# EC516 Contracts and Organisations, LSE

## Convex Games are Balanced

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A TU characteristic function game is *convex* if $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$ for every pair of coalitions $S$ and $T$. It is *balanced* if for every balanced weighting scheme $\delta$,

$$v(N) \geq \sum_S \delta(S) v(S).$$

Theorem. *Every convex game is balanced.*

**Proof.** Without loss of generality we can assume that $v(S) \geq 0$ for all coalitions $s$. (The reason is that we can always add an identical constant to the worth of every coalition without changing anything.)

Now proceed by induction. The theorem is obviously true for any two-person player set. Suppose it is true for every set of players of size $n-1$ or less. Take a balanced weighting scheme $\delta$. Pick any $S$ such that $\delta(s) > 0$, and define $T$ to be the union of all *other* coalitions $W$ such that $\delta(W)$ is also positive.

Notice that $S \cup T$ must equal $N$, otherwise there would be players that belong to no coalition with positive weight, which contradicts balancedness of the weighting scheme.

Create a balanced weighting system for the game restricted to $T$ by using the old weights $\delta$ for all the coalitions that are subsets of $T$. But change only one weight: redefine on weight on $T \cap S$: $\delta'(T \cap S) \equiv \delta(T \cap S) + \delta(S)$. It is easy to see that this is a balanced weighting scheme for the player set $T$. Consequently, because the game restricted to $T$ is convex and because we have the induction hypothesis, the restricted game is also balanced. So

$$\begin{aligned} v(T) &\geq \sum_{W \subseteq T} \delta(W) v(W) + \delta(T \cap S) v(T \cap S) \\ &= \sum_{W \neq S} \delta(W) v(W) + \delta(T \cap S) v(T \cap S). \end{aligned} \tag{1}$$

Consequently, using convexity of the overall game,

$$\begin{aligned} v(N) + v(T \cap S) &= v(S \cup T) + v(T \cap S) \\ &\geq v(S) + v(T) \\ &= v(S) + \sum_{W \neq S} \delta(W) v(W) + \delta(S) v(T \cap S) \\ &= \sum_W \delta(W) v(W) + (1 - \delta(S))[v(S) - v(T \cap S)] \\ &\geq \sum_W \delta(W) v(W), \end{aligned}$$

where the third line uses (1) and the last line uses superadditivity (all convex games are trivially superadditive) and the normalization that the worth of every coalition is nonnegative. This proves balancedness.