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Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division*

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This paper analyzes a sequential game of coalition formation when the division of the coalitional surplus is fixed and the payoffs are defined relative to the whole coalition structure. Gains from cooperation are represented by a valuation which maps coalition structures into payoff vectors. I show that any core stable coalition structure can be attained as a stationary perfect equilibrium of the game. If stationary perfect equilibria may fail to exist in general games, a simple condition is provided under which they exist in symmetric games. Furthermore, symmetric stationary perfect equilibria of symmetric games generate a coalition structure which is generically unique up to a permutation of the players. A general method for the characterization of equilibria in symmetric games is proposed and applied to the formation of cartels in oligopolies and coalitions in symmetric majority games. *Journal of Economic Literature* Classification Numbers : C78, C71.

1. INTRODUCTION

Since the publication of *Theory of Games and Economic Behavior*, the study of endogenous formation of coalitions has been one of the most intriguing and challenging problems open to game theorists. Many solution concepts such as Von Neumann and Morgenstern's stable sets (Von Neumann and Morgenstern, 1944) and Aumann and Maschler's bargaining set (Aumann and Maschler, 1964)

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were in fact primarily designed as ways to solve the problem of joint determination of a coalition structure and the allocation of the coalitional surplus among coalition members. While these approaches proved fruitful in the study of many situations of cooperation, they mostly rely on the assumption that gains from cooperation can be defined independently of the coalitions formed by external players.[1] Using the terminology introduced by Shubik (1982), cooperative game theory has focused on games with orthogonal coalitions which are well-suited to situations of pure competition but fail to capture the effects of externalities among coalitions. The objective of this paper is to propose a model of formation of coalitions in nonorthogonal games where payoffs depend on the whole coalition structure.

The presence of externalities among coalitions introduces a new difficulty in the study of endogenous coalition formation. When players decide to form a coalition, they must take into account the reaction of external players to the formation of the coalition. The sequential model analyzed in this paper addresses this problem by explictly describing a procedure in which individual players, when deciding to form a coalition, consider the consequences of their actions on the behavior of the other players. However, to keep the analysis tractable and concentrate on the role played by externalities on the formation of the coalition structure, I do not model the allocation of the coalitional surplus among members of a coalition, and assume instead that the coalitional worth is distributed according to a fixed sharing rule. Gains from cooperation are then represented by a valuation which maps coalition structures into vectors of individual payoffs.

Arguably, the assumption that payoffs are determined by a fixed rule is very restrictive and may seem a high price to pay for allowing externalities among coalitions. But valuations arise naturally in two distinct categories of economic models and the study of coalition formation in games represented by a valuation may appear fruitful in the resolution of these models.

First, valuations are considered in the models of coalition formation studied by Myerson (1978), Shenoy (1979), Hart and Kurz (1983) and Aumann and Myerson (1988). In these models, the formation of coalitions is viewed as a two-stage process where players form coalition in the first stage and decide on the allocation of the coalitional surplus, given a fixed coalition structure, in the second stage. Hence, at the time coalitions are formed, players evaluate the payoffs they receive in each coalition structure according to a fixed rule. The exact characterization of the rule employed in the second stage depends on the situations considered in the different models. In Myerson (1978)'s threats and settlement game, the fair settlement function assigns to each collection of coalitions (not necessarily a coalition structure) a unique vector of payoffs. Shenoy (1979) uses as an evaluation rule Aumann and Drèze (1974)'s extension

[1] Two important exceptions are Thrall and Lucas (1963)'s study of games in partition function form and Aumann and Drèze (1974)'s analysis of games with fixed coalition structures.

of the Shapley Value to games with fixed coalition structures. In Hart and Kurz (1983)'s analysis, players evaluate coalition structures according to a different extension of the Shapley Value first analyzed by Owen (1977). In Aumann and Myerson (1988)'s study of formation of links among players, the valuation used is Myerson (1977)'s extension of the Shapley Value to games with cooperation graphs of players.[2]

Second, valuations emerge in various applications of Game Theory to Industrial Organization and Public Economics involving competing coalitions of economic agents. The study of the formation of cartels in oligopolies leads to a natural definition of a valuation representing, for each cartel structure, the payoffs obtained by the firms belonging to the different cartels.[3] Similarly, the formation of associations of firms which agree to share some common resource but behave as competitors on the market can be analyzed with the use of a valuation.[4] The analysis of the provision of local public goods in a spatial setting where members of a community can benefit from the public goods provided in neighboring communities also requires the use of a valuation.[5] As a final example, the formation of customs unions allowing national firms to compete in a market characterized by the existence of different customs unions also leads to the definition of a valuation.

Cooperative solution concepts for games represented by a valuation were introduced by Shenoy (1979) and Hart and Kurz (1983) in their models of endogenous coalition formation.[6] To predict which coalitions will be formed, they propose different definitions of stability of coalition structures.[7] The variety of stability concepts accounts for the fact that, in games described by a valuation, the payoffs obtained by members of a blocking coalition depend on the reaction of the external players. The solution concepts range from the core stability concept, which supposes a very optimistic conjecture about the reaction of the external players since players deviate if there exists a coalition structure in which they are better off to the α stability concept which is based on pessimistic conjectures since a coalition only deviates when it is guaranteed to obtain a higher

[2] In Myerson (1978) and Hart and Kurz (1983), the emphasis is put on the axiomatic derivation of a reasonable valuation rather than on the first stage game of coalition formation. This paper, by contrast, focuses on the game of coalition formation.

[3] Salant *et al*. (1983) were the first to point out in a simple model the problems of cartel formation in oligopolies. Yi and Shin (1995) contains a very complete description of the derivation of the valuation in the cartel problem.

[4] The study of associations of firms, which can be interpreted as Research Joint Ventures or standardization committees, is taken up in a distinct paper (Bloch, 1995).

[5] Guesnerie and Oddou (1981) analyze the provision of local public goods in a model with orthogonal coalitions but discuss the role of externalities among communities.

[6] Hart and Kurz (1983) analyze strong equilibria of a noncooperative game where players simultaneously announce coalitions.

[7] Other concepts of stability of coalition structures are surveyed in Greenberg (1995).

payoff independently of the reaction of the other players. The study of stable coalition structures raises three important difficulties. First, the definitions of stability rely on ad hoc assumptions on the behavior of the other players after a coalition has deviated. Second, all definitions of stability assume that external players react to the formation of a coalition in a myopic way. Hence, when a coalition forms, its members do not take into account the final result of their decisions but only the immediate reaction of the other players. Finally, even the less restrictive definition of stability (α stability) may not be useful, since α stable coalition structures fail to exist in situations which are not easily characterized. (Hart and Kurz (1984) give an example of a game without stable structure which is otherwise well-behaved.)

By contrast, in this paper, I explicitly model the formation of coalitions as a noncooperative sequential process in the spirit of Rubinstein (1982)'s alternating offers bargaining game and its extensions to n players by Selten (1981) and Chatterjee *et al*. (1993). Players are ranked according to an exogenous rule of order. The first player starts the game by proposing the formation of a coalition. If all prospective members accept the proposal, the coalition is formed. If one player rejects the proposal, she becomes the initiator in the next round. The important feature of the game is that, once a coalition is formed, the game is only played among the remaining players and that established coalitions may not seek to attract new members nor break apart. Hence, by agreeing to group in a coalition, players commit to stay in that coalition.

I restrict my attention to stationary strategies and establish the following properties of stationary perfect equilibria. I first show that, if the game always admits a subgame perfect equilibrium, stationary perfect equilibria may fail to exist. A sufficient condition for the game to admit a stationary perfect equilibrium is that the valuation and all its restrictions to smaller sets of players admit core stable structures. Furthermore, any core stable coalition structure can be reached as a stationary perfect equilibrium of the extensive form game of coalition formation, provided that the set of stationary perfect equilibria is nonempty. I then study the restricted class of symmetric games where all players are ex ante identical. In this class of games, using a result due to Ray and Vohra (1995), I provide a simple condition under which symmetric stationary perfect equilibria exist, and I show that the coalition structures they generate are generically unique up to a permutation of the players. Furthermore, I provide a general method for the characterization of the coalition structures generated by symmetric stationary perfect equilibria in symmetric games. This method is used to derive equilibrium coalition structures in two situations: the formation of cartels in a symmetric oligopoly and the symmetric majority games discussed by Hart and Kurz (1984).

The game analyzed here is similar to games of coalition formation proposed by Selten (1981), Chatterjee *et al*. (1993), Moldovanu (1992) and Winter (1993) in the context of games in coalitional form. The games they analyze have the same sequence of moves as the one described above. The crucial difference between their games and mine stems from differences in the action spaces. By fixing the division of the payoffs, I restrict the actions of the agents to announcements of coalitions whereas they study a more general framework where players announce both a coalition and the division of the coalitional worth. A further difference is due to the underlying specification of gains from cooperation since they do not allow for externalities among coalitions. Given these differences, the results they obtain are not directly comparable to mine.

Different extensive form procedures of coalition formation in games represented by a valuation were proposed by Aumann and Myerson (1988) and Shin and Yi (1995). The procedure in Aumann and Myerson (1988) is defined for games where players evaluate cooperation graphs rather than coalition structures. The particular feature of cooperation graphs where coalition members need not unanimously agree to admit new members leads them to define a game where links can be formed at any stage. This approach cannot easily be applied to situations where gains from cooperation accrue when coalitions are formed, rather than bilateral links among players. Yi and Shin (1995) analyze games based on a "matching procedure." Players announce coalitions and coalitions are formed whenever all its members have made identical announcements. In general, the equilibria they obtain are very different from the equilibria of the infinite horizon game analyzed in this paper.

The paper is organized as follows. The game of sequential formation of coalitions is introduced and the equilibrium concept defined in Section 2. In Section 3, I analyze the relations between stationary perfect equilibria and stability concepts for coalition structures in games described by a valuation. Section 4 is devoted to the analysis of symmetric games. I present applications of the model to the formation of cartels in oligopolies and of coalitions in symmetric majority games in Section 5. My concluding remarks and some directions for future research appear in Section 6.

2. SEQUENTIAL FORMATION OF COALITIONS

In this section, I introduce the sequential game of coalition formation and the equilibrium concept that I will use. The set of players is denoted N, with cardinality n. The index i will refer to the players. A *coalition* T is a nonempty subset of players. A *coalition structure* π is a partition on the set N. The set of all coalition structures is denoted by Π. For any subset K of N, the set of partitions on K is denoted Π_K with typical element π_K.

Gains from cooperation are described by a *valuation* v which maps the set of coalition structures Π into vectors of payoffs in $\Re^n$. The component $v_i(\pi)$ denotes the payoff obtained by player i if the coalition structure π is formed. I assume that payoffs are normalized so that any player, by opting to leave the game can get a strictly positive payoff. Formally, $\forall i \in N$, $\min_{\pi \supset \{\{i\}\}} v_i(\pi) > 0$.

A *rule of order* ρ is an ordering of the players, which is used to determine the order of moves in the sequential game of coalition formation.

The sequential game of coalition formation is defined by the exogenous specification of the valuation v and of the rule of order ρ. To emphasize this dependence, I denote the game of coalition formation by $\Gamma(v, \rho)$.

The game $\Gamma(v, \rho)$ proceeds as follows. The first player according to the rule of order ρ starts the game by proposing the formation of a coalition T to which she belongs. Each prospective member responds to the proposal in the order determined by ρ. If one of the player rejects the proposal, she must make a counteroffer and propose a coalition T' to which she belongs. If all members accept, the coalition is formed. All members of T then withdraw from the game, and the first player in $N \setminus T$ starts making a proposal.[8]

This game describes in the simplest way a procedure where coalitions are formed *in sequence*. The main characteristic of the game is that, once a coalition has been formed, the game is only played among the remaining players. The extensive form thus embodies a high degree of commitment of the players. When players agree to join a coalition, they are bound to remain in that coalition. They can neither leave the coalition nor propose to change the coalition at later stages of the game. Figure 1 depicts the extensive form of the game with three players.

A *history* h^t at date t is a list of offers, acceptances and rejections up to period t. At any point in the game $\Gamma(\rho, v)$, a history h^t determines

- a set $\hat{K}(h^t)$ of players who have already formed coalitions
- a coalition structure $\pi_{\hat{K}(h^t)}$ formed by the players in $\hat{K}(h^t)$
- an ongoing proposal (if any) $\hat{T}(h^t)$
- a set of players who have already accepted the proposal
- a player who moves at period t.

Player i is called *active* at history h^t is it is her turn to move after the history h^t. The set of histories at which player i is active is denoted H_i.

A *strategy* σ_i for player i is a mapping from H_i to her set of actions, namely

$$\sigma_i(h^t) \in \{\text{Yes, No}\} \qquad \text{if } \hat{T}(h^t) \neq \emptyset$$

$$\sigma_i(h^t) \in \{T \subset N \setminus \hat{K}(h^t), i \in T\} \qquad \text{if } \hat{T}(h^t) = \emptyset.$$

When $\hat{T}(h^t) \neq \emptyset$, player i is a respondent to the offer $\hat{T}(h^t)$ and she can choose to accept or reject it. If $\hat{T}(h^t) = \emptyset$, either a coalition has just formed and player i is the first player in $N \setminus \hat{K}(h^t)$ according to the rule of order ρ, or player

[8] Each time a coalition T is proposed, the order of responses is fixed by ρ independently of the history or the identity of the proposer. Hence, for example, if player 2 proposes the formation of a coalition $\{1, 2, 3\}$, player 1 responds first and player 3 responds after player 1.

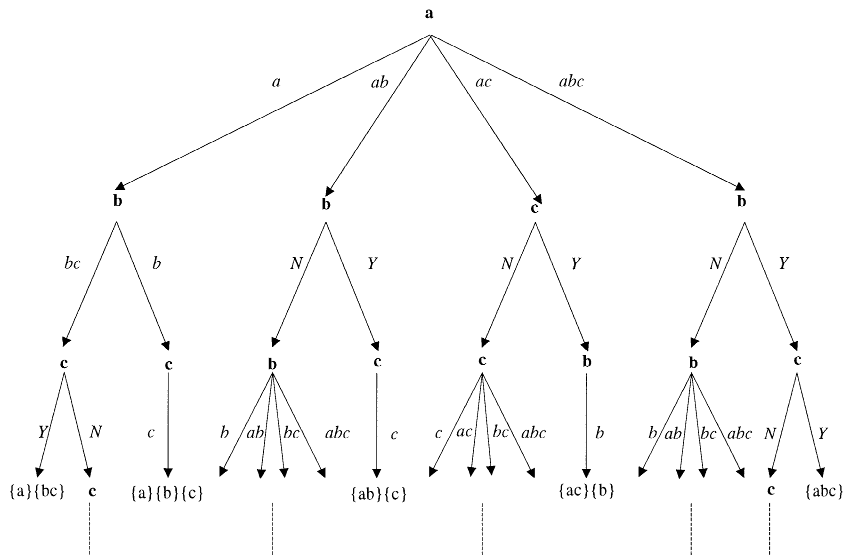

FIG. 1. The game Γ.

i has just rejected an offer. In both cases, it is her turn to propose a new coalition T which must be a subset of the remaining players to which she belongs.

I restrict my attention to strategies which only depend on the payoff-relevant part of the history. For a player i active at history h^t, the only payoff-relevant features of the history are the set K of players who left the game, the partition π_K representing the coalitions they formed and the current offer T. In particular, the set of players who have already accepted the offer T is uniquely determined by the rule of order ρ.

A strategy σ_i is *stationary* if it only depends on the state $s = (K, \pi_K, T)$ where K is a (possibly empty) subset of N, π_K is a partition of K and T is a (possibly empty) subset of $N \setminus K$. Formally, letting $\mathcal{T}(i, K)$ define the collection of subsets of $N \setminus K$ to which player i belongs, a stationary strategy is a mapping from the set of states at which player i is active, S_i, to a set of actions, where

$$\sigma_i(K, \pi_K, T) \in \{\text{Yes, No}\} \text{ if } T \neq \emptyset$$

$$\sigma_i(K, \pi_K, \emptyset) \in \mathcal{T}(i, K).$$

Any strategy profile $\sigma = \{\sigma_i\}_{i \in N}$ determines an outcome $(\pi(\sigma), t(\sigma))$ of the game. If the game ends in a finite number of periods, $\pi(\sigma)$ is a coalition structure on the set N, and $t(\sigma)$ is the period at which the agreement has been reached.

I assume that players do not discount the future. In the case of an infinite play of the game, players who have not formed a coalition receive a payoff of zero. More precisely, suppose that a subset $N \setminus K$ of the players does not reach an agreement in a finite number of periods. Payoffs are then given by

$$v_i(\pi(\sigma)) = 0 \qquad \text{for all players in} \quad N \setminus K$$

$$v_i(\pi(\sigma)) = \max_{\pi_K \subset \pi} v_i(\pi) \qquad \text{for all players in} \quad K.$$

DEFINITION 2.1. A *subgame perfect equilibrium* σ^* is a strategy profile such that $\forall i \in N, \forall h^t \in H_i, \forall \sigma_i, v_i(\pi(\sigma_i^*(h^t), \sigma_{-i}^*)) \geq v_i(\pi(\sigma_i(h^t), \sigma_{-i}^*))$.

DEFINITION 2.2. A *stationary perfect equilibrium* σ^* is a subgame perfect equilibrium where $\forall i \in N, \sigma_i^*$ is a stationary strategy.

A coalition structure π generated by a subgame perfect equilibrium is called an *equilibrium coalition structure* (ECS). Coalition structures generated by stationary perfect equilibria are called *stationary equilibrium coalition structures* (SECS). The set of stationary equilibrium coalition structures is denoted $SECS(v, \rho)$

Remark 2.3. Since every player obtains a higher payoff by leaving the game than by disagreeing forever, an infinite play of the game cannot be part of a subgame perfect equilibrium. Hence, the concept of an equilibrium coalition structure is well defined.

The payoffs of the game described above are not continuous at infinity. Hence the existence of a subgame perfect equilibrium is not guaranteed. To circumvent this difficulty, I first show that any subgame perfect equilibrium of the game with sufficiently high discounting is a subgame perfect equilibrium of the game $\Gamma(v, \rho)$. To be more precise, let $\Gamma_\delta(v, \rho)$ denote the game where strategies and moves are defined as above but payoffs are given by: $v_i(\sigma) = \delta_i^{t(\sigma)} v_i(\pi(\sigma))$.

PROPOSITION 2.4. *There exists* $\underline{\delta} \in (0, 1)$ *such that, if* $\forall i, \delta_i > \underline{\delta}$*, any subgame perfect equilibrium of* $\Gamma_\delta(v, \rho)$ *is a subgame perfect equilibrium of* $\Gamma(v, \rho)$.

Proof. Observe first that, since Π is finite, the set of payoffs of the game, $v(\Pi)$ is finite. Hence, the set of possible coalition structures formed in $\Gamma_\delta(v, \rho)$ is finite. In particular, this implies that, as δ varies continuously from 0 to 1, the strategy profiles of the game can only lead to a finite number of coalition structures. Hence, there exists a $\underline{\delta}$ such that, for all $\delta, \delta' > \underline{\delta}$, if σ^* is a subgame perfect equilibrium of $\Gamma_\delta(v, \rho)$, then σ^* is a subgame perfect equilibrium of $\Gamma_{\delta'}(v, \rho)$.

Consider now $\delta' > \underline{\delta}$, and let σ^* be a subgame perfect equilibrium of $\Gamma_{\delta'}(v, \rho)$. Then, for any player i, any history h^t in H_i, any strategy σ_i and any $\delta \in [\delta', 1)$,

$$\delta_i^{t(\sigma_i^*(h^t), \sigma_{-i}^*)} v_i(\pi(\sigma_i^*, \sigma_{-i}^*)) \geq \delta_i^{t(\sigma_i(h^t), \sigma_{-i}^*)} v_i(\pi(\sigma_i, \sigma_{-i}^*)).$$

Taking limits as δ goes to 1,

$$v_i(\pi(\sigma_i^*(h^t), \sigma_{-i}^*)) \geq v_i(\pi(\sigma_i(h^t), \sigma_{-i}^*)).$$

Hence, $\sigma*$ is a subgame perfect equilibrium of $\Gamma(v, \rho)$. ■

COROLLARY 2.5. *For any valuation v and any rule of order ρ, there exists a subgame perfect equilibrium of the game* $\Gamma(v, \rho)$.

Proof. Fix a $\delta > \underline{\delta}$. The game $\Gamma_\delta(v, \rho)$ is a finite action game of perfect information and is continuous at infinity. Hence, by a result of Fudenberg and Levine (1983) (Corollary 4.2, p. 262), the game $\Gamma_\delta(v, \rho)$ has a subgame perfect equilibrium. From Proposition 2.4, any subgame perfect equilibrium of $\Gamma_\delta(v, \rho)$ is a subgame perfect equilibrium of $\Gamma(v, \rho)$. ■

By imposing stationarity, I require that strategies only depend on the payoff-relevant part of the history. In the framework analyzed here, the payoff-relevant part of the history is summarized by the state s characterizing the coalition structure formed by the previous players and the ongoing offer. Chatterjee *et al.* (1993) and Moldovanu (1992) show that, when players bargain over the division of the coalitional worth, the set of nonstationary perfect equilibria may be very large, and stationarity is a useful restriction to refine the set of subgame perfect equilibria. A striking aspect of the game analyzed here is that *stationary perfect equilibria may fail to exist*. This point is illustrated by the following example.

EXAMPLE 2.6. $N = \{a, b, c\}$, and ρ defines $a < b < c$.

π	$v_a(\pi)$	$v_b(\pi)$	$v_c(\pi)$
$a\|b\|c$	1	1	1
$ab\|c$	3	2	1
$ac\|b$	2	1	3
$a\|bc$	1	3	2
abc	1	1	1

In this example, player a wants to form a coalition with player b, player b with player c, and player c with player a.

To show that the game $\Gamma(v, \rho)$ does not admit any stationary equilibrium coalition structure, observe first that the three coalition structures $\{\{a, b, c\}\}$, $\{\{a\}, \{b\}, \{c\}\}$ and $\{\{a\}, \{b, c\}\}$ cannot be supported by any equilibrium since player a would benefit from deviating and offering the formation of the coalition $\{a, c\}$ which player c would accept. The two other coalition structures

$\{\{a, b\}, \{c\}\}$ and $\{\{a, c\}, \{b\}\}$ can be supported by equilibria in nonstationary strategies but not by a stationary perfect equilibrium. For $\{\{a, b\}, \{c\}\}$ to be supported by a stationary perfect equilibrium, it must be that player c rejects the offer $\{b, c\}$. But, in equilibrium, player c will only reject the offer $\{b, c\}$ if player a accepts the offer $\{a, c\}$. By stationarity, player b accepts the offer $\{a, b\}$ irrespective of the history of rejections which have preceded it. Hence, since player b always accepts the offer $\{a, b\}$, player a cannot accept the offer $\{a, c\}$. Similarly, the coalition structure $\{\{a, c\}, \{b\}\}$ is only supported by a strategy prescribing that player b rejects the offer $\{a, b\}$, implying that player c accepts the offer $\{b, c\}$. Since, by stationarity, player a always accepts the offer $\{a, c\}$, player c should reject the offer $\{b, c\}$. Hence, the game $\Gamma(v, \rho)$ does not admit any stationary perfect equilibrium.

However, the coalition structures $\{\{a, b\}, \{c\}\}$ and $\{\{a, c\}, \{b\}\}$ can be supported by equilibria in nonstationary strategies.[9] To support these coalition structures as equilibria, one only needs to allow players to condition their actions on the number of times they have received an offer. Consider first the coalition structure $\{\{a, b\}, \{c\}\}$ and the following strategies. Player a always accepts the offer $\{a, b\}$ and proposes $\{a, b\}$. She rejects $\{a, b, c\}$ and accepts $\{a, c\}$ when, in the history h^t, she has made the offer $\{a, b\}$ to player b an *odd* number of times. Player b accepts $\{b, c\}$ and proposes $\{b, c\}$. She rejects $\{a, b, c\}$ and only accepts $\{a, b\}$ if, in the history h^t, the offer $\{a, b\}$ has been made by player a an *odd* number of times. Player c accepts $\{a, c\}$ and proposes $\{a, c\}$. She rejects $\{a, b, c\}$ and only accepts $\{b, c\}$ if, in the history h^t, player a has made the offer $\{a, b\}$ an *even* number of times. These strategies form a subgame perfect equilibrium of the game (in nonstationary strategies), and are depicted in Figure 2. A strategy profile supporting the coalition structure $\{\{a, c\}, \{b\}\}$ can be constructed in a similar way.

In Example 2.6, the three players play a symmetric role. Hence, no change in the rule of order can guarantee the existence of a stationary perfect equilibrium. Moreover, Example 2.6 is generic, since the nonexistence of a stationary perfect equilibrium is robust to small variations of the valuation. Nonexistence of stationary perfect equilibria is thus a robust phenomenon in games with more than three players.

Note however that the nonexistence of a stationary perfect equilibrium in pure strategies in Example 2.6 is linked to the fixed sharing rule. If players were allowed to bargain freely over the worth of the coalition in a game with transferable utility, the nonexistence result would disappear.

The central feature of Example 2.6 is the disagreement among players over the coalitions which should be formed. A similar problem was noted by Shenoy (1979) in Apex games, where a single big player faces a number of small play-

[9] These strategies are closely related to strategies constructed by Shaked to support any division of the payoffs in a three-person bargaining game (Sutton (1986)).

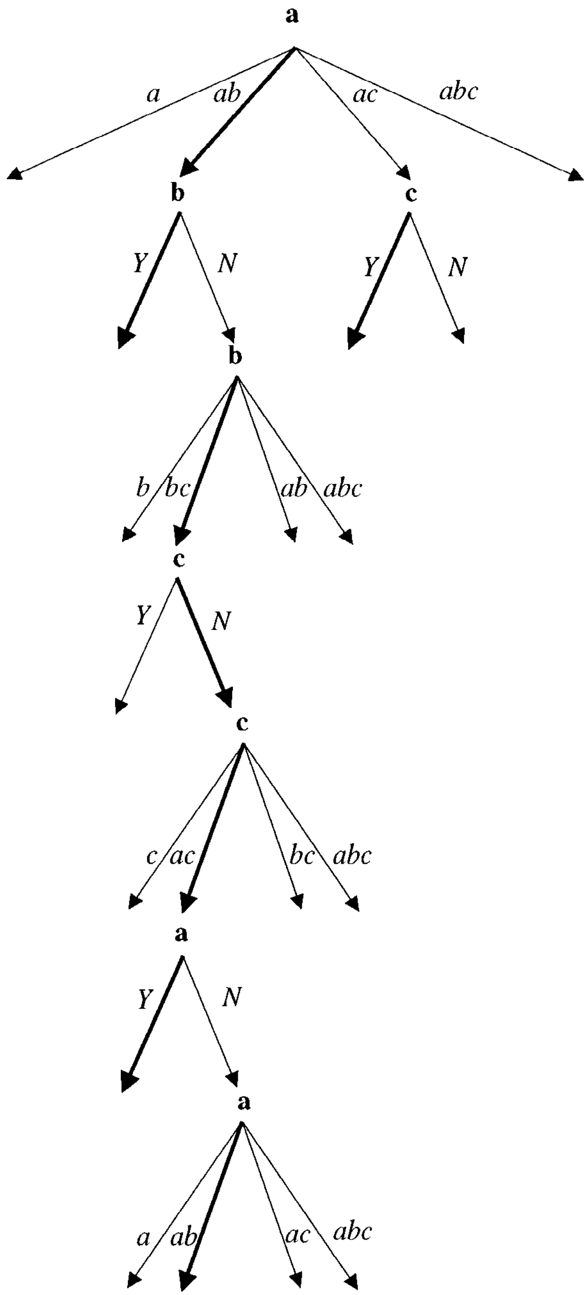

FIG. 2. Nonstationary equilibrium strategies supporting the coalition structure $\{\{a, b\}, \{c\}\}$.

ers (Example 7.5, p. 150). The preferred coalition for the big player is the grand coalition, since it offers her the possibility of diluting the power of the small players. Small players, on the other hand, would rather form a two-member coalition with the big player. This disagreement among players about the coalition which should be formed leads, as in Example 2.6, to the nonexistence of a stationary perfect equilibrium. This suggests that a sufficient condition for the existence of an equilibrium coalition structure is high degree of unanimity among players about the coalitions they wish to form. While this point is not pursued here, the class of symmetric games analyzed in Section 4 provides an example of games where players unanimously agree on the coalitions they want to belong to.

3. STABLE COALITION STRUCTURES

In this section, I compare the equilibrium coalition structures with coalition structures satisfying cooperative concepts of stability. Concepts of stability in games with externalities require a specification of the reaction of external players to the formation of a coalition, and different assumptions on the behavior of external players give rise to different definitions of stability. Kurz (1988) distinguishes five models of reaction of the external players. The core stability concept, first introduced by Shenoy (1979), is based on the following dominance relation. A coalition structure π dominates a coalition structure π' if there exists a coalition in π whose members receive strictly higher payoffs than in π'. A coalition structure is called *core stable* if it belongs to the core of the dominance relation. In effect, this definition of stability is very restrictive, since it assumes that, when a group of players deviate, they consider that external players react in such a way as to maximize the payoff of deviating players.

Hart and Kurz (1983) propose four models of reaction of the external players. In the γ model, coalitions which are left by some members dissolve. In the δ model, members of coalitions which lose members remain together and form smaller coalitions. The last two stability concepts are based on the β and the α cores.[10] In the β model, a group K of players deviates if, for any possible reaction of the external players, namely any coalition structure $\pi_{N\setminus K}$ of $N \setminus K$, there exists a coalition structure of K, π_K, such that all members of K are better off in the new coalition structure $\pi = \pi_{N\setminus K} \cup \pi_K$. In the α definition, a group K of players deviates if there exists a coalition structure π_K such that, whatever the reaction of the external players, members of K are better off forming the coalition structure π_K.

Letting, for any fixed valuation v, the sets of Core stable, γ stable, δ stable, β stable and α stable coalition structures be denoted by $CC(v)$, $C\gamma(v)$, $C\delta(v)$,

[10] See Aumann (1967) for a complete description of the α and β core concepts.

$C\beta(v)$ and $C\alpha(v)$, the following lemma is easily established.[11]

LEMMA 3.1. *For any valuation* v,$CC(v) \subset (C\gamma(v) \cup C\delta(v)) \subset C\beta(v) \subset C\alpha(v)$.

I will focus here on the two extreme concepts of core and α stability.[12] Formally, a coalition structure π is *core stable* if there does not exist a coalition K and a coalition structure π' such that $K \in \pi'$ and $\forall i \in K, v_i(\pi') > v_i(\pi)$. A coalition structure π is α *stable* if there does not exist a coalition K and a partition π'_K on K such that, $\forall i \in K, \forall \pi_{N\backslash K} \in \Pi_{N\backslash K}, v_i(\pi'_K \cup \pi_{N\backslash K}) > v_i(\pi)$.

The next proposition shows that, when the set of stationary equilibrium coalition structures is nonempty, it contains the set of core stable structures.

PROPOSITION 3.2. *Assume that there exists a rule of order* ρ *such that* $SESC(v, \rho) \neq \emptyset$. *Then* $CC(v) \subset SECS(v, \rho)$.

Proof. Let $\tilde{\rho}$ denote one rule of order for which $SECS(v, \tilde{\rho}) \neq \emptyset$. Let $\tilde{\pi}$ denote a coalition structure in $CC(v)$. To prove the proposition, I construct a stationary perfect equilibrium $\tilde{\rho}$ of the game $\Gamma(v, \tilde{\rho})$ such that $\pi(\tilde{\sigma}) = \tilde{\pi}$. I denote by $T(i)$ the coalition to which player i belongs in the coalition structure $\tilde{\pi}$. A partition π_K of a subset K of the players is called a *subpartition* of $\tilde{\pi}$ if it is formed by the union of elements of $\tilde{\pi}$. The set of all subpartitions of $\tilde{\pi}$ is denoted $Sub(\tilde{\pi})$. Pick a stationary perfect equilibrium $\tilde{\tilde{\sigma}}$ of the game $\Gamma(v, \tilde{\rho})$. A stationary strategy $\tilde{\sigma}_i$ for player i is then constructed as follows.

Assume that a subset K of players, where $i \notin K$, has already formed a coalition structure π_K.

$$\text{If} \quad \pi_K \notin Sub(\tilde{\pi}), \quad \tilde{\sigma}_i(K, \pi_K, \cdot) = \tilde{\tilde{\sigma}}_i(K, \pi_K, \cdot)$$

$$\text{If} \quad \pi_K \in Sub(\tilde{\pi}), \quad \tilde{\sigma}_i(K, \pi_K, \phi) = T(i)$$

$$\tilde{\sigma}_i(K, \pi_K, T(i)) = \text{Yes}$$

$$\tilde{\sigma}_i(K, \pi_K, T') = \text{Yes if } v_i(\pi(T')) > v_i(\tilde{\pi})$$

$$\tilde{\sigma}_i(K, \pi_K, T') = \text{No if } v_i(\pi(T')) \leq v_i(\tilde{\pi}),$$

where $\pi(T')$ is the coalition structure generated by $\tilde{\tilde{\sigma}}$ after the coalition T' has been formed.

[11] Hart and Kurz (1983) derive the last three inclusions of the Lemma. The first inclusion is immediate, once one reinterprets the core stability concept in terms of reaction of the external players to the deviation of a group of players.

[12] The absence of coincidence between α stable structures and equilibrium coalition structures can be extended to the intermediate concepts of β, γ, and δ stability.

The strategy $\tilde{\sigma}$ prescribes that player i follows her part of a stationary perfect equilibrium $\tilde{\tilde{\sigma}}$ if a coalition structure π_K off the equilibrium path has been formed, and that she forms the coalition $T(i)$ otherwise.

It remains to check that $\tilde{\sigma}$ is a subgame perfect equilibrium of the game $\Gamma(v, \tilde{\rho})$. Observe first that, since $\tilde{\tilde{\sigma}}$ is a stationary perfect equilibrium profile, the strategy profile $\tilde{\sigma}$ is a subgame perfect equilibrium if a coalition structure off the equilibrium path has been formed. Suppose now that the previous players have formed a coalition structure π_K in $Sub(\tilde{\pi})$. To check that $\tilde{\sigma}$ is a subgame perfect equilibrium on the equilibrium path, consider the possible deviations for player i.

Player i can deviate by announcing a coalition structure $T' \neq T(i)$ when it is her turn to announce a coalition. However, since $\tilde{\pi}$ is a core stable structure, there exists a player j in T' such that $v_j(\pi(T')) \leq v_j(\tilde{\pi})$. Hence, any coalition T' different from $T(i)$ will be rejected.

If now player i receives an offer $T(i)$, any deviation will lead to the formation of the coalition $T(i)$, since any different offer by player i will be rejected by some player.

Finally, suppose that player i receives an offer $T' \neq T(i)$. If $v_i(\pi(T')) \leq v_i(\tilde{\pi})$, she cannot benefit from accepting the offer. If all other members of T' accept the offer, the coalition T' is formed and player i obtains a payoff $v_i(\pi(T'))$, whereas, by rejecting the offer, player i obtains the payoff $v_i(\tilde{\pi})$. If $v_i(\pi(T')) > v_i(\tilde{\pi})$, player i should accept the offer, since her rejection would lead to the formation of the structure $\tilde{\pi}$, whereas her acceptance may either secure the formation of $\pi(T')$, if no player following player i rejects the offer T', or yield the formation of $T(i)$, if some player following player i rejects the offer T'.

Since player i has no incentive to deviate from her strategy $\tilde{\sigma}_i$, the strategy profile $\tilde{\sigma}$ forms a subgame perfect equilibrium of the game $\Gamma(v, \tilde{\rho})$. Furthermore, by construction, $\pi(\tilde{\sigma}) = \tilde{\pi}$. Hence, $CC(v) \subset SECS(v, \tilde{\rho})$. ■

In the statement of Proposition 3.2, I require the set of stationary perfect equilibria to be nonempty. This assumption is needed to show that, once a coalition structure is formed off the equilibrium path, the game still admits a stationary perfect equilibrium. The following example shows that the assumption cannot be relaxed.

EXAMPLE 3.3. $N = \{a, b, c, d\}$.

π	$v_a(\pi)$	$v_b(\pi)$	$v_c(\pi)$	$v_d(\pi)$
$abcd$	5	5	5	5
$a\|bc\|d$	1	3	2	1
$a\|b\|cd$	1	1	3	2
$a\|bd\|c$	1	2	1	3
Others	1	1	1	1

The game of Example 3.3 admits a unique core stable structure, the grand coalition which Pareto dominates any other coalition structure. However, the subgame following the formation of the coalition $\{a\}$ is identical to the game in Example 2.6 and does not admit any stationary perfect equilibrium.

The difficulty illustrated by Example 3.3 can be alleviated by assuming that, in addition to the valuation v, all restrictions of the valuation to subsets of the players admit a core stable structure.[13] Since payoffs depend on the whole coalition structure, the restriction of the valuation v to a subset K of the players must entail a description of the partition formed by the external players.

The *restriction* of the valuation v to a subset K of the players relative to the coalition structure $\pi_{N\setminus K}$ is defined as follows. $v(K, \pi_{N\setminus K})\colon\ \Pi_K \to \Re^{|K|}$, where $v(K, \pi_{N\setminus K})_i(\pi_K) = v_i(\pi_K \cup \pi_{N\setminus K})$.

LEMMA 3.4. *Let v be a valuation such that $CC(v) \neq \emptyset$, and, for any restriction v' of v, $CC(v') \neq \emptyset$. Then, for any rule of order ρ, $SECS(v, \rho) \neq \emptyset$.*

Proof. Let ρ be a fixed rule of order. I construct a stationary perfect equilibrium strategy profile. For any restriction v' of v to a subset K of the players, relative to the coalition structure $\pi_{N\setminus K}$, pick a core stable structure. This core stable structure is denoted by $CS(\pi_{N\setminus K})$, and, for any player i in K, $T(i, \pi_{N\setminus K})$ denotes the coalition player i belongs to in $CS(\pi_{N\setminus K})$.

Construct a stationary strategy profile σ as follows.

$$\sigma_i(N \setminus K, \pi_{N\setminus K}, \emptyset) = T(i, \pi_{N\setminus K})$$

$$\sigma_i(N \setminus K, \pi_{N\setminus K}, T(i, \pi_{N\setminus K})) = \text{Yes}$$

$$\sigma_i(N \setminus K, \pi_{N\setminus K}, T') = \text{Yes} \qquad \text{if } v_i(CS(\pi_{N\setminus K} \cup T')) > v_i(CS(\pi_{N\setminus K}))$$

$$\sigma_i(N \setminus K, \pi_{N\setminus K}, T') = \text{No} \qquad \text{if } v_i(CS(\pi_{N\setminus K} \cup T')) \leq v_i(CS(\pi_{N\setminus K})).$$

[13] This requirement is very similar to the condition of total balancedness for games without externalities.

To show that σ forms a subgame perfect equilibrium, consider all possible deviations for player i.

If player i proposes a coalition $T' \neq T(i, \pi_{N\setminus K})$, one of the members of T' will reject the offer, since $CS(\pi_{N\setminus K})$ is a core stable structure. Hence, player i cannot benefit from announcing a coalition different from $T(i, \pi_{N\setminus K})$. Similarly, by rejecting the offer $T(i, \pi_{N\setminus K})$, player i cannot obtain a higher payoff since the only coalition she can announce is the coalition $T(i, \pi_{N\setminus K})$.

Suppose now that player i receives an offer T' off the equilibrium path. By the same argument as in Proposition 3.2, she should accept the offer only if the payoff she receives in the final coalition structure is higher than the payoff she receives in $CS(\pi_{N\setminus K})$. The final coalition structure obtained after the formation of T', given the construction of the strategies, is the coalition structure $CS(\pi_{N\setminus K} \cup T')$. Hence, no deviation from the strategy σ_i can be profitable and the constructed strategy profile σ is a stationary perfect equilibrium. ■

Proposition 3.2 and Lemma 3.4 immediately lead to the following corollary.

COROLLARY 3.5. *Let v be a valuation such that $CC(v) \neq \emptyset$, and, for all restrictions v' of v, $CC(v') \neq \emptyset$. Then, for any rule of order ρ, $CC(v) \subset SECS(v, \rho)$.*

Lemma 3.4 provides a sufficient condition for the existence of an equilibrium coalition structure. Corollary 3.5 shows that any core stable structure of a valuation v whose restrictions also admit core stable structures can be reached as the outcome of a stationary perfect equilibrium of the game of coalition formation. In the case of α stability, no such result can be expected. The following example shows that the sets of stationary equilibrium coalition structures and of α stable structures may be nonempty and disjoint.

EXAMPLE 3.6. $N = \{a, b, c, d, e\}$.

π	$v_a(\pi)$	$v_b(\pi)$	$v_c(\pi)$	$v_d(\pi)$	$v_e(\pi)$
$ab\|cd\|e$	4	4	3	3	1
$a\|bc\|d\|e$	1	5	5	4	1
$ae\|bc\|d$	1	5	5	4	1
$a\|bc\|de$	1	1	1	5	5
$ac\|b\|de$	1	2	1	1	1
$a\|b\|c\|de$	1	2	1	1	1
Others	1	1	1	1	1

The game admits three α stable structures $\{\{a\}, \{b, c\}, \{d\}, \{e\}\}$, $\{\{ae\}, \{bc\}, \{d\}\}$ and $\{\{a\}, \{bc\}, \{de\}\}$. To check that the coalition structure $\{\{a\}, \{b, c\}, \{d\}, \{e\}\}$ is α stable, observe that the only players who have an incentive to deviate are players d and e, who may want to form a coalition. However, their deviation is prevented by the formation of the coalition structure $\{\{a, c\}, \{b\}\}$ by the three other players. The coalition structure $\{\{ae\}, \{bc\}, \{d\}\}$ is α stable for the same reason. The structure $\{\{a\}, \{bc\}, \{de\}\}$ is α stable because the only two profitable deviations can be prevented by the external players. If players a and b form the coalition $\{a, b\}$, the three other players can react by forming the structure $\{\{c\}, \{d\}, \{e\}\}$, inducing a payoff of 1 for the two deviating players. If player b decides to break the coalition with player c, the four external players can form the coalition $\{a, b, c, d\}$ which yields a payoff of 1 for player b.

These three coalition structures are the only α stable structures of the game. The coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$ is not α stable since players b, c and d can deviate and form the structure $\{\{b, c\}, \{d\}\}$ in which they are guaranteed to obtain higher payoffs. All other coalition structures are Pareto dominated by the coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$ and hence are not α stable.

I now claim that the unique stationary equilibrium coalition structure of the game, independently of the rule of order ρ, is the coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$. Two cases must be distinguished, one where ρ assigns as the first player a or b, one where c, d or e are chosen to start the game. If player a starts the game, player a should offer the formation of a coalition $\{a, b\}$. This offer will be accepted by player b, since, if player b were to form the coalition $\{b, c\}$, players d and e would form a coalition, inducing a payoff of 1 for player b. Given that players a and b have formed a coalition, player c should offer to form a coalition with player d, who will accept. Hence, in equilibrium, the coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$ is formed. The same line of reasoning applies when player b starts the game.

If now player c starts the game, she should offer the formation of the coalition $\{c, d\}$, since the offer $\{b, c\}$ will be rejected by player b. This offer will be accepted by player d. In fact, player d has no incentive to form the coalition $\{d, e\}$ since this induces player b to form the coalition $\{b\}$. Once the coalition $\{c, d\}$ is formed, players a and b form the coalition $\{a, b\}$, yielding the coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$. A similar line of reasoning applies to the cases where d and e start the game.

Example 3.6 is robust to small variations of the valuation. Hence there exists a class of valuations v, such that $SESC(v, \rho) \neq \emptyset$, $C\alpha(v) \neq \emptyset$ and $SECS(v, \rho) \cap C\alpha(v) = \emptyset$.

The absence of coincidence between the sequential game of coalition formation and the model of α stability stems from two countervailing forces in the definitions of deviations. On the one hand, deviations in the sequential model are *easier* to obtain, because the external players who have already formed a coalition cannot freely react to the deviation. This suggests that there may exist α stable structures which cannot be outcomes of subgame perfect equilibria of the game. In Example 3.6, the coalition structures $\{\{a\}, \{b, c\}, \{d\}, \{e\}\}$ and $\{\{ae\}, \{bc\}, \{d\}\}$ are not stationary equilibrium structures, because, once players a, b and c have left the game, players d and e can deviate and form the coalition $\{d, e\}$. Similarly, the coalition structure $\{\{a\}, \{bc\}, \{de\}\}$ cannot be obtained in a stationary perfect equilibrium, because b has an incentive to deviate after the coalition $\{d, e\}$ has been formed.

On the other hand, deviations in the sequential model are *harder* to obtain, because group deviations are not allowed, and players look forward to the final consequences of their deviations. Hence stationary equilibrium coalition structures are not necessarily α stable. In Example 3.6, the coalition structure $\{\{a, b\}, \{c, d\}, \{e\}\}$ is not α stable, because players b, c and d may deviate jointly and form the coalition structure $\{\{a\}, \{b, c\}, \{d\}, \{e\}\}$.

4. SEQUENTIAL FORMATION OF COALITIONS IN SYMMETRIC GAMES

In this section, I analyze the formation of coalitions in the restricted class of symmetric games. Symmetric games are described by valuations where all players are ex ante identical. Hence, the payoffs received by the players only depend on coalition sizes and not on the identity of the coalition members.

Formally, let p denote a *permutation* of the players in N. For any coalition structure π of N, let $p\pi$ denote the coalition structure obtained by permuting the players according to p. A valuation v is *symmetric* if and only if $\forall i \in N$, $v_i(\pi) = v_{pi}(p\pi)$.

A *symmetric game* is a game described by a symmetric valuation. Observe that in symmetric games all members of a coalition receive the same payoff and payoffs only depend on the sizes of the coalitions. An important feature of symmetric games is that two coalition structures which only differ by the distribution on the players in the coalitions generate the same payoff distribution. This leads to the notion of *equivalence* of coalition structures in symmetric games.

Two coalition structures π and π' are called *equivalent* if there exists a permutation p of the players in N such that $\pi' = p\pi$. Two equivalent partitions are said to be equal up to a permutation of the players. The equivalence class of a coalition structure π is denoted by $eq(\pi)$. If the valuation v is symmetric, two equivalent partitions generate the same distribution of payoffs. Hence, in symmetric games, the study of coalitions can be restricted to the study of equivalence classes of partitions. An equivalence class of partitions can be identified with a list of coalition sizes, that is a sequence of positive integers adding up to n. I assume that the rule of order ρ is fixed, and let the players be indexed by the ordered set $I = 1, 2, \ldots, n$. This can be done without loss of generality, since any coalition structure emerging as an equilibrium of the game $\Gamma(v, \rho')$, for $\rho' \neq \rho$, is equivalent to a coalition structure generated by an equilibrium of the game $\Gamma(v, \rho)$. Since the rule of order ρ is fixed, the game Γ will only be indexed by the valuation v.

Since in a symmetric game, all players are ex ante identical, I restrict my attention to symmetric equilibria where all players adopt similar strategies. Formally, a strategy profile $\sigma = \{\sigma_i\}_{i \in N}$ is called *symmetric* if and only if (i) at any two states $s = (K, \pi_K, T)$, $s' = (K, \pi_K, T')$ with $|T| = |T'| \neq 0$, for any two players $i \in T$, $j \in T'$, $\sigma_i(s) = \sigma_j(s')$ and (ii) at any state $s = (K, \pi_K, \emptyset)$, for any two players $i, j \notin K$ $|\sigma_i(s)| = |\sigma_j(s)|$. In words, a strategy profile is symmetric if, at any state, all responders adopt the same strategy and all proposers announce coalitions of the same size. The set of coalition structures supported by symmetric stationary perfect equilibria is denoted $SSECS(v)$.

I first show that, in a symmetric game, any symmetric stationary perfect equilibrium coalition structure can be reached as the outcome of a finite game of choice of coalition sizes. Furthermore, under a simple condition proposed by Ray and Vohra (1995), any equilibrium outcome of the game of choice of coalition sizes can be obtained as a symmetric stationary equilibrium coalition structure of the sequential game of coalition formation. Using this equivalence, I derive a sufficient condition under which a symmetric game admits a symmetric stationary equilibrium coalition structure and prove that this structure is generically unique.

The game of choice of coalition sizes $\Delta(v)$ is described as follows. Player 1 starts the game and chooses an integer k_1 in the interval $[1, n]$. Player $k_1 + 1$ then moves and chooses an integer k_2 in the set $[1, n - k_1]$. Player $k_1 + k_2 + 1$ chooses at the next stage an integer k_3 in the set $[1, n - k_1 - k_2]$. The game continues until the sequence of integers $(k_1, k_2, \ldots, k_j, \ldots, k_J)$ satisfies $\sum k_j = n$. The game for three players is depicted in Fig. 3.

A strategy τ_i for player i in the game $\Delta(v)$ is a mapping from the set Π_{i-1} to the set of integers in the interval $[1, n - i - 1]$. In words, for any coalition structure π_{i-1} of the first $i - 1$ players, player i chooses a coalition size $\tau_i(\pi_{i-1})$. All players need not be called to announce coalition sizes in the game. Observe, however, that, for any strategy profile τ, a single coalition structure $\pi(\tau)$ is formed. The payoffs received by the players are then given by $v_i(\pi(\tau))$.

A strategy profile τ^* is a *subgame perfect equilibrium* if and only if for all players i, for all coalition structures π_{i-1} in Π_{i-1} and for all strategies τ_i, $v_i(\pi(\tau_i^*(\pi_{i-1}), \tau_{-i}^*)) \geq v_i(\pi(\tau_i(\pi_{i-1}), \tau_{-i}^*))$. As before, a coalition structure generated by a subgame perfect equilibrium τ^* is called an *equilibrium coalition structure* of the game $\Delta(v)$. The set of equilibrium structures of $\Delta(v)$ is denoted $ECS'(v)$.

LEMMA 4.1. *For any symmetric valuation v, $ECS'(v) \neq \emptyset$.*

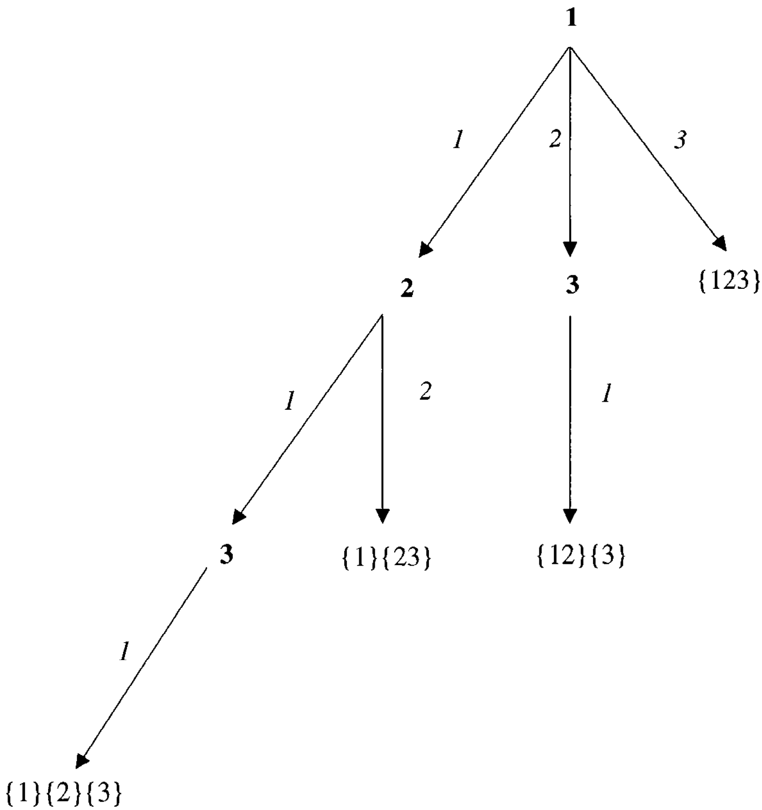

FIG. 3. The game Δ.

Proof. The game $\Delta(v)$ is a finite game of perfect information with perfect recall. Hence it admits a subgame perfect equilibrium in pure strategies. ■

In the next proposition, I show that any symmetric stationary equilibrium coalition structure of the game $\Gamma(v)$ can be reached as an equilibirum coalition structure of the game $\Delta(v)$, up to a permutation of the players.

PROPOSITION 4.2. *For any coalition structure π in $SSECS(v)$ there exists a coalition structure π' equivalent to π such that $\pi' \in ECS'(v)$.*

Proof. Let σ be the symmetric stationary perfect equilibrium of the game $\Gamma(v)$ supporting the coalition structure π. I first show that the strategy profile σ cannot involve any delay and that all offers prescribed by σ are accepted. Suppose to the contrary that some player i rejects an offer T with

$|T| \geq 2$ at some state $s = (K, \pi_K, T)$. Since the strategy profile σ is symmetric, $|\sigma_i(K, \pi_K, \emptyset)| = |T|$ and for all players $j \in \sigma_i(K, \pi_K, \emptyset)$, we have $\sigma_j(K, \pi_K, \sigma_i(K, \pi_K, \emptyset)) = \sigma_i(K, \pi_K, T) = \text{No}$. Hence offers are continuously rejected and the play of the game is infinite yielding a payoff of 0 to player i. Since however $\min_{\pi \supset \{\{i\}\}} v_i(\pi) > 0$, player i has an incentive to deviate and leave the game. This shows that, at a symmetric equilibrium σ, all offers are accepted. Hence the strategy σ can be described by a list of offers made by the players at all states where they are proposers.

As a second step, I show that we can assume without loss of generality that, at any two equivalent states $s = (K, \pi_K, \emptyset)$ and $s' = (K', \pi'_K, \emptyset)$ where $|K| = |K'|$ and the two coalition structures π_K and $\pi_{K'}$ are equivalent, $|\sigma_i(s)| = |\sigma_i(s')|$. To see this first reorder the players according to a rule of order $\hat{\rho}$ consistent with the order in which the coalition structure π is formed. Now, for any set K with $i \notin K$, let $\hat{K}$ denote the first $\hat{\rho}$-elements in $N \setminus \{i\}$ and, for any partition π_K of K, let $\hat{\pi_K}$ denote the equivalent partition of $\hat{K}$. Construct then the strategy $\hat{\sigma_i}$ as follows. At any state $s = (K, \pi_K, \emptyset)$, let $\hat{\sigma_i}(s)$ be a subset of $N \setminus K$ containing i such that $|\hat{\sigma_i}(K, \pi_K, \emptyset)| = |\sigma(\hat{K}, \hat{\pi_K}, \emptyset)|$. In words, I select for any state $s = (K, \pi_k, \emptyset)$ a particular representative of the equivalence class $eq(\pi_K)$ and $\hat{\sigma_i}$ assigns the action chosen for this representative state to the entire equivalence class. Clearly, the strategy $\hat{\sigma}$ satisfies the condition that sets of the same cardinality are chosen at two equivalent states. Furthermore, given the particular order $\hat{\rho}$ chosen, $\pi(\hat{\sigma}) = \pi(\sigma)$. It remains to show that $\hat{\sigma}$ forms a subgame perfect equilibrium of the game $\Gamma(v)$. To see this, consider a state $s = (K, \pi_K, \emptyset)$ and note that, since the strategy $\hat{\sigma}$ is played, any action of player i induces a unique partition of the set $N \setminus K$. Now suppose by contradiction that $\hat{\sigma_i}$ is not an optimal choice, i.e. that there exists a strategy $\tilde{\sigma_i}$ inducing a partition $\pi_{\tilde{N \setminus K}}$ such that $v_i(\pi_K \cup \pi_{\tilde{N \setminus K}}) > v_i(\pi_K \cup \pi_{\hat{N \setminus K}})$ where $\pi_{\hat{N \setminus K}}$ is the coalition induced by $\hat{\sigma_i}$. Next consider a permutation $\hat{p}$ of the players such that $\hat{p}K = \hat{K}$. Since the game is symmetric, $v_i(p(\pi_K \cup \pi_{\tilde{N \setminus K}})) = v_i(\pi_K \cup \pi_{\tilde{N \setminus K}}) > v_i(\pi_K \cup \pi_{\hat{N \setminus K}}) = v_i(p(\pi_K \cup \pi_{\hat{N \setminus K}}))$, contradicting the fact that σ_i is an optimal choice at $(\hat{K}, \hat{\pi_K}, \emptyset)$.

Since we may assume, by the preceding step, that the strategy σ assigns sets of the same cardinality at any two equivalent states, we are ready to construct a strategy profile τ in the game $\Delta(v)$ as follows. For any player i and any coalition structure π_{i-1} of the preceding players, let $\tau_i(\pi_{i-1}) = |\sigma_i(K, \pi_K, \emptyset)|$. To show that τ forms a subgame perfect equilibrium of the game $\Delta(v)$, suppose by contradiction that player i has a profitable deviation $\tau'_i \neq \tau_i$ after the coalition structure π_{i-1} is formed. I claim that this implies that player i has a profitable deviation from σ_i in the game $\Gamma(v)$. To see this, suppose that a coalition structure π_K equivalent to π_{i-1} has been formed and let player i reject any offer T such that $|T| \neq \tau'_i$ and propose the formation of a coalition T' of size τ'_i. Since τ'_i is a profitable deviation in the game $\Delta(v)$ and letting π' denote the coalition structure induced by the choice τ_i', we must have $v_i(\pi') > v_i(\pi)$. Now, by symmetry, for all players j in T', $v_j(\pi') = v_i(\pi') > v_i(\pi) = v_j(\pi)$, so that player i's offer is accepted. ■

While Proposition 4.2 guarantees that any symmetric equilibrium can be obtained as an equilibrium outcome of the game of choice of coalition sizes, it does not imply that the equilibrium coalition structures of the game Δ form symmetric stationary equilibrium outcomes of the sequential game of coalition formation. In fact, as noted by Ray and Vohra (1995), a stronger condition is needed for this assertion to hold : the coalitions formed in the game Δ must have the property that the players' payoffs are decreasing in the order in which coalitions are formed.

PROPOSITION 4.3 (Ray and Vohra (1995)). *Let π be an equilibrium coalition structure of the game $\Delta(v)$ with the property that players' payoffs are decreasing in the order in which coalitions are formed. Then there exists a coalition structure π' equivalent to π such that $\pi' \in SSECS(v)$.*

Proof. Let τ be the subgame perfect equilibrium supporting τ. Define a strategy σ_i for player i in the game $\Gamma(v)$ as follows. At any state $s = (K, \pi_K, \emptyset)$ let player i announce a subset T of $N \setminus K$ with $|T| = \tau_j(\pi_{j-1})$ for the coalition structure π_{j-1} equivalent to π_K. At any state $s = (K, \pi_K, T)$ with $T \neq \emptyset$, let $\sigma_i(s) =$ Yes if $|T| = \tau_j(\pi_{j-1})$ and $\sigma_i(s) =$ No otherwise. This strategy profile is symmetric and yields a coalition structure $\pi(\sigma)$ equivalent to π. It remains to show that it forms a stationary perfect equilibrium of the game $\Gamma(v)$. First consider player i's possible deviation at a state $s = (K, \pi_K, \emptyset)$ when it is her turn to make an offer. If she makes any offer T' such that $|T'| \neq \tau_j(\pi_{j-1})$ and $|T'| \geq 2$, her offer will be rejected. Hence player i will belong to a coalition formed later in the game and, by assumption, her payoff is lower than the one she obtains in coalition T. By the same reasoning, player i has no incentive to reject an offer T where $|T| = \tau_j(\pi_{j-1})$. Finally, consider player i's response to an offer T' with $|T'| \neq \tau_j(\pi_{j-1})$. By rejecting the proposal and offering to form a coalition T of size $|T| = \tau_j(\pi_{j-1})$, she can secure the formation of the coalition structure π. Since τ is a subgame perfect equilibrium of the game of choice of coalition sizes, $v_i(\pi) \geq v_i(\pi_K \cup \tilde{\pi}_{N\setminus K})$ for any other coalition structure $\tilde{\pi}_{N\setminus K}$ induced by the formation of a coalition T' at state $s = (K, \pi_K, \emptyset)$. Hence no player has any incentive to deviate from the strategy prescribed by σ. ■

Propositions 4.2 and 4.3 provide a sufficient condition on the underlying valuation v for the equivalence between the symmetric stationary perfect equilibrum outcomes of the sequential game of coalition formation and the subgame perfect equilibrium outcomes of the game of choice of coalition sizes. This result is formally stated in the next corollary.

COROLLARY 4.4. *Suppose that, in the game $\Delta(v)$, players' payoffs are decreasing in the order in which coalitions are formed. Then, for any coalition structure π in $SSECS(v)$ and any coalition structure π' in $ECS'(v)$, $eq(\pi) = eq(\pi')$.*

Hence, under a simple condition, the game of choice of coalition sizes provides an easy method for the construction of equilibrium coalition structures in symmetric games. The exact nature of the restriction that players' payoffs are decreasing in the order in which coalitions are formed is difficult to interpret. Ray and Vohra (1995) provide an example where the condition is violated and the subgame perfect equilibrium outcome of the game of choice of coalition sizes does not form a symmetric stationary perfect equilibrium of the sequential game. However, in most economic applications of coalitions with externalities, including the formation of cartels and of coalitions in majority games discussed in this paper, this condition is satisfied. The equivalence result of Corollary 4.4 can now be used to establish several important properties of equilibrium coalition structures in symmetric games.

COROLLARY 4.5. *Let v be a symmetric valuation such that, in the game $\Delta(v)$, players' payoffs are decreasing in the order in which coalitions are formed. Then $SECS(v) \neq \emptyset$.*

Proof. Follows from Lemma 4.1 and Corollary 4.4. ■

Corollary 4.4 also leads to a simple sufficient condition for the uniqueness of symmetric stationary equilibrium coalition structures in symmetric games. A valuation v is called *strict* if, for any player i, and for any two different partitions π and π', $v_i(\pi) \neq v_i(\pi')$. In a game described by a strict valuation, every agent receives different payoffs in different coalition structures. The next proposition shows that the strictness condition is sufficient to guarantee the uniqueness of the equilibrium coalition structure in the game $\Delta(v)$.

PROPOSITION 4.6. *Let v be a strict symmetric valuation. Then the game $\Delta(v)$ has a unique equilibrium coalition structure.*

Proof. The proof is by induction on the number n of players. If $n = 1$, the game $\Delta(v)$ has a unique subgame perfect equilibrium. Suppose now that, for any $n' < n$, the game admits a unique subgame perfect equilibrium, and consider the first player's choices in a game with n players. For any choice of an integer k, the continuation game has less than n players, and thus admits a unique subgame perfect equilibrium $\tau^*(\{k\})$. Since the valuation is strict, there exists a unique k^*, such that

$$v_1(\{k^*\} \cup \pi(\tau^*(\{k^*\})) > v_1(\{k\} \cup \pi(\tau^*(\{k\})) \qquad \forall k \neq k^*.$$

Hence the n player game admits a unique subgame perfect equilibrium. ■

Proposition 4.6 implies that, if the valuation is strict, all equilibrium coalition structures of the game $\Gamma(v)$ are equivalent. Hence I obtain the following corollary.

COROLLARY 4.7. *Let v be a strict symmetric valuation such that, in the game $\Delta(v)$, players' payoffs are decreasing in the order in which coalitions are formed. Then the game $\Gamma(v)$ has a unique symmetric stationary equilibrium coalition structure, up to a permutation of the players.*

5. APPLICATIONS

In this section, I apply the sequential model of coalition formation to two particular symmetric situations. I first analyze the formation of cartels in a symmetric Cournot oligopoly. The second application is based on Hart and Kurz (1984)'s study of endogenous coalition formation in symmetric majority games. In both applications, I derive the subgame perfect equilibrium of the game of choice of coalition sizes. It is straightforward to check that players' payoffs are decreasing in the order in which coalitions are formed, so that the equivalence result of Corollary 4.4 can be applied.

5.1. *Cartels in a Symmetric Cournot Oligopoly*

It has long been noted that the formation of cartels in oligopolies involves a fundamental instability (See Stigler (1968)), since, once a cartel has been formed, members of the cartel obtain a lower profit than outsiders, and hence have an incentive to leave the cartel. Salant *et al*. (1983) analyze this instability in a simple symmetric Cournot oligopoly with linear demand and homogeneous goods, and show that there exists a minimal profitable size of the cartel which is never lower than four fifths of the members of the industry. This cartel is however (intuitively) unstable since members of the cartel would prefer to stay out and let the other firms form a cartel. In the sequential model analyzed here, firms have the power to commit to stay out of the cartel. Hence, the unique equilibrium coalition structure predicts that firms choose to remain outside of the cartel, until the remaining firms form the cartel of minimal profitable size.

More precisely, consider a Cournot oligopoly where firms face a linear inverse demand curve, $D = \alpha - \sum q_i$, where q_i is the quantity produced by each firm i. All firms are assumed to have a constant marginal cost of λ. Suppose that K cartels have formed on the market, and that the structure of cartels is given by $\pi = \{T_1, T_2, \dots, T_k, \dots, T_K\}$. Straightforward computations show that, in equilibrium, each cartel will produce $q_i^*(\pi) = (\alpha - \lambda)/(K + 1)$.[14] Hence, firm

[14] It is important to note that the equilibrium quantity produced by each cartel only depends on the number of cartels on the market.

i in the cartel $T(i)$ of size $t(i)$ obtains a payoff of

$$P_i^*(\pi) = \frac{(\alpha - \lambda)^2}{t(i)(K+1)^2}.$$

The problem of cartel formation can thus be summarized by the valuation defined by $v_i(\pi) = P_i^*(\pi)$.

PROPOSITION 5.1. *Any equilibrium of the game of cartel formation is characterized by* $\pi^* = (T_1^*, \{j\}_{j \notin T_1^*})$ *where* t_1^* *is the first integer following* $(2n + 3 - \sqrt{4n+5})/2$. *(If* $\sqrt{4n+5}$ *is an integer,* t_1^* *can take on the two values* $(2n+3-\sqrt{4n+5})/2$ *and* $(2n+5-\sqrt{4n+5})/2$.)

Proof. See the Appendix. ■

5.2. *Coalitions in Symmetric Majority Games*

In their study of endogenous coalition formation, Hart and Kurz (1983) advocate a two-stage approach, where players evaluate their payoffs, in any coalition structure, according to a fixed rule (Owen (1977)'s extension of the Shapley Value to games with coalition structures), and play a game of coalition formation using the value as their expected payoff. Owen (1977)'s value differs from Aumann and Drèze (1974)'s value in assuming that players bargain over the worth of the grand coalition, as opposed to the worth of the coalition they belong to in the coalition structure. The formation of a coalition is thus interpreted as a way for the players to modify the environment in which they bargain over the worth of the grand coalition.[15]

Owen (1977)'s value is computed, for any game in coalitional function form w, any coalition structure π and any player i as

$$\phi_i(w, \pi) = E(w(\mathcal{P} \cup i) - w(\mathcal{P})),$$

where the expectation is taken over any random order which is consistent with the coalition structure π (i.e. ranks consecutively members of any coalition in the coalition structure) and $\mathcal{P}$ is the set of predecessors of i according to the random order.

Hart and Kurz (1984) apply Owen (1977)'s value to analyze the formation of coalitions in different types of games in coalitional function form. We consider here only symmetric majority games.

DEFINITION 5.2. A symmetric majority game $M(n, m)$ is defined as follows. The number n is the total number of players, and the integer m (the majority) is

[15] The axiomatic derivation of the two different values are given in Aumann and Drèze (1974) and Hart and Kurz (1983). The differences are thouroughly discussed in Hart and Kurz (1983).

any integer in the interval $[(n+1)/2, n]$. The coalitional function is given by

- $w(T) = 0$ if $t < m$
- $w(T) = 1$ for $t \geq m$,

where T is any coalition, and t denotes the cardinality of coalition T.

To compute the Owen value in the symmetric majority game $M(n, m)$, let me consider a coalition structure π containing K coalitions, $\pi = \{T_1, T_2, \ldots, T_k, \ldots T_K\}$. The total number of random orders consistent with the coalition structure π is $K!t_1!t_2!\ldots t_k!\ldots t_K!$, where t_k denotes the number of elements of the coalition T_k. It is then clear that for the incremental value of player i to be positive, it must be that player i is ordered at position m in the random order. Denoting by $T(i)$ the coalition player i belongs to and letting $\omega_i(\pi)$ denote the number of orderings of the coalitions in π such that a member of the coalition $T(i)$ is at position m, I obtain the following simple expression for the Owen value

$$\phi_i(\pi) = \frac{\omega_i(\pi)}{t(i)K!}.$$

Hence I can now define the valuation $v_i(\pi) = \phi_i(\pi)$. The characterization of the equilibrium coalition structures is made difficult by the lack of structure of the function $\omega_i(\pi)$. In the absence of general characterization results, Table I describes the equilibrium coalition structures of any symmetric majority game with $n \leq 10$.[16]

The equilibrium coalition structures of symmetric majority games are not easily interpreted. When the majority required to win (m) is small, it appears that the minimal winning coalition forms, members of the winning coalition all obtain $1/m$ and external members, who obtain 0, organize themselves freely. When the number of votes required to win increases, the share of any member of the winning coalition decreases and it may become profitable to form smaller coalitions. This effect explains why the minimal winning coalition does not necessarily form in the symmetric majority games $M(5, 4)$, $M(6, 5)$, $M(7, 6)$, $M(8, 6)$, $M(9, 7)$, $M(9, 8)$ and $M(10, 8)$. However, if all votes are required to win, the only equilibrium coalition structures are the grand coalition and the coalition consisting of singletons. In fact, in that case, the probability to win is independent of the size of the coalition, and players should always try to form the smallest coalitions. Hence, the only possible equilibrium coalition structure are the coalition consisting of singletons and the grand coalition which yield the same payoff of $1/n$ to all players. Finally, it should be noted that Hart and Kurz (1984) observed that the majority game $M(10, 8)$ has no α stable coalition structure. However, in my framework, an equilibrium coalition structure exists for this game.

[16] The computations leading to the characterization of the coalition structures are not reproduced here and are available from the author.

TABLE I. Coalition Structures in Symmetric Majority Games

$n = 3$

$m = 2$	$m = 3$
$ab\|c$	$a\|b\|c$
	abc

$n = 4$

$m = 3$	$m = 4$
$abc\|d$	$a\|b\|c\|d$
	$abcd$

$n = 5$

$m = 3$	$m = 4$	$m = 5$
$abc\|d\|e$	$ab\|cd\|e$	$a\|b\|c\|d\|e$
$abc\|de$	$abcd\|e$	$abcde$

$n = 6$

$m = 4$	$m = 5$	$m = 6$
$abcd\|e\|f$	$abc\|de\|f$	$a\|b\|c\|d\|e\|f$
$abcd\|ef$		$abcdef$

$n = 7$

$m = 4$	$m = 5$	$m = 6$	$m = 7$
$abcd\|e\|f\|g$	$abcde\|f\|g$	$abcdef\|g$	$a\|b\|c\|d\|e\|f\|g$
$abcd\|ef\|g$	$abcde\|fg$	$ab\|cd\|ef\|g$	$abcdefg$
$abcd\|efg$		$abc\|def\|g$	

6. CONCLUSIONS

In this paper, I analyze a sequential noncooperative game of coalition formation when the rule of payoff division is fixed and payoffs depend on the whole coalition structure. The extensive form of the game is closely related to the extensive forms proposed by Selten (1981), Chatterjee *et al*. (1993) and Moldovanu (1992) for games of coalitional bargaining. I show that any core stable structure can be obtained as the outcome of a stationary perfect equilibrium, provided that the set of stationary perfect equilibria is nonempty. I analyze games described by symmetric valuations and provide a condition under which, when all the players

TABLE I. *Continued.*

$n = 8$

$m = 5$	$m = 6$	$m = 7$	$m = 8$
$abcde \mid f \mid g \mid h$	$abcdef \mid g \mid h$	$abcdefg \mid h$	$a \mid b \mid c \mid d \mid e \mid f \mid g \mid h$
$abcde \mid fg \mid h$	$abcdef \mid gh$		$abcdefgh$
$abcde \mid fgh$	$abc \mid def \mid g \mid h$		
	$abc \mid def \mid gh$		

$n = 9$

$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$
$abcde \mid f \mid g \mid h \mid i$	$abcdef \mid g \mid h \mid i$	$abcd \mid efg \mid hi$	$ab \mid cd \mid ef \mid gh \mid i$	$a \mid b \mid c \mid d \mid e \mid f \mid g \mid h \mid i$
$abcde \mid fg \mid h \mid i$	$abcdef \mid gh \mid i$	$abcd \mid efg \mid h \mid i$	$abcdefgh \mid i$	$abcdefghi$
$abcde \mid fg \mid hi$	$abcdef \mid ghi$			
$abcde \mid fgh \mid i$				
$abcde \mid fghi$				

$n = 10$

$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
$abcdef \mid g \mid h \mid i \mid j$	$abcdefg \mid h \mid i \mid j$	$abcdefgh \mid i \mid j$	$abcdefghi \mid j$	$a \mid b \mid c \mid d \mid e \mid f \mid g \mid h \mid i \mid$
$abcdef \mid gh \mid i \mid j$	$abcdefg \mid hi \mid j$	$abcdefgh \mid ij$		$abcdefghij$
$abcdef \mid gh \mid ij$	$abcdefg \mid hij$	$abcd \mid efg \mid h \mid i \mid j$		
$abcdef \mid ghi \mid j$		$abcd \mid efgh \mid ij$		
$abcdef \mid ghij$				

are identical ex ante, the game admits a symmetric equilibrium coalition structure which is generically unique up to a permutation of the players. I also provide examples to show that stationary perfect equilibria may fail to exist in general valuations and that the noncooperative approach followed here is unrelated to standard cooperative game-theoretic solution concepts.

The determination of the equilibrium coalition structure in the sequential game of coalition formation is driven by two basic features of the extensive form. First, the exogenous rule of order imposes a fixed order of moves by players in the game. Depending on the valuation, players may have an advantage in moving first, second or in any other position in the game. The rule of order thus creates an asymmetry among players which is determined outside the game. An important direction for future research is to eliminate this asymmetry and to explore conditions under which the equilibrium of the extensive form game is independent of the rule of order. This line of research has been pursued by Moldovanu and Winter (1995) in the context of games of coalitional bargaining. They show that order independent equilibria only exist when the underlying game in characteristic function form, as well as all its restrictions, have nonempty cores.

The second important feature of the extensive form is the commitment power of the players. I assume that, by accepting the offer to join a coalition, players are bound to remain in that coalition whatever coalition structure the other players may form. This implies that coalitions are formed one after another and that coalitions may not compete to attract members. In fact, this sequential structure of the process of coalition formation is the feature of the extensive form which guarantees the existence of an equilibrium. Extensive form games where players do not commit to stay in a coalition can easily be constructed. The existence and characterization of equilibria in these games constitutes a difficult but important area for future research.

Finally, the model analyzed in this paper assumes that the coalitional worth is divided according to a fixed sharing rule. While this approach greatly simplifies the analysis, it clearly restricts the applicability of the model. The study of extensive form procedures allowing players to bargain over the worth of coalitions seems to me to be the foremost topic for future research.

APPENDIX: PROOF OF PROPOSITION 5.1

The proof consists in three steps. In the first two steps, I explicitly construct the stationary perfect equilibria of the game. Observe first that the only payoff-relevant part of any history of the game is the *number of coalitions which have already been formed*. To fix notations, let K be the number of coalitions already formed, and m be the number of remaining players in the game, after a given history.

Step 1. After a given history, suppose that K coalitions have been formed, and that m players remain in the game. Suppose furthermore that, if a coalition of size μ is formed, the remaining $m - \mu$ players remain isolated. Then the optimal choice of μ is given by:

$$\mu^* = 1 \qquad \text{if} \quad m \leq (K+1)^2$$

$$\mu^* = m \qquad \text{if} \quad m \geq (K+1)^2$$

Given that the remaining $m - \mu$ players form singletons, the optimal number of players in a coalition, μ^*, solves:

$$\max F(\mu) = \frac{(\alpha - \lambda)^2}{\mu(K + m - \mu + 2)^2}$$

$$\text{subject to } 1 \leq \mu \leq m.$$

The function $1/\mu(K+m-\mu+2)^2$ is strictly decreasing for $1 \leq \mu \leq (K+m+2)/3$, and strictly increasing for $(K+m+2)/3 \leq \mu \leq m$. Hence, the optimal choice μ^* is either 1 or m. Now,

$$F(1) = \frac{(\alpha - \lambda)^2}{(K+m+1)^2}$$

$$F(m) = \frac{(\alpha - \lambda)^2}{m(K+2)^2}.$$

Solving the quadratic in m, I obtain:

$$F(1) \leq F(m) \text{ if and only if } m \geq (K+1)^2.$$

Step 2. The game admits two stationary perfect equilibria, given by *Strategy* 1.

If $m < (K+1)^2$	choose $\mu = 1$
If $(K+1)^2 \leq m < (K+2)^2 + 1$	choose $\mu = m$
If $(K+2)^2 + 1 \leq m$	choose $\mu = 1$

Strategy 2.

If $m \leq (K+1)^2$	choose $\mu = 1$
If $(K+1)^2 < m \leq (K+2)^2 + 1$	choose $\mu = m$
If $(K+2)^2 + 1 < m$	choose $\mu = 1$.

The two equilibria only differ in the rules chosen to break ties. In the first equilibrium, if a player is indifferent between forming a cartel of size m or forming a singleton, she chooses to form a cartel. In the second equilibrium, she chooses to remain isolated. In the remainder of the proof, I focus on strategy 1, and show that *given that ties are broken according to the rule that indifferent players choose to form coalitions*, strategy 1 is the unique stationary perfect equilibrium of the game.

The proof is by induction on the number of remaining players in the game. If $m = 2$, the player before last chooses whether to form a cartel of size 2 or to remain isolated, in which case the last player remains isolated as well. Since $K \geq 0$, $2 < (K+2)^2 + 1$. Hence strategy 1 prescribes that a cartel is formed if and only if $2 \geq (K+1)^2$, and by Step 1, this is the unique optimal strategy for the player before last.

Suppose now that, for any $m' < m$, strategy 1 is the unique equilibrium strategy. Consider the different possibilities with m players remaining in the game.

If $m < (K+1)^2$, then $\forall m' < m, m' < (K+2)^2$. Hence, whatever coalition the player forms, all subsequent players choose to remain isolated. Then, by Step 1, the unique optimal strategy is to choose to form a singleton.

If now $(K+1)^2 \leq m < (K+2)^2 + 1$, similarly, $\forall m' < m, m' < (K+2)^2$. Hence, irrespective of the coalition formed by the player, the subsequent players choose to remain isolated and, by Step 1, since $m \geq (K+1)^2$, the player should choose to form a coalition of size m.

Finally, when $m \geq (K+2)^2 + 1$, different possibilities have to be considered. The player may either choose to form a coalition μ such that $(m-\mu) \geq (K+2)^2$, in which case the remaining players form a coalition, or a coalition μ such that $(m-\mu) < (K+2)^2$, in which case the remaining players choose to remain separate.

When the coalition size μ is such that $(m-\mu) < (K+2)^2$, the player's payoff is given by:

$$F(\mu) = \frac{(\alpha-\lambda)^2}{\mu(K+2+m-\mu)^2}.$$

From Step 1, since $m > (K+1)^2$, the optimal choice of coalition size is $\mu^* = m$.

In the case where μ is chosen small enough, other players form a coalition later in the game. Given the specification of the strategy, after the formation of the coalition of size μ, a group of players will choose to remain separate, and the last players will form a single coalition. The number of players who choose to remain isolated, ν, is the unique integer satisfying:

$$(K+2+\nu)^2 \leq (m-\mu-\nu) < (K+3+\nu)^2 + 1.$$

A simple computation shows that ν is the first integer following:

$$\nu^* = \frac{\sqrt{9+4(K+m-\mu)} - (2K+5)}{2}.$$

Hence, the payoff to a player who chooses a coalition of size μ where $m-\mu \leq (K+2)^2$ is given by:

$$G(\mu) = \frac{(\alpha-\lambda)^2}{\mu(K+3+\nu^*)^2},$$

or

$$G(\mu) = \frac{(\alpha-\lambda)^2}{\mu(\sqrt{9+4(K+m-\mu)}+1)^2}.$$

The optimal value μ^* is thus the minimum over the interval $[1, m-(K+2)^2]$ of the function

$$H(\mu) = \mu(\sqrt{9+4(K+m-\mu)}+1)^2.$$

Next consider the derivative H' of H,

$$\begin{aligned} H'(\mu) &= (\sqrt{9+4(K+m-\mu)}+1)(\sqrt{9+4(K+m-\mu)} \\ &\quad + 1 - \frac{4\mu}{\sqrt{9+4(K+m-\mu)}}. \end{aligned}$$

A study of the sign of H' shows that the function H is increasing up to the value $\mu = (16K+16m+35+\sqrt{73+32K+32m})/32$, and decreasing thereafter. Hence, the optimal choice of μ, μ^*, is either $\mu^* = 1$, or $\mu^* = m-(K+2)^2$. Now, a simple computation shows that the choice $\mu^* = m-(K+2)^2$ is dominated by $\mu^* = m$.

To complete this step, it suffices to show that $\mu^* = 1$ is the optimal choice, that is that $H(1) \leq m(K+2)^2$.

$$\begin{aligned} H(1) &= \tfrac{1}{4}(\sqrt{9+4(K+m-1)}+1)^2 \\ &= \tfrac{1}{2}(3+2K+2m+\sqrt{9+4(K+m-1)}) \\ &< \tfrac{1}{2}(3+2K+2m+9+4(K+m-1)) \\ &< (3K+3m+4). \end{aligned}$$

Hence,

$$\begin{aligned} m(K+2)^2 - H(1) &> m(K+2)^2 - (3K+3m+4) \\ &> m(K^2+4K+1) - 3K - 4 \\ &> (K+2)^2(K^2+4K+1) - 3K - 4 \\ &> 0. \end{aligned}$$

Step 3. The coalition structure generated by the stationary perfect equilibria corresponding to strategies 1 and 2 is given by $\pi^* = (T_1^*, \{j\}_{j \notin T_1^*})$ where t_1^* is the first integer following $(2n+3-\sqrt{4n+5})/2$. (If $\sqrt{4n+5}$ is an integer, t_1^* can take on the two values $(2n+3-\sqrt{4n+5})/2$ and $(2n+5-\sqrt{4n+5})/2$.)

When $K = 0$, strategy 1 prescribes that the first player forms a singleton. In fact, singletons will continue to be formed as long as $m \geq (n-m+2)^2+1$. The unique coalition formed will comprise t^* members, where t^* is the unique integer such that

$$(n-t^*+1)^2 \leq t^* < (n-t^*+2)^2+1.$$

It is easy to show that t^* is the first integer following $((2n+3) - \sqrt{4n+5})/2$. The only possible difficult arises when there exist two integers, t_1^* and t_2^* such that $t_1^* = (n - t_1^* + 1)^2$ and $t_2^* = (n - t_2^* + 2)^2 + 1$. Then, strategy 1 prescribes that a coalition of size t_1^* is formed whereas strategy 2 induces a coalition of size t_2^*. ■

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