption needs. The middle class earn \$3,000 per year, of which \$2,000 is spent and the rest saved. The rich earn \$10,000 per year, and consume 80% of it, saving the rest.

(a) Calculate the overall savings rate in Sonrisa if 30% of the people are poor and 60% are in the middle class.

The safest way to do this question is to find out total income and total savings in an economy of, say, 100 people in the proportions given, and then divide the latter by the former. There will be 30 poor people, 60 middle-class, and 10 rich people. So total income is \$ $(500 \times 30 + 3000 \times 60 + 10,000 \times 10)$, which is \$ 295,000. Total savings is \$ $(0 \times 30 + 1000 \times 60 + 2000 \times 10)$, which is \$80,000. It follows that the overall rate of savings in the economy is $\frac{80{,}000}{295{,}000}$, which is approximately 0.27, or 27%. *Note well that the overall rate of savings is affected by the distribution of income, because people in different income categories save different amounts.*

(b) Suppose that all growth occurs by moving people from the poor category to the middle-class category. Will the savings rate rise over time or fall? Using the Harrod–Domar model and assuming that population growth is zero and all other variables are exogenous, predict whether the resulting growth rate will rise or fall over time.

It should be obvious that this kind of growth leads to an increase in the savings rate because the middle-class saves a positive fraction of income, while the poor do not. Using the formula for the Harrod-Domar model, the rate of growth should accelerate.

(c) Outline another growth scenario with positive growth where the rate of growth changes over time in a way opposite to that of (b).

Consider a scenario where growth occurs by shifting people from the middle-class to the rich. Because the rich save a lower fraction of their income than the middle-class do, this should bring down the rate of savings over time, and therefore the rate of growth.

(d) Understand well that this question asks you about how growth *rates* are changing. In the simple Harrod–Domar model, the growth rate is constant over time because the savings rate is presumed to be unchanging with the level of income. Do you understand why matters are different here?

Recap this for your own understanding.

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(8) A well-known measure of economic inequality is the log variance. It is measured by

$$\mathrm{LV} = \frac{1}{n}\sum_{i=1}^{m} n_i[\ln y_i - \ln \mu]^2.$$

Show that this measure satisfies anonymity, the population principle and the relative income principle, but fails to be Lorenz-consistent.

Anonymity and the population principle are trivial to check. For the relative income principle, use the formula that $\ln(\lambda y) = \ln \lambda + \ln y$ for all positive λ and y and apply it to the formula for the log variance.

What follows is more subtle and you don't need to prepare it for an exam.

To show that the Pigou-Dalton principle is violated, all we need is an example. So suppose that there are just three incomes y_1, y_2 and y_3, with population shares $1-2n$, n and n respectively, where $n < 1/4$. μ tis mean income, given by $(1-2n)y_1 + ny_2 + ny_3$. Then the formula for the log variance reduces to

$$\mathrm{LV} = \left[(1-2n)(\ln y_1 - \ln \mu)^2 + n(\ln y_2 - \ln \mu)^2 + n(\ln y_3 - \ln \mu)^2\right]$$

I want you to think of n as small enough so that μ is always smaller than the smaller of the two incomes, which is y_2. Now, if a transfer of ϵ is made from the y_2-group to the y_3-group, then

$$\mathrm{LV} = \left[(1-2n)(\ln y_1 - \ln \mu)^2 + n(\ln\{y_2 - \epsilon\} - \ln \mu)^2 + n(\ln\{y_3 + \epsilon\} - \ln \mu)^2\right]$$

Differentiate this with respect to ϵ:

$$\frac{d\mathrm{LV}}{d\epsilon} = 2n\left[-\frac{\ln\{y_2 - \epsilon\} - \ln \mu}{y_2 - \epsilon} + \frac{\ln\{y_3 + \epsilon\} - \ln \mu}{y_3 + \epsilon}\right],$$

and evaluate the result for $\epsilon \simeq 0$:

$$\frac{d\mathrm{LV}}{d\epsilon}_{|\epsilon=0} = 2n\left[-\frac{\ln y_2 - \ln \mu}{y_2} + \frac{\ln y_3 - \ln \mu}{y_3}\right].$$

Now, to get our example going, we fix y_2 and make y_3 really big. All along, we adjust n (make it smaller) so that the mean income stays absolutely constant as we move y_3. It is a fact of life that $(\ln y)/y \to 0$ as $y \to \infty$, so that means that $[\ln y_3 - \ln \mu]/y_3$ converges to 0 as $y_3 \to \infty$. For large enough y_3, therefore, the derivative of LV with respect to ϵ is *negative* evaluated at $\epsilon \simeq 0$. But now we have our required example: the Dalton transfer from relatively poor to relatively rich can *reduce* inequality as measured by LV, which is what we set out to show.