Professor Debraj Ray
19 West 4th Street, Room 608
email: debraj.ray@nyu.edu, homepage: `https://debrajray.com/`
Webpage for course: `https://pages.nyu.edu/debraj/Courses/23UGDev/index.html`

Econ-UA 323

Answers to Sample Examination 1

[a] In the Solow model, *faster* technical progress must *lower* steady state output per effective unit of labor, but increases output per capita.

True. In the Solow model, the long run per-capita growth rate is given by the rate of technical progress. That goes up with technical progress, so output per capita must increase. However, the steady state output per *effective* unit of labor must fall. Recall that with a Cobb-Douglas production function,

$$\hat{y}^* = A^{1/(1-\alpha)} \left(\frac{s}{n+\delta+\pi} \right)^{\alpha/(1-\alpha)}.$$

This makes it very clear that as π goes up, $\hat{y}^*$ must fall.

To show that output per capita actually goes up, draw the diagram for per-capita income over time (in logs) done in class. Show that technical progress swivels that line so that countries move to a higher output per capita and a higher growth rate of that output.

[This sort of answer gets full credit but you can get partial credit even if you don't write down the formula. You can get extra credit if you write the formula without assuming Cobb-Douglas and prove the assertion using that. You get credit for drawing the diagram and explaining the swivel. All this is just to give you an idea of how we plan to grade.

[b] The larger the share of capital in national income, the larger is the variation in steady state incomes that can be explained by differences in savings rates in the Solow model.

True. In the Solow model written here for simplicity without technical progress, we know that

$$(1+n)k(t+1) = sAk(t)^\alpha + (1-\delta)k(t),$$

where α is the share of capital in national income. In steady state, $k(t) = k(t+1) = k^*$, so that

$$k^* = \left(\frac{sA}{n+\delta} \right)^{1/1-\alpha}.$$

Therefore per-capita output is given by

$$y^* = A \left(\frac{sA}{n+\delta} \right)^{\alpha/1-\alpha}.$$

For two countries that differ *only* in their savings rates, the ratio of their steady state incomes is therefore given by

$$\frac{y_1^*}{y_2^*} = \left(\frac{s_1}{s_2} \right)^{\alpha/1-\alpha}.$$

If α is larger, then so is $\alpha/(1-\alpha)$, and so a given increase in the savings rate will translate into a larger variation in incomes.

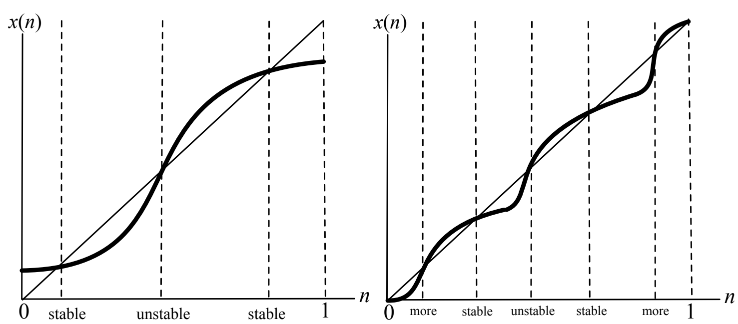
The above is a perfect answer. But if you just know the formula for the ratio, which is the last line, that will get you full credit too, depending on how well you exposited the argument. If you don't know the formula at all, then you will get less credit depending on how well you explain the problem otherwise.

[c] A Cobb-Douglas production function that has increasing returns to scale must also have increasing returns to at least one of its inputs.

False. Example: $Y = AK^{3/4}L^{3/4}$. Explain that this has increasing returns to scale (why?) but diminishing returns to each input (why?). Both explanations will need to be given for full credit. Apply the general definition of in creasing returns given in class to do the first, and for the second, you can use (for instance) calculus to show that the marginal product of each input is declining.

[d] A complementarity map with an unstable equilibrium *must* have at least *two* stable equilibria elsewhere (it may have more).

True. Draw a complementarity map with range of n equal to $[0, 1]$ and range of $x(n)$ also confined to $[0, 1]$. If there is an unstable equilibrium, there is a crossing of the 45 degree line of $x(n)$ "from below." (Explain why.) But because the $x(n)$ line must "end" at 1 or below, and start at 0 or above, this must generate at least two more left-right crossings "from above," which are both stable. There could be more, but there must be at least two. See diagram:



[e] This *could be* a conceivable mobility matrix, with 4 relative income categories on the rows and columns, showing country transitions from one category to another category.

	1	2	3	4
1	80%	10%	10%	0%
2	20 %	70%	10%	10%
3	10%	10%	70%	20%
4	0%	10%	10%	80%

False. The middle two rows don't add up to 100% as they must for every mobility matrix.

(2) (20 points) Verticalia is a new community in the town of Flatland that is inviting residents to move in. The cool thing about Verticalia is that everyone will live in a mega highrise and get to hang out with everyone else. This "hanging-out" gives rise to *networking payoffs*, given by

$$P = An,$$

where $A > 0$ is a positive parameter and n is the fraction of people in Flatland that move into Verticalia. But the problem is that more people moving in will also raise per-person rents in Verticalia because of congestion. Rent is given by the function

$$R = 10 + 5n.$$

Assume that the net payoff a person receives in Verticalia is networking payoff minus rent, and that there are no other costs or benefits other than these.

Meanwhile, living in Flatland *outside* Verticalia has its own payoff. We simply assume that payoff to be equal to zero for every resident.

(i) [4 points] Show that *no one* living in Verticalia; i.e., $n = 0$, is always an equilibrium, no matter how small or large the value of A is.

If payoff is given by networking payoff minus rent, then it is equal to

$$P - R = An - 5n - 10.$$

If $n = 0$, then this payoff is negative. So if no one expects to live in Verticalia, no one will. This is independent of however large A might be.

(ii) [4 points] Describe precisely the values of A for which $n = 0$ is the *only* equilibrium.

For n to be the only equilibrium, it must be the case that the net payoff to living in Verticalia is negative for all fractions n; i.e.,

$$An - 5n - 10 < 0$$

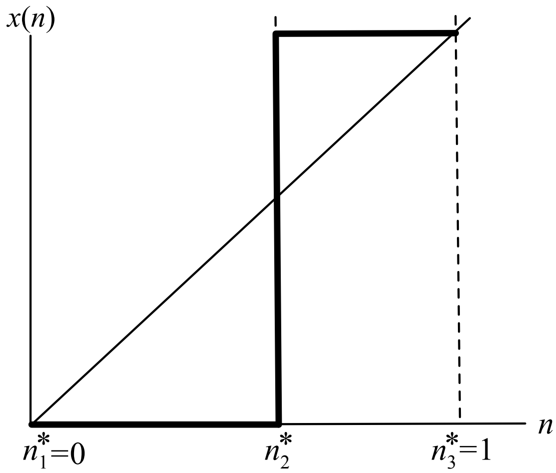
for all n between 0 and 1. Solving this inequality, we see this will happen when $A < 15$.

(iii) [7 points] If the conditions in part (ii) are not met, show that there are multiple equilibria and describe all of them, including the unstable one.

First, if A is *strictly* larger than 15, then at $n = 1$, the net payoff is strictly positive, so that everyone expects everyone else to live in Verticalia, then everyone will. Moreover, this is stable, because even if $n < 1$ but close enough to it, the net payoff to living in Verticalia

$$An - 5n - 10$$

will be strictly positive. So the system will push towards $n = 1$. The usual complementarity map is useful here:



Where is the unstable point n_2^* located? It is at the point where An - 5n - 10 = 0, or

$$n_2^* = \frac{10}{A-5}.$$

Because $A > 15$, it is easy to check that $0 < n_2^* < 1$. Note how the complementarity map cuts the 45^0 line at n_2^* "from below," so you will need to argue that this intermediate equilibrium is unstable.

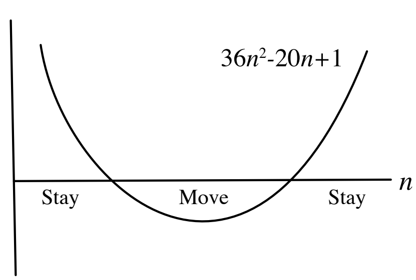
There is one "knife-edge" case that we left out, and if you include it you will get a small amount of extra credit. When A *exactly* equals 15, then there is one stable equilibrium at $n = 0$. For every $0 < n < 1$, the net payoff is striictly negative, and only *at* $n = 1$ does it become zero. So now the unstable equilibrium coincides with $n = 1$.

(iv) [5 points] Now change the rent function to $R = 1 + 36n^2$, and assume that $A = 20$. Show that there are now just two stable equilibria, one in which no-one lives in Verticalia, and another in which just half of them do.

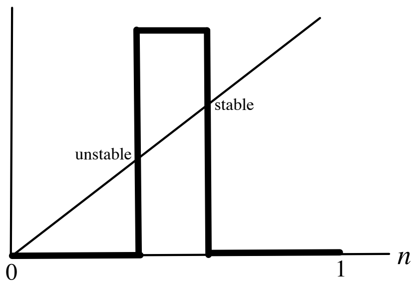
Under these conditions, the conditions for wanting to move to Verticalia are given by $20n - 36n^2 - 1 \geq 0$ or equivalently

$$36n^2 - 20n + 1 \leq 0.$$

See diagram, which plots this quadratic equation:



So the relevant "complementarity map" looks like this:



Indeed, we don't have a complementarity throughout any more! For small values of n there is a complementarity, as the networking effect dominates the rent congestion effect. For large values of n it is the other way around as the congestion effect dominates the networking effect and the map jumps *down*. These jumps happen at the roots of the quadratic, shown in the previous diagram. So all we have to do is solve for the roots, which are given by

$$\frac{20 \pm \sqrt{400 - 4.36.1}}{2.36} = \frac{20 \pm \sqrt{256}}{72},$$

which yields the solutions $1/18$ for the lower unstable root and the solution $1/2$ for the upper stable root.

This question is deliberately harder than the others.

(3) (20 points)

(a) [5 points]Derive the per-capita capital accumulation equation in the Solow model with Cobb-Douglas constant returns to scale production function and *without* technical progress, and show that it leads to a unique steady state.

Completely standard; follow class notes.

(b) [5 points] Adapt the technology so that you can move to the Harrod-Domar model. Use a version of the Solow steady state diagram to show that (barring some knife-edge values of the parameters) either per-capita capital grows without bound or it shrinks all the way down to zero.

The idea here is that the production function is linear so when you draw the Solow diagram that we did in class, we have two straight flat lines. If the $sA + (1 - \delta)$ line lies above the $(1+n)$ line, output will grow forever. If it lies below, then output will constantly shrink over time to zero. If you point out the knife-edge case where the two lines coincide, and do a good job of explaining that too, you would get some extra credit.

(c) [5 points] Define $g(t)$ to be the growth rate of per-capita capital, and prove that

$$sA = (1 + n)(1 + g) - (1 - \delta),$$

where the parameters in the equation have their usual meaning.

Completely standard; follow class notes. Take the equation

$$(1 + n)k(t + 1) = sy(t) + (1 - \delta)k(t),$$

and divide through by $k(t)$; then we see immediately that

$$(1+n)(1+g(t)) = sA + (1-\delta),$$

Since nothing else in the equation depends on time, nor can $g(t)$. It is constant over time. Transposing terms, we see that

$$sA = (1+n)(1+g) - (1-\delta).$$

(d) [5 points] Describe any one way in which the Harrod-Domar model yields different results from the Solow model.

Here the savings rate has a persistent effect on growth rates. The reason is that the Harrod Domar model does not have any diminishing returns, so that a higher rate of savings feeds into a higher rate of economic growth. In contrast, in the Solow model, that effect gets dampened by diminishing returns and ultimately, even though the new steady-state *level* of capital and income per capita are higher, there is no effect on the rate of *growth.*

You could give other differences, but they will all have to rely somehow on this central issue of the Harrod-Domar model allowing for persistent growth even in the absence of technical progress, whereas the Solow model does not.