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Econ-UA 323

Development Economics

Answers to Problem Set 2

(1) This question involves detailed numerical calculations and will not be a typical exam question. But it will help you understand the basic growth model.

The economy of Ping Pong produces its output using capital and labor. The labor force is growing at 2% per year. At the same time, there is "labor-augmenting" technical progress at the rate of 3% per year, so that each unit of labor is becoming more productive.

(a) How fast is the effective labor force growing?

Effective labor grows at the rate of labor force growth *plus* the rate of labor-augmenting technical progress, so the answer is 5% per year. Well, it's a bit different as we discussed in class: effective labor grows by a factor of $(1+n)(1+\pi)$, which equals $1+n+\pi+n\pi$, and we neglect the $n\pi$ term which is relatively small. To illustrate: if $n = 0.02$ and $\pi = 0.03$, then $1+n+\pi+n\pi$ is 1.0505, which implies a growth rate of 5.06%. The 0.06% part is tiny, so the acceptable answer is 5%, but the *real* answer is 5.06%.

(b) Now let's look at production possibilities in Ping Pong. We are going to plot a graph with capital *per unit of effective labor* ($\hat{k}$) on the horizontal axis and output *per effective unit of labor* ($\hat{y}$) on the vertical axis. Here is a description of the "production function" that relates $\hat{y}$ to $\hat{k}$. As long as $\hat{k}$ is between 0 and 3, output ($\hat{y}$) is given by $\hat{y} = (1/2)\hat{k}$. After $\hat{k}$ crosses the level 3, an *additional* unit of $\hat{k}$ only yields one-seventh *additional* units of $\hat{y}$. This happens until $\hat{k}$ reaches 10. After that, each additional unit of $\hat{k}$ produces only one-tenth additional units of $\hat{y}$. (To draw this graph, you may want to measure the $\hat{y}$ axis in larger units than the $\hat{k}$ axis; otherwise, the graph is going to look way too flat.) On a graph, plot this production function. What are the capital–output ratios at $\hat{k} = 2$, 6, and 12? Note that the answers you get in the case $\hat{k} = 6$ and 12 are different from what happens at the margin (when you increase capital by one unit). Think about why this is happening.

Note: in class, we talked about the output-capital ratio rather than the capital-output ratio. These are just flip sides of the same coin! If θ is the output-capital ratio, the capital-output ratio is nothing but its reciprocal, which is $1/\theta$.

I am going to skip the graph which you should be able to do without a problem. Let's calculate capital-output ratios. At $\hat{k} = 2$, total output is $\hat{y} = 1$. So the ratio of capital to output is 2. (And the output-capital ratio is just its reciprocal: 1/2; you should be totally comfortable with either concept.) At $\hat{k} = 6$, we have to figure out what total output is. The first three units of $\hat{k}$ produce $y = 1.5$ units of output. The next three produce an *additional* $3/7$ units of output. So output per effective labor when $\hat{k} = 6$ is $(3/2) + (3/7) = 27/14$. The capital-output ratio is, therefore, $6 \times (14/27)$, which is approximately 3. True to the discussion in class, the capital-output ratio *rises* with $\hat{k}$, and correspondingly the output-capital ratio falls with $\hat{k}$.

Similarly, you can work out the capital-output ratio for $k = 12$. Omitted here, as the way of calculating it is exactly the same.

Note that these answers are *different* from the "marginal" capital-output ratio in each of the relevant regions of the production function. For instance, at $\hat{k} = 6$, each additional unit of output is requiring 7 units of capital, *not 3*. The average ratio is smaller than the marginal ratio, because the former includes capital applied in the earlier phase of the production function, where its marginal product is higher. The difference between marginal and average capital output ratios comes from diminishing returns to physical capital in production.

(c) Now let us suppose that Ping Pong saves 20% of its output and that the capital stock is perfectly durable and does not depreciate from year to year. If you are told what $\hat{k}(t)$ is, describe precisely how you would calculate $\hat{k}(t+1)$. In your formula, note two things: (i) convert all percentages to fractions (e.g., $3\% = 0.03$) before inserting them into the formula and (ii) remember that the capital–output ratio *depends* on what the going value $\hat{k}(t)$ is, so that you may want to use the symbol $1/\theta$ for the capital-output ratio, to be replaced by the appropriate number once you know the value of $\hat{k}(t)$ (as in the next question).

Let's go through the derivation of the Solow model. New capital is simply old capital plus extra investment. But savings equals investment. So capital in period $t+1$ is related to what happens in period t by the equation

$$K(t+1) = K(t) + sY(t)$$

where s is the savings rate, and $Y(t)$ is income in period t. (**There is no depreciation**.) Now, we divide by the effective labor force $e(t)L(t)$ at time t. Remember that $\hat{k}(t) = K(t)/e(t)L(t)$ and $\hat{y}(t) = Y(t)/e(t)L(t)$ for all t. So we get

$$\frac{K(t+1)}{e(t)L(t)} = \hat{k}(t) + s\hat{y}(t)$$

(This is all a rehash of stuff done in the lectures.) Now we play with the left-hand side: $[K(t+1)/e(t)L(t)] = [K(t)/e(t+1)L(t+1)] \times [e(t+1)L(t+1)/e(t)L(t)]$, which is just $\hat{k}(t+1) \times 1.05$ (using part (a)). Substitute this in the equation above to get:

$$(1.05)\hat{k}(t+1) = \hat{k}(t) + s\hat{y}(t).$$

To finish the formula, we know that $\hat{k}(t)$ and $\hat{y}(t)$ are linked by whatever the capital-output ratio is at date t. This ratio is not a constant but varies as described in part (b) and in class. If we use the notation $\theta(t)$ for the *output-capital* ratio as we did in class, then $\hat{k}(t) = \hat{y}(t)/\theta(t)$. Using this in the formula above, and recalling that $s = 1/5$, we get

$$(1.05)\hat{k}(t+1) = \hat{k}(t)\left[1 + \frac{\theta(t)}{5}\right]. \quad (1)$$

Now we have a formula that can precisely compute $k(t+1)$, given any value of $k(t)$.

(d) Now, using a calculator if you need to and starting from the point $\hat{k}(t) = 3$ at time t, calculate the value of $\hat{k}(t+1)$. Likewise do so if $\hat{k}(t) = 10$. From these answers, can you guess in what range the long-run value of $\hat{k}$ in Ping Pong must lie?

At $\hat{k}(t) = 3$, the value of $\hat{y}(t)$ is $3/2$, which means that θ at that point is $1/2$ So, using our trusty formula (1), we see that $(1.05)\hat{k}(t+1) = 3[1 + (1/10)]$, which will give you a value of $\hat{k}(t+1)$ that *exceeds* 3, what we started with in the previous period. The idea, as discussed in class, is that capital is more productive at the margin when its level is low, so that the economy tends to accumulate capital more quickly than effective labor, raising $\hat{k}$.

At $\hat{k}(t) = 10$, figure the value of $\theta(t)$: you will see that it is $1/4$. Using our formula (1) again, $(1.05)k(t+1) = 10[1 + (1/20)]$, which means that $\hat{k}(t+1)$ is also 10. By a stroke of luck, we have found the steady-state ratio $\hat{k}^*$.

You should appreciate the point that this is just pure luck. What would have happened had you started with $\hat{k}(t) > 10$?

(e) Calculate the long-run value of $\hat{k}$ in Ping Pong. (Hint: You can do this by playing with different values or, more quickly, by setting up an equation that tells you how to find this value.)

Let's try to set up an equation. At the steady state value of $\hat{k}^*$, we will have $\hat{k}(t+1) = \hat{k}(t) = \hat{k}^*$. Use this in the formula: you see that $\hat{k}^*$ drops out (it appears on both sides of the equation), so that $1.05 = [1 + (\theta^*/5)]$, where θ^* is the output-capital ratio at the steady state. Solve this equation to see again that $\theta^* = 1/4$. Unlike the previous problem, this answer is not luck but comes from the insight of using $\hat{k}(t) = \hat{k}(t+1)$ in our formula.

(2) Let's study the production function $Y = AK^aL^b$ in more detail.

(a) We say that a production function satisfies *increasing returns to scale* if a proportional increase in all inputs raises output by *more* than that proportion. Prove carefully that this happens if and only if $a + b > 1$.

A production function satisfies *increasing returns to scale* if a proportional increase in all inputs raises output by *less* than that proportion. In algebra, we have that *for every* $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

Applying this inequality to the Cobb-Douglas form it means that

$$A(\lambda K)^a(\lambda L)^b > \lambda AK^aL^b$$

for $\lambda > 1$. But the above holds if and only if

$$\lambda^{a+b} > \lambda$$

for $\lambda > 1$, which happens if and only if $a + b > 1$.

(b) Using basic calculus, find a formula for the marginal product of labor. Show that if there is constant returns to scale (so $b = 1 - a$), the marginal product depends *only* on the ratio of capital to labor. But this is no longer true if $a + b \neq 1$. Explain why intuitively.

Using derivatives, we see that the marginal product of labor is given by

$$\text{MPL } = AbK^aL^{b-1}.$$

When $a + b = 1$, this reduces to

$$\text{MPL } = Abk^a,$$

where $k = K/L$. Notice that MPL only depends on the ratio of capital to labor in this case. Intuitively, when we scale up capital and labor, there are two effects. One is that labor goes up, and by diminishing returns to an input, this tends to lower the marginal product of labor. But the other effect is that capital goes up and that tends to increase the marginal product of labor. With constant returns to scale the two effects *just* cancel out.

(c) Notice that if the production function has constant returns to scale, it *automatically* generates diminishing returns to each input. Show this with algebra but then also try to explain it intuitively.

Under constant returns to scale, we have $a + b = 1$. Because each input is productive (it is an input after all!), we know that each of a and b is strictly positive. But that must also mean — because $a + b = 1$ — that each of them must also be strictly less than 1. Voilà, diminishing returns to inputs.

Intuitively, constant returns to scale means that as you scale up all inputs in proportion, output goes up by exactly that proportion. But because each input enhances the marginal product of the other inputs, it follows that if we just scale up a subset of the inputs in some proportion, output must rise by *less* than that proportion, which gives us diminishing returns to every subset of inputs.

(3) (This could be part of a typical exam question.) Suppose that the country of Xanadu saves 20% of its income and has a constant capital–output ratio of 4. Assume capital does not depreciate.

(a) Using the Harrod–Domar model, calculate the rate of growth of total GNP in Xanadu.

Neglecting depreciation as we've been asked to do, the Harrod-Domar model leads us to the equation: $g = s\theta$, where g is the aggregate growth rate, s is the rate of savings, and θ is the output-capital ratio. Here $s = 1/5$ and $\theta = 1/4$ (because the capital-output ratio is 4). So $g = 1/20$, or 5% per year.

(b) If population growth were 3% per year and Xanadu wanted to achieve a growth rate *per capita* of 4% per year, what would its savings rate have to be to get to this growth rate?

We know that the per-capita growth rate is the *aggregate* growth rate minus the population growth rate. Therefore, if the required per-capita growth rate is 4% and the population growth rate is 3%, the required aggregate growth rate is 7% per year, or 7/100. Using the

Harrod-Domar equation, we see, therefore, that the required rate of savings is g/θ, which in this case is $(7/100) \times 4$, or 28% of income.

(c) Now go back to the case where the savings rate is 20% and the capital–output ratio is 4. Imagine, now, that the economy of Xanadu suffers violent labor strikes every year, so that whatever the capital stock is in any given year, a quarter of it goes unused because of these labor disputes. If population growth is 2% per year, calculate the rate of per capita income growth in Xanadu under this new scenario.

The trick in this problem is to calculate what is, *effectively*, the capital-output ratio in Xanadu because of the labor problems. Let's use a symbol for the capital output ratio; say $\gamma = 1/\theta$. Then γ is the amount of capital you need to produce a single unit of output. With the strikes, you effectively end up using more than that. How much more? Well, it must be $\gamma \times (4/3)$. After all, if you take away a quarter of this, you will get back exactly γ. So the effective capital-output ratio is now $4 \times (4/3) = 16/3$. Using this in the Harrod-Domar equation with a rate of savings at $1/5$, we see that $g = 3/80$, which is 3.75% per year. Subtract the population growth rate, to get per-capita growth at 1.75% per year.

(4) (Again, a possible exam question, this time on the Solow model.) A country has a production function which depends on capital and labor, and it is given by

$$Y = 100K^{1/3}L^{2/3}.$$

(a) Describe per-capita production y as a function of the per-capita capital stock k.

Divide both sides of the production function by L; then:

$$\frac{Y}{L} = \frac{100K^{1/3}L^{2/3}}{L} = 100\left(\frac{K}{L}\right)^{1/3},$$

so that, remembering that $y = Y/L$ and $k = K/L$, we have:

$$y = 100k^{1/3}.$$

(b) Assume that capital does not depreciate at all, and that there is no technical progress. Let the savings rate be given by s and the rate of growth of population (=labor) be given by n. Show that steady state output per-capita is given by

$$y^* = 1000\sqrt{s/n}, \tag{2}$$

describing precisely all the steps that lead to this conclusion.

Let us recall the Solow equation for capital accumulation, which is:

$$K(t+1) = K(t) + sY(t),$$

where we use the assumption that there is no depreciation. Dividing through by $L(t)$ on both sides:

$$\frac{K(t+1)}{L(t)} = \frac{K(t)}{L(t)} + s\frac{Y(t)}{L(t)} = k(t) + sy(t),$$

and because $K(t+1)/L(t) = [L(t+1)/L(t)][K(t+1)/L(t+1)] = (1+n)k(t+1)$, we get

$$(1+n)k(t+1) = k(t) + sy(t) = k(t) + 100sk(t)^{1/3},$$

where the last equality uses part (a). In steady state k^*, we therefore have

$$(1+n)k^* = k^* + 100sk^{*1/3},$$

and solving this system for its positive solution, we have:

$$k^* = 100^{3/2}\left(\frac{s}{n}\right)^{3/2}.$$

It follows that

$$y^* = 100k^{*1/3} = 100\left[100^{3/2}\left(\frac{s}{n}\right)^{3/2}\right]^{1/3} = 100^{3/2}\left(\frac{s}{n}\right)^{1/2} = 1000\sqrt{s/n}.$$

(c) Provide intuitive economic reasoning to explain why equal percentage increases in savings rates and population growth rates appear to nullify each other in equation (2), in the sense of leaving per-capita income unchanged.

In the equation derived in part (b), notice how s and n appear to cancel each other out as far as their joint effect on steady state incomes are concerned. An increase in savings rate will increase the per-capita capital stock, both directly by adding to tomorrow's capital stock and indirectly, by adding to production that permits a further boost to the per-capita capital stock. But population growth does exactly the opposite: it reduces the per-capita capital stock both directly by first diluting the per-capita stock tomorrow (it is spread over more people) and indirectly by reducing per-capita production. These two effects run in opposite directions.

(5) In class, we discussed how the Solow model fails to adequately account for per-capita income differences across countries. There is a related problem: the model also appears to predict too high a discrepancy between the rates of return to capital across a developed and developing country. To appreciate this problem, suppose that the production function in both countries is given by

$$Y = K^a L^{1-a}.$$

(a) Show, using calculus and a little algebra, that the rate of return r to capital is given by

$$r = ak^{a-1} = ay^{(a-1)/a},$$

where k and y have the usual meanings.

There are two inputs, K and L, and a Cobb-Douglas production function — the same one — for two countries, given by

$$Y = K^a L^{1-a}.$$

The return to capital is just its marginal product, which is given — taking derivatives — by

$$r = aK^{a-1}L^{1-a} = ak^{a-1}.$$

But we know that $y = k^a$, so it follows that

$$y^{(a-1)/a} = k^{a-1}.$$

Using this in the first equation gives us the desired answer.

(b) Show that if one country is 15 times richer than another and $a = 0.3$, then the poorer country will have a rate of return to capital over 200 times that of the richer country.

Just plug in the values and use the formula. You will get in fact that the poorer country has over 500 times the rate of return compared to the richer country, If you use the approximation $a = 1/3$, you will get a ratio of 225.

(c) Suggest ways of resolving this puzzle. That is, what forces could still allow the 15-times gap in incomes and yet give you a smaller gap between the two rates of return on physical capital? Question (8) at the end is an optional question which is *not* required for this problem set, but it bears on this discussion and I encourage you to try.

There are various discrepancies that can resolve at least part of the difference:

(i) There is human capital, which is in low supply in developing countries and will tend to drag down the rate of return to physical capital.

(ii) The rate of return in developing countries also includes the risk factors associated with political change or revolutions, in which case capital could get expropriated. This fear factor will lower the rate of return to capital in developing countries.

(iii) There could be technological differences between the two countries, so that we cannot use the same production function for both.

(6) (These could be typical true-false questions in exam.) *Discuss* whether the following statements are true or false. In each case, just saying "true" or "false" is not enough. Provide an argument for truth, or simply a counterexample if you think it's false.

(a) The Harrod–Domar model states that a country's per capita growth rate *depends* on its rate of savings, whereas the Solow model states that it does not.

True. Here write down the Harrod-Domar equation. And then go on to mention that in the Solow model, long-run growth rate is determined simply by the exogenous rate of technical progress. The savings rate only determines long-run capital stocks per-capita and the *level* of per-capita output, not its rate of growth.

(b) According to the Harrod–Domar model, if the capital–output ratio in a country is high, that country will grow faster.

False. Simply write down the Harrod-Domar equation and argue that an increase in the capital-output ratio must *lower* the rate of growth. (Of course, an increase in the *output*-capital ratio must increase the rate of growth.)

(c) To understand if there is convergence in the world economy, we must study countries that are currently rich.

False. Studying countries that are *currently* rich introduces a bias towards convergence, as you are simply selecting *ex post* countries that were successful and so similar. You can mention Baumol's study as an example of this kind of mistake.

(d) Middle-income countries are more likely to change their relative position in world rankings of GNP than poor or rich countries.

True. Our investigation of the mobility matrix across countries shows that both very poor and very rich countries are unlikely to change world rankings all that much. In contrast, countries that were middle-income in 1960 have shown remarkable changes. A large fraction of them have become dramatically richer, while a large fraction have also become dramatically poorer.

(e) In the Solow model, a change in the population growth rate has no effect on the long-run rate of per capita growth.

True. In the Solow model, population growth only changes the *level* ofn long-run per-capita income. Here you may draw a quick diagram that describes the steady state in the Solow model and show what happens as population growth increases. Then point out that in the long-run, the rate of growth in the Solow model is just the rate of technical progress.

(f) In the Solow model, output per head goes down as capital per head increases, because of diminishing returns.

False. Draw the production function relating output per capita to capital per head. Of course output per capita increases as capital per capita increases. The point is that it does so at a diminishing rate, but it increases nevertheless.

(g) A Cobb-Douglas production function that has increasing returns to scale must also have increasing returns to at least one of its inputs.

False. Example: $Y = AK^{3/4}L^{3/4}$. Explain that this has increasing returns to scale (why?) but diminishing returns to each input (why?).

(h) A country which has been growing steadily at 10% per year and now has a per-capita income of \$100,000 would have a per-capita income of \$12,500 approximately 22 years ago.

True. Let income be y 21 years ago. Because the rate of growth is a steady 10% per year, income would double in 7 years, then double again in another 7, and once more in another 7. Then income today is given by

$$8y = 100,000$$

which gives $y = 12,500$.

Optional Practice Problems (do not need to be handed in)

(7) This is a review question and all it does is encourage you to go over the class notes and the text material.

A. Review of Solow. Suppose that the production function is given by $Y(t) = AK(t)^a L(t)^{1-a}$, where A is a fixed technological parameter. There is no technical change. Assume a fixed rate of depreciation δ and a constant rate of growth of population n. Explicitly solve for the *steady-state* value of the per capita capital stock and per capita income. How do these values change in response to a rise in (a) the technological parameter A, (b) the rate of saving s, (c) a, (d) δ, the depreciation rate, and (e) the population growth rate n?

This is just a review of material in class that will help you understand how the steady state in the Solow model is described. We have the equation

$$Y(t) = AK(t)^a L(t)^{1-a}$$

describing how total output is produced with capital and labor. In the first step, we transform this into a per-capita magnitude by dividing through by the labor force L (there is no technical progress here so that labor is just the same as effective labor). If we define $y = \frac{Y}{L}$ and $k = \frac{K}{L}$, we see that

$$y(t) = Ak(t)^a.$$

Therefore the equation describing the Solow model is

$$(1+n)k(t+1) = (1-\delta)k(t) + sAk(t)^a. \tag{3}$$

In the steady state k^*, $k(t) = k(t+1) = k^*$. Consequently,

$$(1+n)k^* = (1-\delta)k^* + sAk^{*a}. \tag{4}$$

Now we solve this equation (4) to figure out what the value of k^* must be:

$$(n+\delta)k^{*1-a} = sA,$$

or that

$$k^* = \left[\frac{sA}{n+\delta}\right]^{1/(1-a)}. \tag{5}$$

Now using equation (5), you should be able to easily tell the direction in which k^* moves, in response to all the changes asked about in the question.

B. Review of Harrod-Domar. Recall the basic accumulation equation

$$(1+n)k(t+1) = sy(t) + (1-\delta)k(t)$$

In this problem we're interested in looking at a case in which there is *no* diminishing returns in production, so that

$$y = Ak.$$

(a) Draw a diagram to convince yourself that in this case, there is no positive limit capital stock as in the Solow model: either k grows without bound or it shrinks all the way down to zero.

The idea here is that the production function is linear so when you draw the Solow diagram that we did in class, we have two flat lines. To get to this point, look at equation (3) in Part A, and divide both sides by first $k(t)$ and then $1+n$ to get

$$\frac{k(t+1)}{k(t)} = \frac{(1-\delta) + sAk(t)^{a-1}}{1+n}. \tag{6}$$

In your review, notice that the term $Ak(t)^{a-1}$ is just the output-capital ratio, which we denoted in class by θ. Notice how whenever $a < 1$, so that capital and input are both crucial to production, this object decreases as $k(t)$ increases, and try and understand fully why that is the case.

With that firmly in mind, now take a to 1 to arrive at the Harrod-Domar model. Notice how the output-capital ratio morphs into just A, so that equation (6) changes to

$$\frac{k(t+1)}{k(t)} = \frac{(1-\delta)+sA}{1+n}. \tag{7}$$

From this equation (7), we can see that If $(1-\delta)+sA$ line lies above $1+n$, per-capita capital (and therefore per-capita output) will grow forever. If it lies below, then output will shrink over time to zero.

(b) Define $g(t)$ to be the growth rate of per-capita capital; that is

$$g(t) = [k(t+1)/k(t)] - 1.$$

Prove that this is also the growth rate of per-capita output at date t.

$y = Ak$ at every date, so in particular:

$$y(t) = Ak(t),$$

and

$$y(t+1) = Ak(t+1).$$

Dividing both sides of the second equation by the first, we see that

$$y(t+1)/y(t) = k(t+1)/k(t),$$

which is what we needed to prove.

(c) Show that $g(t)$ is the same value at every date, and that

$$sA = (1+n)(1+g) - (1-\delta) \simeq n + g + \delta.$$

This is the *Harrod-Domar equation*. See text for more details.

ust use equation (7) to do this:

$$\frac{k(t+1)}{k(t)} = \frac{(1-\delta)+sA}{1+n}.$$

The left hand side is just "the growth rate of capital plus one," which is also "the growth rate of output plus one" in the Harrod-Domar model, as we have seen in part (b) of this problem. Calling that growth rate g, we have

$$1+g = \frac{(1-\delta)+sA}{1+n},$$

or

$$sA = (1+g)(1+n) - (1-\delta) = n + g + \delta,$$

because ng is very small relative to the other variables (recall why).

(d) Note that in the Solow model, the savings rate only affects the limit value of per-capita output, *but does not affect the rate of growth of that output in the long run.* But in the Harrod-Domar variant, it does affect the growth rate. Discuss why.

Here the savings rate has a persistent effect on growth rates. The reason is that the Harrod Domar model does not have any diminishing returns, so that a higher rate of savings feeds into a higher rate of economic growth. In contrast, in the Solow model, that effect gets dampened by diminishing returns and ultimately, even though the new steady-state *level* of capital and income per capita are higher, there is no effect on the rate of *growth*.

In the Harrod-Domar model, the value A is known as the *output-capital ratio*. It is the amount of flow output generated by one fixed unit of the capital stock. Its reciprocal, $1/A$, is more familiarly known as the *capital-output ratio*. The best way to think about the Harrod–Domar equations is to attach some numbers to them, as we did in Problem 3.

(8) Think of the three-input model with unskilled and skilled labor (as well as physical capital) mentioned in class: $Y(t) = AK(t)^a H(t)^b U(t)^{1-a-b}$, where H is skilled labor and U is unskilled labor. One useful feature of this model is that it simultaneously explains how rates of return to physical capital as well as the wage rate for unskilled labor might be low for developing countries. But there is a problem with this argument.

(a) Using the Cobb-Douglas with three inputs instead of two, show that such a model predicts that the rate of payment to *human capital* must be higher in developing countries.

(b) Adapt the Cobb-Douglas specification in part (a) to allow for differences in technology across developed and developing countries. Now it is possible to generate situations in which the return to every input is lower in developing countries. Which input is likely to have the lowest return (in a relative sense)?

To gain some intuition for this problem, go back to the answer for problem (5), and recall the rate of return to capital:

$$r = ak^{a-1}.$$

By taking derivatives with respect to the labor input, we can figure out what the wage is:

$$w = (1-a)K^a L^{-a} = k^a.$$

Notice how k influences the rate of return to capital negatively. That makes sense by diminishing returns, and mathematically the way we see it is to remember that $a - 1 < 0$, so r is related negatively to k. But in the wage equation, it is obvious that w is related *positively* to k (the more capital there is to work with, the higher is the productivity of labor). This has an interesting implication:

If two countries have the same overall production function using just capital and labor, and one country has a higher wage rate, then it has a lower rate of return to physical capital.

This is what we worked with in question (5): in fact we saw that the richer country has a *much* lower return to capital. One way to try and fix that is given by this problem; also see our three-input model in the text and in class, which is related. Suppose there are three inputs, K, H (human capital or skilled labor), and U (unskilled labor). Then it is possible to "explain" why a poor country has a lower unskilled wage *and* a lower rate of return to physical capital, but the explanation comes at a price: the return to skilled labor must be higher in the poor country, if the two countries have the same production function. This counterfactual prediction is a way of trying to "theoretically prove" that the two countries must have different production functions.

Now for the details. We have

$$Y = K^a H^b U^{1-a-b}$$

Let us use k to denote K/U and h to denote H/U, per-capita versions relative to unskilled labor.

First let us calculate the rate of return to unskilled labor — call it w_u for unskilled wage. This is the derivative of the production function with respect to U, so that

$$w_u = (1-a-b)K^a H^b U^{-a-b} = (1-a-b)k^a h^b. \tag{8}$$

Likewise, the rate of return to physical capital — call it r — is given by

$$r = aK^{a-1}H^b U^{1-a-b} = ak^{a-1}h^b. \tag{9}$$

And the rate of return to human capital — call it w_h — is given by

$$w_h = bK^a H^{b-1} U^{1-a-b} = bk^a h^{b-1}. \tag{10}$$

Say that the index 1 stands for the developing country and 2 for the rich country. Because we are trying to explain how *both* the wage and the return to physical capital could be lower in country 1, we are imposing $w_u(1) < w_u(2)$ and $r(1) < r(2)$. This imposes restrictions on how k and h can vary across the two countries. The wage inequality tells us (using (8)) that

$$p^a q^b > 1. \tag{11}$$

where p is the ratio of the rich country's k to that of the poor country: that is, $p = k_2/k_1$, and q is the ratio of the rich country's h to that of the poor country: that is, $q = h_2/h_1$. Likewise, the differences in the rate of return to physical capital tell us, using (9), that

$$p^a q^b > p. \tag{12}$$

Finally, suppose that the model *also* allows for the skilled wage to be lower in the poor country (in fact, we want to prove the opposite but this is what is called a proof by contradiction): then, by the same token, using (10), we have

$$p^a q^b > p. \tag{13}$$

In the rest of the argument, we prove that all three inequalities (11)—(13) cannot simultaneously hold, which will lead to a contradiction, and prove the desired result. First we show that both p and q must be bigger than 1. Suppose not; say $p \leq 1$. Then from (11), it must be that $q > 1$. But (13) implies that $p^a > q^{1-b}$, which cannot be. So $p > 1$. By exactly the same argument involving (11) and (12), we must conclude that $q > 1$.

Now we complete the proof by taking logs in equations (12) and (13). Doing this with (12), we have that

$$a \ln p + b \ln q > \ln p \tag{14}$$

where we note that $\ln p$ and $\ln q$ are both strictly positive because $p > 1$ and $q > 1$ (just shown). Similarly, taking logs in (13), we see that

$$a \ln p + b \ln q > \ln q. \tag{15}$$

Can both these inequalities hold at the same time? They can't, if both $\ln p$ and $\ln q$ are positive, as they must be!Draw a diagram and convince yourself. If you have come this far, you can do it. And you haven't, don't worry about it.