Quantifiers

Quantified sentences in English

- Basic sentences are made by combing a *noun phrase* (NP) with a *verb phrase* (VP).
- Names are noun phrases; but there are others: *all cats*, *some dogs*, *the teacher*, *some black dogs who chase cats*, *most Americans, something, everything*....
  - Words like 'all', 'some', 'the', 'most' are called *determiners*.

- The syntax of FOL works very differently. The work done by determiners (and various other forms of expression) in English is all done by just two symbols, $\forall$ and $\exists$, which correspond roughly to 'Everything' and 'Something', together with *variables* like 'x', 'y', 'z', which correspond roughly to *pronouns* in English.

- Some examples:
  - $\forall x$Meaningless(x) means ‘For every object x, x is meaningless’, or more colloquially, ‘Everything is meaningless’.
  - $\exists x$Omnipotent(x) means ‘For some object x, x is omnipotent’ — ‘Something is omnipotent’ — ‘There is an omnipotent thing’.
  - $\exists x$(Dog(x)$\wedge$Omnipotent(x)) means ‘For some object x, x is a dog and x omnipotent’ — ‘Something is both a dog and omnipotent’ — ‘Some dog is omnipotent’
  - $\forall x$(Man(x)$\rightarrow$Mortal(x)) means ‘For any object x, if x is a man, then x is mortal’ — ‘For any object x, either x isn’t a man or x is mortal’ — ‘All men are mortal’.

Syntax

- So far, we've been looking at sentences that are built up from atomic sentences.
  - But $\exists$x(Dog(x)$\wedge$Omnipotent(x)) is not built up from atomic sentences. 'Dog(x)' is not a sentence at all: the symbol x is a *variable*, not a *name*. 'Dog(x)' is not the sort of thing that can be true or false.
  - Expressions like 'Dog(x)' and '(Dog(x)$\wedge$Omnipotent(x)) are called *well-formed formulas* or *wffs*. All sentences are wffs, but not all wffs are sentences.

- Now that we’ve introduced the quantifiers, we’re in a position to give a precise account of the syntax of FOL. Let’s first deal with the kind of language that doesn’t contain any function symbols.
    - A *variable* is one of the letters t, u, v, w, x, y, z, with or without a numerical subscript.
    - A *term* is a variable or an individual constant.
    - An *atomic wff* consists of an n-ary predicate together with a list of *n* terms, separated by commas and surrounded by parentheses.

- We define the notion of wff as follows:
  1. If P is a wff, ¬P is a wff
  2. If P1... Pn are all wffs, (P1∧...∧Pn) and (P1∨...∨Pn) are both wffs.
  3. If P and Q are wffs, (P→Q) and (P↔Q) are both wffs.
  4. If P is a wff and *v* is a variable, then ∀vP and ∃vP are both wffs, and all occurrences of v inside P are said to be *bound*.
  5. Nothing else is a wff.
- A *sentence* is a wff that contains no free (unbound) variables.
  - By convention, we can leave off the outermost parentheses.

- Which of the following wffs are sentences?
  - $\exists x Dog(x)$
  - $\exists x Dog(x) \wedge Omnipotent(x)$
  - $\exists x (Dog(x) \wedge Omnipotent(x))$
  - $\forall x (Dog(x) \rightarrow \exists y (Flea(y) \wedge IsOn(y,x)))$
  - $\forall x (Dog(fido))$

Semantics for the quantifiers

- In different versions of FOL, the quantifiers have different *domains*. For example, in the language of arithmetic, the domain of the quantifiers is the natural numbers, so $\forall x(\mathrm{Even}(x) \lor \mathrm{Odd}(x))$ is true.
- In the blocks language of Tarski's World, the domain of the quantifiers comprises the blocks in the given world.
- Playing the game.

For next week

- Read: chapter 9; optionally, chapters 10 and 11.
- Do: exercises 8.31, 8.33, 8.34 and 8.37 (don't forget to look back at the informal proofs you gave in last week's homework); 8.26 - 8.28 (you may use Taut Con to justify an instance of Excluded Middle); 9.1, 9.2, 9.6. (10% per exercise.)